Sufficient Conditions for Hamiltonian Graphs

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Abstract

Hamiltonian graphs are named after Sir, Willian Rowan Hamilton, an Irish mathematician, who introduced the problems of finding a cycle in which all vertices of a graph appear exactly once except for the starting and ending vertex that appears twice. In this paper, sufficient conditions for a graph to be Hamiltonian graph are illustrated. Moreover, some applications on Hamiltonian Graph are studied.

Key words: path, cycle, Hamilton path. Hamilton cycle, spanning cycle, sufficient condition.

Introduction

A path that contains every vertex of G is called a Hamilton path of G; similarly a Hamilton cycle of G is a cycle that contains every vertex of G. Such paths and cycles are named after Hamilton (1856), who described, in a letter to his friend Graves, a mathematical game on the dodecahedron in which one person sticks five pins in any five consecutive vertices and the other is required to complete the path so formed to a spanning cycle. A graph is hamiltonian if it contains a Hamilton cycle. The dodecahedron is Hamiltonian; the Herschel graph is nonhamiltonian, because it is bipartite and has odd number of vertices. Although no useful and sufficient conditions for the existence of Hamilton cycle are known, quite a few sufficient conditions have been found. If a graph G with $\nu \geq 3$ vertices and the clousure of G, c(G), is complete, then G is Hamiltonian. Now, we will discuss the sufficient conditions found by Dirac in 1952, Ore in 1962 and Chvatal in 1972.

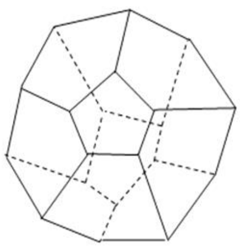

Figure1 The dodecahedron

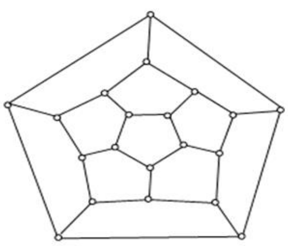

Figure 2 The corresponding graph

Some Theorems on the Sufficient Conditions for a Hamiltonian Graph

We now discuss sufficient conditions for a graph G to be hamiltonian, since a graph is hamiltonian if and only if its underlying graph is hamiltonian, it suffices to limit our discussion to simple graph. We start with a result due to Dirac (1952).

Theorem (1)Dirac (1952). If G is a simple graph with $\upsilon \geq 3$ and $\delta \geq \frac{\upsilon}{2}$, then G is hamiltonian.

Proof. By contradiction. Suppose that the theorem is false, and let G be a maximal nonhamiltonian simple graph with $\nu \geq 3$ and $\delta \geq \frac{\upsilon}{2}$. Since $\nu \geq 3$, G cannot be complete. Let u and v be nonadjacent vertices in G.

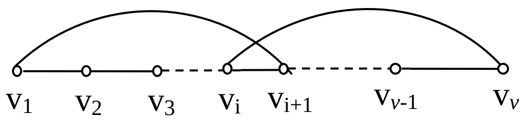

Figure 3 A Hamilton Graph

By the choice of G, G + uv is hamiltonian. Moreover, since G is nonhamiltonian, each Hamilton cycle of G + uv must contain the edge uv. Thus there is a Hamilton path $v_1v_2\ldots v_\nu$ in G with origin $u = v_1$ and terminus $v = v_\upsilon$. Set

$S = \{v_i \mid uv_{i+1} \in E\}$ and $T = \{v_i \mid v_i v \in E\}$.

Since $v_\upsilon \notin S \cup T$ we have $|S \cup T| < \upsilon$. Furthermore $|S \cap T| = 0$ since if $S \cap T$ contained some vertex v_i , then G would have the Hamilton cycle $v_1v_2 \ldots v_iv_\nu v_{\nu-1} \ldots v_{i+1}v_i$ contrary to assumption.

Using $|S \cup T| < \upsilon$ and $|S \cap T| = 0$ we obtain

$d(u) + d(v) = |S| + |T| = |S \cup T| + |S \cap T| < \upsilon$.

But this contradicts the hypothesis that $\delta \geq \frac{\upsilon}{2}$. ■

Bondy and Chvatal (1974) observed that the proof of theorem 1 can be modified to yield stronger sufficient conditions than that obtained by Dirac. The basis of their approach is the following condition.

Dirac’s condition is immediate corollary. A stronger sufficient condition than that of Dirac was modified by Ore.

Theorem (2)Ore (1962). Let G be a simple graph and let u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq \nu$. Then G is hamiltonian if and only if G + uv is hamiltonian.

Proof. If G is Hamiltonian then, trivially, so too is G + uv. Conversely, suppose that G + uv is hamiltonian but G is not. Then, as in the proof of Theorem 1, we obtain

$d(u) + d(v) = |S| + |T| = |S \cup T| + |S \cap T| < \nu$.

But this contradicts hypothesis

$d(u) + d(v) \geq \nu$. ■

Ore theorem motivates the following definition.

Definition(1). The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. We denote the closure of G by c(G).

Lemma(1). The *closure* of the graph G, c(G) is well defined.

Proof. Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequences of edges added to G in obtaining G_1 and G_2, respectively. We shall show that each e_i is an edge of G_2 and each f_j is an edge of G_1. If possible, let $e_{k+1} = uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_n$ that is not an edge of G_2. Set $H = G + \{e_1, e_2, \ldots, e_k\}$. It follows from the definition of G_1 that $d_H(u) + d_H(v) \geq \nu$. By the choice of e_{k+1}, H is a subgraph of G_2.

Therefore $d_{G_2}(u) + d_{G_2}(v) \geq \nu$.

This is contradiction, since u and v are nonadjacent in G_2. Therefore each e_i is an edge of G_2 and, similarly, each f_j is an edge of G_1.

Hence $G_1 = G_2$, and c(G) is well defined. ■

Example(1). This example illustrates the construction of the closure of a graph G on six vertices. In this example, c(G) is not complete.

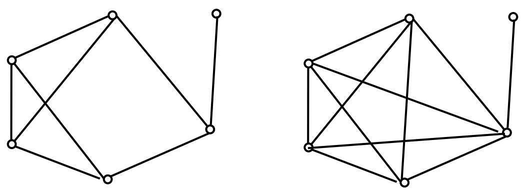

Figure 4 c(G) is not complete

Example(2). This example illustrates the construction of the closure of a graph G on six vertices. In this example, c(G) is complete.

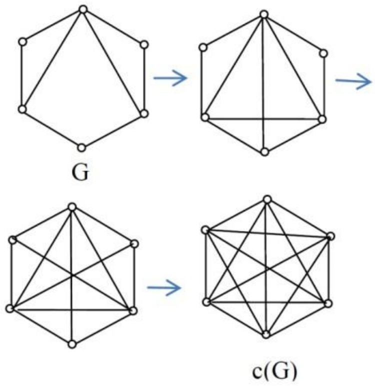

Figure 5 The closure of a graph

Theorem (3). A simple graph is hamiltonian if and only if a closure is hamiltonian.

Proof. Let a simple graph G be hamiltonian. Therefore, G contains Hamilton cycle. There is no effect on the Hamilton cycle by adding edges. So, the closure of G is also hamiltonian.

Conversely, suppose that the closure of G is hamiltonian and G is not. Let u and v be adjacent vertices in c(G) but not in G. Therefore, $d(u) + d(v) \geq \nu$. Since c(G) is hamiltonian and G is nonhamiltonian. Each Hamilton cycle contains the edge uv. Then, there is a Hamilton path $v_1v_2 \ldots v_\nu$ in G with origin $u = v_1$ and terminus $v = v_\nu$. Set $S = \{v_i \mid uv_{i+1} \in E\}$ and

$T = \{v_i \mid v_iv \in E\}$. Since $v_\upsilon \notin S \cup T$ we have $|S \cup T| < \nu$. Furthermore $|S \cap T| = 0$. Since if $S \cap T$ contains some vertex v_i, then G would have the Hamilton cycle $v_1v_2 \ldots v_iv_\nu\ v_{\nu-1} \ldots v_{i+1}\ v_1$. It is contrary to the assumption.

Using $|S \cup T| < \nu$ and $|S \cap T| = 0$, we obtain $d(u) + d(v) = |S| + |T| = |S \cup T| + |S \cap T| < \nu$.

This is contradiction to $d(u) + d(v) \geq \nu$. Our supposition is false. Therefore, if the closure of G is hamiltonian, a simple graph G is hamiltonian. ■

Corollary (1). Let G be a simple graph with $\nu \geq 3$. If c(G) is complete, then G is hamiltonian.

Corollary 1 can be used to deduce various sufficient conditions for a graph to be hamiltonian in terms of its vertex degrees. For example, since c(G) is clearly complete when $\delta \geq \frac{\upsilon}{2}$, Dirac’s condition is an immediate corollary. A more general condition than that of Dirac was obtained by Chvatal (1972).

Theorem(4)(Chvatal (1972)). Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of m less than $\nu/2$ for which $d_m \leq m$ and $d_{\nu-m} \leq \nu - m$. Then G is hamiltonian.

Proof. Let G satisfy the hypothesis of the theorem. We shall show that its closure c(G) is complete, and the conclusion will then follow from corollary 1. We denote the degree of a vertex v in c(G) by $d'(v)$. Assume that c(G) is not complete, and let u and v be two nonadjacent vertices in c(G) with $d'(u) \leq d'(v)$ and $d'(u) + d'(v)$ as large as possible; since no two nonadjacent vertices in c(G) can have degree sum or more, we have $d'(u) + d'(v) < \nu$. Now denote by S the set of vertices in $V - \{v\}$ which are nonadjacent to v in c(G), and by T the set of vertices in $V - \{u\}$ which are nonadjacent to u in c(G). Clearly $|S| = \nu - 1 - d'(v)$ and $|T| = \nu - 1 - d'(u)$. Furthermore, by the choice of u and v, each vertex in S has degree at most $d'(u)$ and each vertex in $T \cup \{u\}$ has degree at most $d'(v)$. Setting $d'(u) = m$ and using $d'(u) + d'(v) < \nu$ and

$|S| = \nu - 1 - d'(v)$ and $|T| = \nu - 1 - d'(u)$, we find that c(G) has at least m vertices of degree at most m and least ν –m vertices of degreeless than $\nu - m$.

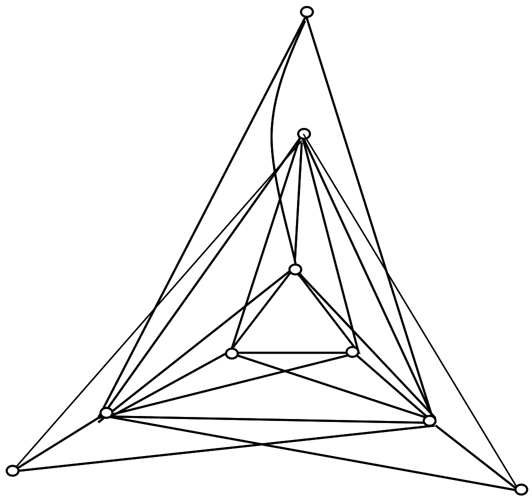

Figure 6 A Hamiltonian graph

Because G is a spanning subgraph of c(G), the same is true of G ; therefore $d_m \leq m$ and $d_{\nu-m} < \nu - m$. But this is contrary to hypothesis since, by $d'(u) \leq d'(v)$ and $d'(u) + d'(v) < \nu$, $m < \nu/2$. We conclude that c(G) is indeed complete and hence, by Corollary 1, that G is hamiltonian. ■

Finding Hamiltonian Graph on the Sufficient Conditions

Example 3(i). The graph with four vertices satisfies Dirac's Theorem.

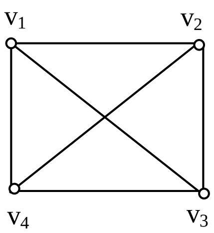

Figure 7 A graph with a Hamiltonian

Clearly, it is a Hamilton cycle. And, it satisfies Dirac's Theorem since $\nu = 4$, $\nu \geq 3$ and

$\delta(v) = 3 \geq \frac{\upsilon}{2}$.

Example 3(ii). The graph with five vertices does not satisfy Dirac's Theorem.

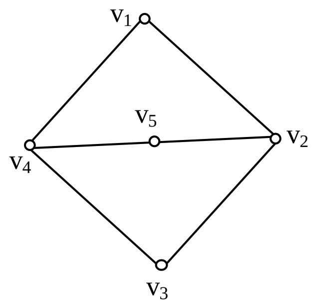

Figure 8 A graph with nonhamiltonian

Figure 8 does not satisfy Dirac's Theorem. It is not Hamiltonian since $\nu = 5$, $\nu \geq 3$ and

$\delta(v) = 2 \leq \frac{\upsilon}{2}$.

Example 4(i). The graph with five vertices satisfies Ore's Theorem.

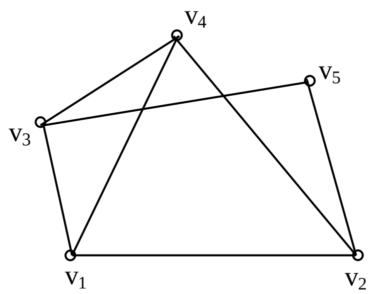

Figure 9 A graph with a Hamiltonian

It is a Hamiltonian for the graph of Figure 9. And, it satisfies Ore's Theorem since $\nu = 5$, such that for every pair of distinct nonadjacent verticesu,v$\in$G, $d(u) + d(v) \geq \nu$, then G is Hamiltonian.

Example 4(ii). The graph with five vertices does not satisfy Ore's Theorem.

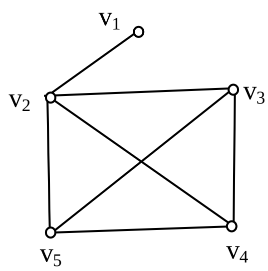

Figure 10 A graph with nonhamiltonian

It is nohamiltonian for the graph of Figure 10. And, it does not satisfy Ore's Theorem since $\nu = 5$, such that for every pair of distinct nonadjacent vertices u, v $\in$ G, $d(u) + d(v) \leq \nu$, then G is nonhamiltonian.

Example 5(i). The graph with six vertices satisfies Chvatal's Theorem.

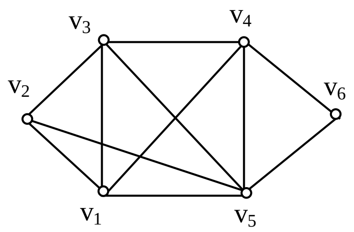

Figure 11 A graph with Hamiltonian

G is Hamiltonian since the degree sequence of G is (2, 3, 4, 4, 4, 5). $\nu = 6$, $\nu \geq 3$. There is no value m less than 3 for which $d_m \leq m$ and $d_{\nu-m} \leq \nu - m$.

Example 5(ii). The graph with six verticesdoes not satisfy Chvatal's Theorem.

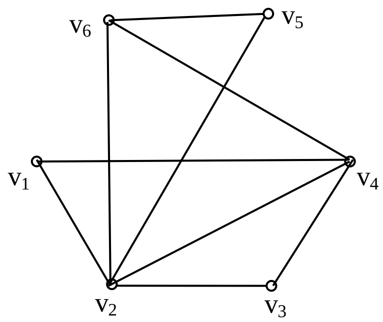

Figure 12 A graph with nonhamiltonian

G is nonhamiltonian since the degree sequence of G is (2, 2, 2, 3, 4, 5). $\nu = 6$. There is a value m less than 3 for which $d_m \leq m$ and $d_{\nu-m} \leq \nu - m$.

Some Applications on Hamiltonian Graph

The Knight's Tour. In chess, the knight's move consists of moving two squares horizontally or vertically and then moving one square in the perpendicular direction. For example, in Figure 13 a knight on the square marked K can move to any of the squares marked X. A knight's tour of an n×n board begins as some square, visits each square exactly once making legal moves, and returns to the initial square. The problem is to determine for which n a knight's tour exists.

	X		X	
X				X
		K		
X				X
	X		X	

Figure 13 The knight's legal moves in chess

We let the squares of the board, alternately colored black and white in the usual way, be the vertices of the graph and we place an edge between two vertices if the corresponding squares on the board represent a legal move for the knight. We denote the graph as GK_n. Then there is a knight's tour on the n ×n board if and only if GK_n has a Hamiltonian cycle.

We show that if GK_n has a Hamiltonian cycle, n is even. To see this, note that GK_n is bipartite. We can partition the vertices into set V_1, those corresponding to the white squares, and V_2, those corresponding to the black squares ; each edge is incident on a vertex in V_1 and V_2 .

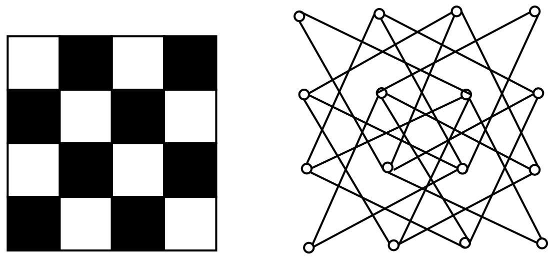

Figure 14 A 4×4 chessboard and the graph GK_4

Since any cycle must alternate between a vertex in V_1 and one in V_2, any cycle in GK_nmust have even length. But since a Hamiltonian cycle must visit each vertex exactly once, a Hamiltonian cycle in GK_n must have length n^2. Thus n must be even.

In view of the preceding result, the smallest possible board that might have a knight's tour is the 2 ×2 board, but it does not have a knight's tour because the board is so small the knight has no legal moves. The next smallest board that might have a knight's tour is the 4 ×4 board, although, as we shall show, it too does not have a knight's tour.

We argue by contradiction to show that GK_4 does not have a Hamiltonian cycle. Suppose that GK_4 has a Hamiltonian cycle C = (v_1 , v_2 , ..., v_{17}) . We assume that v_1 corresponds to the upper - left square. We call the eight squares across the top and bottom *outside squares*, and we call the other eight squares *inside squares*. Notice that the knight must arrive at an outside square from an inside square and that the knight must move from an outside square to an inside square. Thus in the cycle C, each vertex corresponding to an outside square must be preceded and followed by a vertex corresponding to an inside square. Since there are equal numbers of outside and inside squares, vertices v_i where i is odd correspond to outside squares, and vertices v_i where i is even correspond to inside squares. But looking at the moves the knight makes, we see that vertices v_i where i is odd correspond to white squares, and vertices v_i where i is even correspond to black squares. Therefore, the only outside squares visited are white and the only inside squares visited are black. Thus C is not a Hamiltonian cycle. This contradiction completes the proof that GK_4 has no Hamiltonian cycle. This argument was given by Louis Posa when he was a teenager.

The graph GK_6 has a Hamiltonian cycle. This fact can be proved by simply exhibiting one .

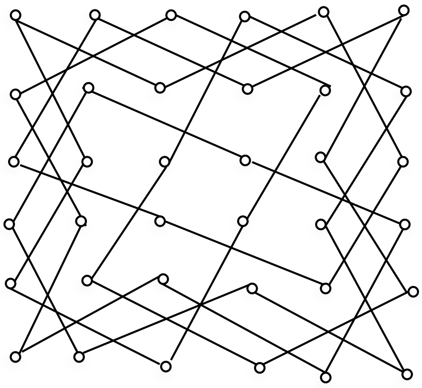

Figure 15 GK$_6$

It can be shown using elementary methods that GK$_4$ has a Hamiltonian cycle for all even $n \geq 6$. The proof explicity constructs Hamiltonian cycle for certain smaller boards and then pastes smaller boards together to obtain Hamiltonian cycles for the larger boards.

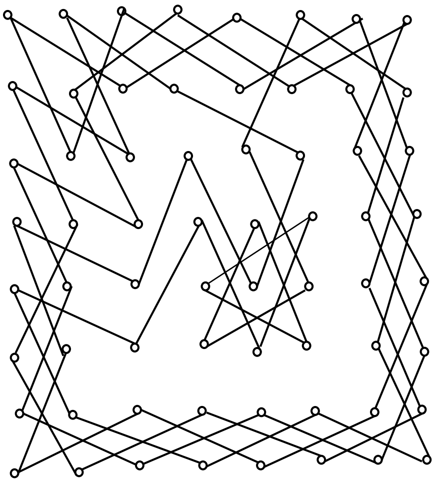

Figure 16 GK$_8$

The Travelling Salesperson Problem

We can solve the travelling salesperson problem in the following traffic system. The travelling salesperson problem is related to the problem of finding a Hamiltonian cycle in a weighted graph. The weights on the edges are given. From given a weighted graph G, we must find a minimum-length Hamiltonian cycle in G. We think of the vertices in a weighted graph as cities and the edges weights as distances.

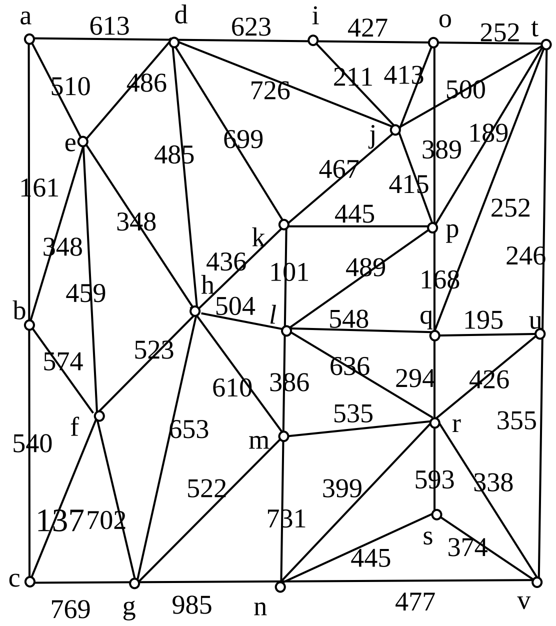

Figure 17 The traffic system

The above traffic system contains a Hamiltonian cycle.

161+540+137+702+522+386+101+445+168+294+ 399+445+374+355+246+252+413+211+623+485+ 348+510 = 8117 miles.

The Hamiltonian cycle is (ab, bc, cf, fg, gm, ml,lk, kp, pq, qr, rn, ns, sv, vu, ut, to, oj, ji, id ,dh, he, ea). Its minimum weight is 8117 miles.

Finding the Shortest Road

In the following figure, we can find the shortestroad in which the salesperson can visit each city. The shortest road is Mahlaing, Wundwin, ThaPhayWa, Meiktila, Thazi, Payagazu, Pyawbwe, Yamethin, Hkabu, ThitSoneGyi, AhLeYwa, SganMaNge, Sedo, KyaykTan, Mahlaing.

24.01+9.54+9.14+11.41+13.89+14.82+11.44+7.84+11. 53+21.57+9.26+9.55+8+5 = 167miles.

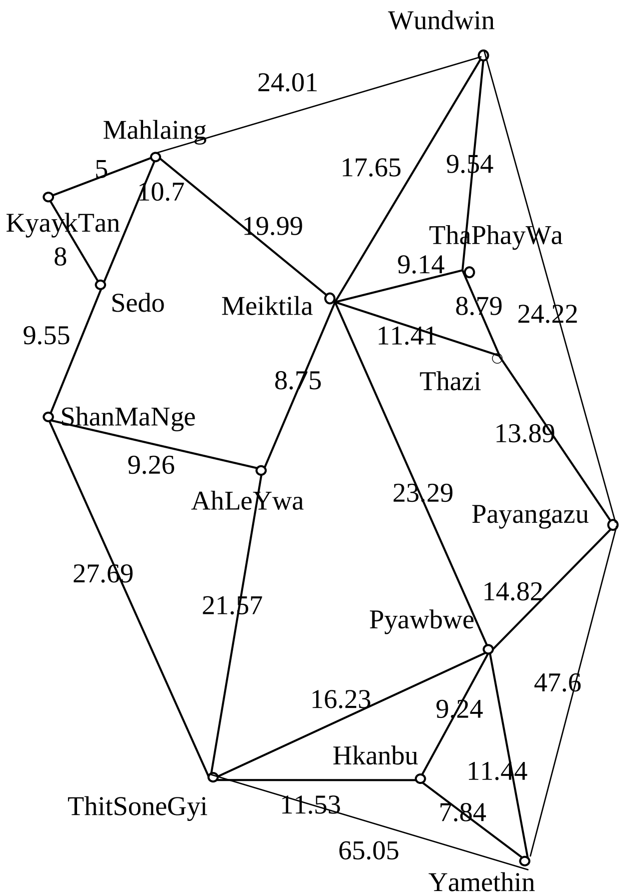

Figure 18 The graph model for the given cities

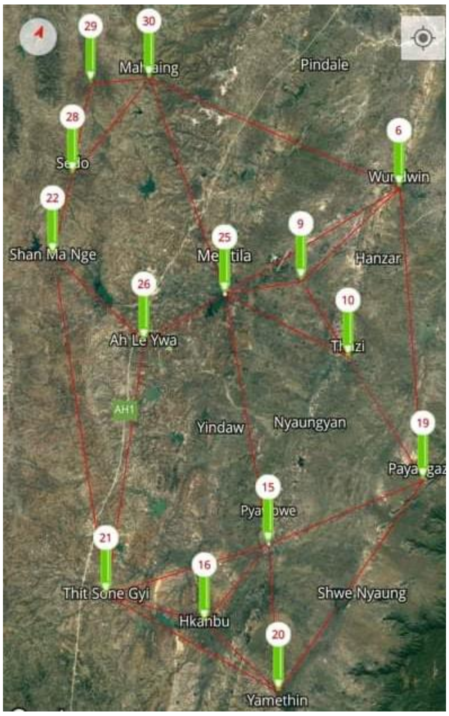

Figure 19 The Shortest road in which the salesperson can visit each city

Conclusion

In this paper, some Theorems on the sufficient conditions for a Hamiltonian graph are presented and considered with examples. Finally, some applications on Hamiltonian graph are studied.

Acknowledgements

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References

Bollobas, B.,(1998) Modern graph Theory,
Springer−Veralg New York, Inc.

Bondy, J. A. and Murty, U.S.R., (1976)Graph Theory with Applications, Macmillan Press Ltd., London.

Douglas, B.W.,(1999) Introduction to Graph Theory, Prentice-Hall of India Private Limited., New Delhi.

Johnsonbaugh. R., (1896) Discrete Mathematics, Macmillan Publishing Company, New York.