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All Authors	Cho Cho Win
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Calculation and Visualization of a Fluid Flow

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By

Dr Cho Cho Win

Lecturer

Department of Physics

University of Mandalay

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Calculation and Visualization of a Fluid Flow

Dr Cho Cho Win*

Abstract

Fluid flow simulations play an important role in many scientific disciplines. For the past two decades, researchers have developed various visualization techniques to enable effective analysis of large scale simulation output. Among the many directions of research, visualization of flow fields remains to be one of the most challenging problems in the field. In this work, the equation of continuity which holds for a fluid in motion is computed. The fundamental equation of motion is also derived for a perfect fluid in motion under the acting of any given system of body forces. The flow velocity of a fluid in general motion can be derived from a vector potential. In this work, the calculation and visualization of fluid flow around a cylinder are also presented.

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PhD, Lecturer, Department of Physics, University of Mandalay

Introduction

For centuries, fluid flow researchers have been studying fluid flows in various ways, and today fluid flow is still an important field of research. The areas in which fluid flow plays a role are numerous. Gaseous flows are studied for the development of cars, aircraft and spacecrafts, and also for the design of machines such as turbines and combustion engines. Liquid flow research is necessary for naval applications, such as ship design and costal protection. In chemistry, knowledge of fluid flow in reactor tanks is important; in medicine the flow in blood vessels is studied. In all kinds of fluid flow research, visualization is a key issue.

Recently, a new type of visualization has emerged: computer-aided visualization . To analyze the results of the complex calculations, computer visualization techniques are necessary. Humans are capable of comprehending much more information when it is presented visually, rather than numerically. One purpose of the visualization of fluid flow simulation data is the verification of theoretical models in fundamental research. Another purpose of fluid flow visualization is the analysis and evaluation of a design. For the design of a car, an aircraft, a harbor, or any other object that is functionally related with fluid flow, calculation and visualization of the fluid flow phenomena can be a powerful tool in design optimization and evaluation.

The increase of computational power has led to an increasing use of computers for numerical simulations and visualizations. In the area of fluid dynamics, computers are extensively used to calculate velocity fields and other flow quantities, using numerical techniques. There are four basic two-dimensional flow fields, from combinations of which all other steady-flow conditions may be modeled. The four flows are uniform parallel stream, source (sink), doublet, and point vortex. These are applied to construct useful flows. The main purpose of this work is to give an introduction to fluid flow visualization with computer graphics.

Equation of Continuity for Fluid

The motion of a fluid is said to be steady when the conditions at every point fixed in space remain unchanged with time i.e., the local time rates

$$\frac{\partial \vec{f}}{\partial t}, \frac{\partial \rho}{\partial t}, \frac{\partial p}{\partial t} \tag{1}$$

are all zero.

Let, dv, be an element of volume within the fixed surface and ρ, the density of the fluid element at time t. Then the mass of the fluid within the surface at time t is

$$m = \int_{v} \rho \, dv, \tag{2}$$

where, V , denotes the volume enclosed within the surface. Its time rate of change

$$= \frac{\partial}{\partial t} \int_{v} \rho \, dv = \int_{v} \frac{\partial \rho}{\partial t} dv \tag{3}$$

The time rate of change considered here is local, for the surface is being kept fixed. The element of volume dv is associated with a fixed point thereof and ρ, is the density of the fluid at that point.

Now , any element, dS of the surface and let, $\hat{n}$, be the unit outward drawn normal at any point of dS and let $\vec{q}$ be the velocity of the fluid particle at time, t, at the point P are considered.

The resolved part of, $\vec{q}$, along the outward drawn normal at P

$$= \vec{q}.\hat{n} \tag{4}$$

so that the inward normal velocity

$$= -\vec{q}.\hat{n}. \tag{5}$$

The mass of fluid entering per unit time across the element dS

$$= -\rho \vec{q}.\hat{n} \, dS \tag{6}$$

Therefore the mass of fluid entering per unit time across the whole surface

$$= -\int_{s} \rho \vec{q}.\hat{n} \, dS. \tag{7}$$

$$= -\int_{v} div(\rho \vec{q}) \, dv, \tag{8}$$

by Gauss's theorem.

Thus $$\int_{v} \frac{\partial \rho}{\partial t} dv = -\int_{v} div(\rho \vec{q}) \, dv$$

$$\int_{v} \left[\frac{\partial \rho}{\partial t} + div(\rho \vec{q}) \right] dv = 0 \tag{9}$$

As the surface S is arbitrary, equation (9) can be deduced that

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{q}) = 0, \qquad (10)$$

which is the required equation of continuity.

The equation of continuity can also be written down in another form so as to involve individual rate of change $d\rho / dt$ instead of the local rate of change $\partial \rho / \partial t$.

$$div(\rho \vec{q}) = \nabla.(\rho \vec{q}) = \rho \nabla.\vec{q} + \vec{q}.\nabla \rho. \qquad (11)$$

From (2.9) and (2.10),

$$\frac{\partial \rho}{\partial t} + \vec{q}.\nabla \rho + \rho \vec{\nabla}.\vec{q} = 0$$

i.e.,
$$\frac{d\rho}{dt} + \rho \vec{\nabla}.\vec{q} = 0 \qquad (12)$$

or
$$\frac{d\rho}{dt} + \rho \, div \, \vec{q} = 0.$$

Equation of Continuity for Incompressible Fluids

For an incompressible fluid, $d\rho / dt = 0$, so that the equation of continuity reduces to

$$div \, \vec{q} = 0, \qquad (13)$$

i.e.,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0;$$

u, v, w being the components of $\vec{q}$ along the co-ordinate axes. Thus in the case of incompressible fluid, the velocity vector $\vec{q}$ of an incompressible fluid is solenoidal.

Irrotational Motion and Scalar Velocity Potential

The vector $\vec{\omega}$ defined by the equality

$$\vec{\omega} = \frac{1}{2} curl \, \vec{q} = \frac{1}{2} \vec{\nabla} \times \vec{q} \qquad (14)$$

is called vorticity. The vector ω is also a function of $\vec{r}$ and t.

The motion is said to be irrotational at any instant if the vorticity $\vec{\omega}$ at the instant is zero. Also a motion is said to be permanently irrotational if the vorticity is zero at every instant. In the case of permanent irrotational motion,

$$\vec{\omega} = \frac{1}{2} curl\, \vec{q} = 0,$$

i.e.,
$$curl\, \vec{q} = 0, \tag{15}$$

so that there exists a scalar point function, say, $\phi(x, y, z, t)$ such that

$$\vec{q} = -\vec{\nabla}\phi. \tag{16}$$

The function, ϕ, is called scalar velocity potential. The equation of continuity of an incompressible fluid for irrotational motion can be written down in terms of the scalar velocity potential. By the equation of continuity for an incompressible fluid is

$$\vec{\nabla}.\vec{q} = 0 \tag{17}$$

Also
$$\vec{q} = -\vec{\nabla}\phi. \tag{18}$$

$\therefore$ we have $\vec{\nabla}.\vec{\nabla}\phi = 0,$ i.e., $\vec{\nabla}^2\phi = 0.$ (19)

This is equivalent to

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{20}$$

in terms of fixed rectangular cartesian axes so that ϕ satisfied the Laplacian equation.

In the general case of motion, irrotational or not, the velocity vector $\vec{q}$ satisfied the equation $\vec{\nabla}.\vec{q} = 0$ if fluid is incompressible. There exists, therefore, a vector A such that

$$\vec{q} = curl\ \vec{A}. \tag{21}$$

This vector, A, is called velocity vector potential. Thus the flow velocity of an incompressible fluid in general motion can be derived from a vector potential.

Stream Lines and Vortex Lines

A curve drawn in the fluid at any given instant such that the direction of the tangent at any point of the curve coincides with that of the velocity of the fluid particle at the point and at the instant is called a stream line.

Again, a vortex line is a curve drawn in the fluid at any given instant such that the direction of the tangent at any point of the curve coincides with that of the vorticity of the fluid particle at the point at the instant.

Thus if, $\vec{t}$, denotes the unit tangent to a stream line, we have

$$\vec{t} \times \vec{q} = 0,$$

for any point of the a stream line, and if, $\vec{t}$, denotes the unit tangent to a vortex line, we have

$$\vec{t} \times \vec{\omega} = 0.$$

Thus at any point of a stream line as well as vortex line, the scalar triple product is

$$[\vec{t}\ \vec{q}\ \vec{\omega}] = 0 \tag{22}$$

For non-steady motion, the patterns of stream lines and vortex lines are different at different times. The path followed by a fluid particle is called a *line of motion* and should be distinguished from stream lines.

Uniform Flow

The scalar potential and stream function (vector potential) that represents a uniform stream in the x direction are first examined. The potentials for a uniform flow in the x direction are

$$\phi = -Ux, \qquad \psi = Uy \tag{23}$$

Thus, applying the irrotationality condition for a scalar function ϕ,

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y} \tag{24}$$

and the continuity equation for the stream function ψ,

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{25}$$

$$\vec{q} = (u, v) = (U, 0) \tag{26}$$

Note that, if ψ is constant, streamlines are parallel to the *x*-axis, as illustrated in Figure (1). Similarly, if ϕ is constant, equipotential lines are parallel to the *y*-axis. The pattern of the potentials for a uniform flow in the *x* direction is shown in the Figure (1).

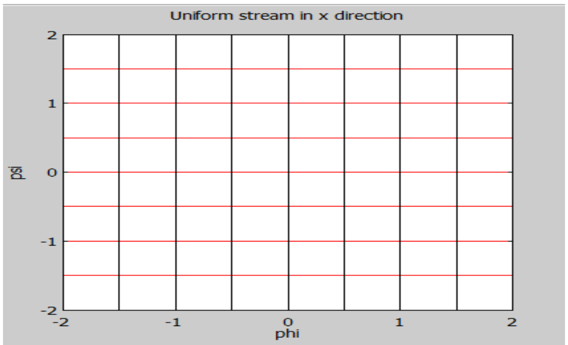

Fig. (1) Vertical lines are equipotentials; horizontal lines are streamlines.

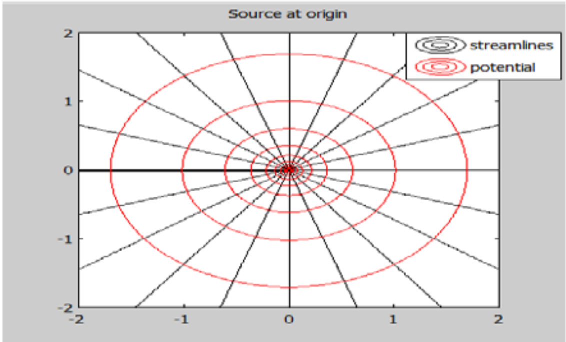

Fig. (2) The potentials of a source

Two-Dimensional Flow from a Source (or towards a Sink)

A source (sink) of strength m(-m) is a point at which fluid is appearing (or disappearing) at a uniform rate of m(-m) m^2s^{-1}. Let us assume this source is located at $r = 0$; hence, in this case, the velocity potential is

$$\phi = -\frac{m}{2\pi}\ln r \tag{27}$$

The flow field for this potential can be interpreted as follows. Consider the analogy of a small hole in a large flat plate through which fluid is welling (the source). If there is no obstruction and the plate is perfectly flat and level, the fluid puddle will get larger and larger, all the while remaining circular in shape. The path that any particle of fluid will trace out as it emerges from the hole and travels outward is a purely radial one; it cannot go sideways because its fellow particles are also moving outward.

Also, its velocity must lessen as it goes outwards. Fluid issues from the hole at a rate of $-m$ m^2s^{-1}. The velocity of flow over a circular boundary of 1-m radius is m/2π ms^{-1}. Over a circular boundary of 2-m radius it is m/(2π x2) m^2s^{-1}, i.e., half as much, and over a circle of diameter 2r the velocity is m/2π r m^2s^{-1}. Therefore, the velocity of flow is inversely proportional to the distance of the particle from the source.

All the previous applies to a sink except that fluid is being drained away through the hole and is moving toward the sink radially, increasing in speed as the sink is approached. Thus the particles all move radially, and the streamlines must be radial lines with their origin at the source (or sink). The stream function that describes the source at the origin of the coordinate system is

$$\psi = \frac{m}{2\pi}\tan^{-1}\frac{y}{x} \qquad \text{or} \qquad \psi = \frac{m}{2\pi}\theta \tag{28}$$

where $\theta = \tan^{-1}(y/x)$. Again, we place the source (for convenience) at the origin of a system of axes, to which the point P is at (x, y) or (r, θ). The velocity potential for this flow is

$$\phi = -\frac{m}{4\pi}\ln(x^2 + y^2) \quad \text{or} \quad \phi = -\frac{m}{2\pi}\ln r \tag{29}$$

Figure (2) is an illustration of the pattern of the potentials for the source. The components of the velocity vector due to the source at $(x, y) = (0,0)$ are

$$u = -\frac{\partial\phi}{\partial x} = -\frac{m}{2\pi}\frac{x}{x^2 + y^2}, \qquad v = -\frac{\partial\phi}{\partial y} = -\frac{m}{2\pi}\frac{y}{x^2 + y^2} \tag{30}$$

$$u_r = -\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{2\pi r}, \qquad u_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = 0 \tag{31}$$

where u_r and u_θ are the radial and circular components of the velocity in polar coordinates.

Doublet Located at (x, y)=(0, 0)

A doublet is a source and sink combination but with the separation infinitely small. A doublet is considered to be at a point, and the definition of the strength of a doublet contains the measure of separation. The strength (μ) of a doublet is the product of the infinitely small distance of separation, and the strength of source and sink. The doublet axis is the line from the sink to the source in that sense.

The doublet in this section points in the -x direction. The potentials that describe this flow field are

$$\phi = \mu\frac{x}{x^2 + y^2} \qquad \text{and} \qquad \psi = -\mu\frac{y}{x^2 + y^2} \tag{32}$$

The components of the corresponding velocity field are

$$u = \mu\frac{y^2 - x^2}{(x^2 + y^2)^2}, \qquad v = -\mu\frac{y}{(x^2 + y^2)^2} \tag{33}$$

The potentials for this flow field are illustrated in Figure (3). The streamlines and velocity potential lines are circles and the flow along the x-axis is in the -x direction.

Line (Point) Vortex

A line vortex can best be described as a string of rotating particles. A chain of fluid particles are spinning on their common axis and carrying around with them a swirl of fluid particles which flow around in circles. A cross-section of such a string of particles and its associated flow show a spinning point outside of which is streamline flow in concentric circles. The flow induced by a line (point) vortex can be interpreted as the flow induced by a straight-line vortex that is infinitely long in the *z* direction. In the (*x*, *y*) plane, it is a point. The potentials for the line (point) vortex, if the point vortex is located at (*x*, *y*) = (0, 0) are

$$\phi = \frac{\Gamma}{2\pi}\tan^{-1}\frac{y}{x}, \qquad \psi = -\frac{\Gamma}{4\pi}\ln(x^2+y^2) \tag{34}$$

The corresponding components of the velocity field are

$$u = \frac{\partial\phi}{\partial x} = -\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}, \qquad v = \frac{\partial\phi}{\partial y} = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} \tag{35}$$

The potentials for this flow field are illustrated in Figure (4).

Since the flow due to a line vortex gives streamlines that are concentric circles, $u_r = 0$ and u_θ is finite. In polar coordinates, the potentials are as follows:

$$\phi = \frac{\Gamma}{2\pi}\theta, \qquad \psi = -\frac{\Gamma}{2\pi}\ln r \tag{36}$$

The corresponding components of the velocity field are

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0, \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r} \tag{37}$$

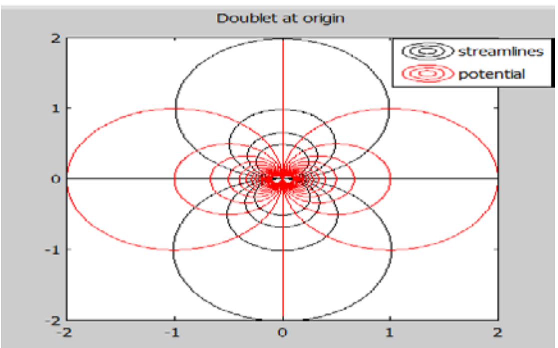

Fig. (3) Velocity potential and stream function for doublet with the strength $\mu = 1$.

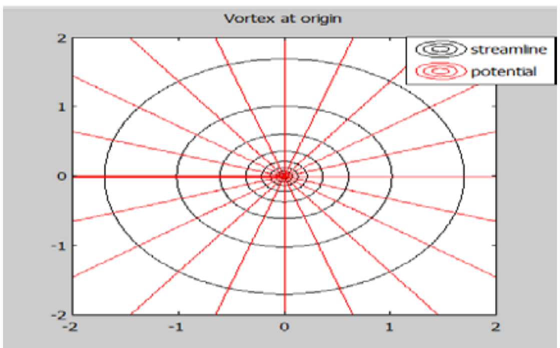

Fig. (4) The potentials of a point vortex

Calculation and Visualization of Flow around a Circular Cylinder

The purpose of this section is to compute the streamlines around a circular cylinder for incompressible, inviscid, irrotational, two-dimensional flow. Streamlines are quantitatively labeled by a constant stream function to specify its position within the flow stream. The stream function is the mass flux per unit time across any stationary line in the fluid. Therefore, streamlines are simply lines along which the stream function is constant.

Fundamentally, the streamline patterns around a circular cylinder can be described by the superposition of uniform flow with a doublet. The streamlines are close together at regions of high velocity. The streamline pattern is useful in the study of fluid flow since it gives an ideal picture of the variation of pressure and velocity around an object.

Laplace's equation in fluid dynamics describes incompressible, inviscid, irrotational, two-dimensional fluid motion through the stream function.

Problem Formulation and Description

Starting from the integral form of continuity relation or mass conservation:

$$\int \frac{\partial \rho}{\partial t} dV + \int \rho\, q.\hat{n}\, dA = 0 \tag{38}$$

The differential form can be obtained to be:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0\,. \tag{39}$$

Assuming steady, incompressible flow and dropping the z – dimension, equation (39) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{40}$$

Two variables (u and v) in equation (40) can be replaced with one, ψ (x, y), known as the Stream Function. Continuity can now be written as:

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0, \text{ where } u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}\,. \tag{41}$$

Taking the curl of the local velocity vector $\vec{q}$,

$$\frac{1}{2}\,\text{curl}\ \vec{q} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \omega \text{, where } \omega \text{ is vorticity.} \tag{42}$$

Setting vorticity to zero results in the two dimensional Laplace equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0\,. \tag{43}$$

Thus, incompressible, inviscid, irrotational, two-dimensional fluid motion can be described by a linear partial differential equation called Laplace’s equation:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{44}$$

It is expressed in polar coordinates:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{45}$$

Laplace’s equation for the stream function is simply derived from the irrotationality condition for an ideal fluid in two dimensions:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{46}$$

whereas Laplace’s equation for the potential function is derived from the equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{47}$$

The velocity components u and v can be expressed in terms of both the stream function and the potential function:

$$u = \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} \tag{48}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \tag{49}$$

Substituting these expressions into the irrotationality condition and the continuity equation respectively, we obtain Laplace’s equation for the stream and potential functions.

The complex potential for flow around a circular cylinder is given by the combination of uniform flow velocity U and a doublet at the origin:

$$w = U\left[z + \frac{a^2}{z}\right] \quad (50)$$

where z = x + iy and a= the length scale.

Finding the solution of the stream function using the complex potential

The potential function and stream function can be described in the same expression by an analytic function called the complex potential w.

$$w = f(z) = \phi + i\psi \quad (51)$$

The analytic function is a convenient method of solving this problem by mapping the potential function and the stream function in a complex plane. Mathematically, all two-dimensional flows are solutions of w = f(z) where z = x + iy. The complex potential for flow around a circular cylinder is modeled as the combined motion $w = f(z) = \phi + i\psi$ of uniform flow velocity U and a doublet at the origin. It is as follows:

$$w = \phi + i\psi = U\left[z + \frac{a^2}{z}\right] \quad (52)$$

where z = x + iy = r(cosθ + isinθ) = $re^{i\theta}$ from Euler's formula.

Substituting this expression for z in w we get:

$$w = U(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}) \quad (53)$$

$$= Ur(\cos\theta + i\sin\theta) + \frac{U}{r}a^2(\cos\theta - i\sin\theta) \quad (54)$$

Multiplying through and collecting real and imaginary terms:

$$w = (Ur + \frac{U}{r}a^2)\cos\theta + i(Ur - \frac{U}{r}a^2)\sin\theta \quad (55)$$

Since $w = \phi + i\psi$

$$u = (Ur + \frac{U}{r}a^2)\cos\theta$$

$$v = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

The solution of the stream function is shown in Figure (5) in polar coordinates. The graph shows the streamlines of incompressible, inviscid, irrotational fluid flow past a circular cylinder. The contours of stream function (blue), ψ are from bottom to top and those of velocity potential (red), ϕ are from left to right.

Discussion

Flow around a circular cylinder is given by a doublet in a uniform horizontal flow. A doublet is a source and sink combination but with the separation infinitely small. The doublet axis is the line from the sink to the source.

Potential flow in two dimensions is simple to analyze by using transformations of the complex plane. The basic idea is to use a function *f*, which maps the physical domain (*x*, *y*) to the transformed domain (ϕ,ψ). While *x*, *y*, ϕ and ψ are all real valued, it is convenient to define the complex quantities

z = x + iy and w = ϕ + iψ .

Now, if we write the mapping *f* as

f (x + iy) = ϕ + iψ or f(z) = w.

The potential flow around a circular cylinder is a classical solution for the flow of an inviscid, incompressible fluid around a cylinder that is transverse to the flow. As shown in Figure (5), far from the cylinder, the flow is unidirectional and uniform. The flow has no vorticity and thus the velocity field is irrotational and can be modeled as a potential flow.

The computation of incompressible, inviscid, irrotational fluid flow around a cylinder is important because it can then be used to transform the flow around a cylinder into the flow around a wing or airfoil. The idea behind airfoil analysis using flow around a circular cylinder is to relate the flow field around one shape which is already known to the flow field around an airfoil.

Conclusion

The aim of this work is to present the calculation and visualization of fluid flow around a cylinder. The continuity relation or mass conservation and irrotational condition are used for

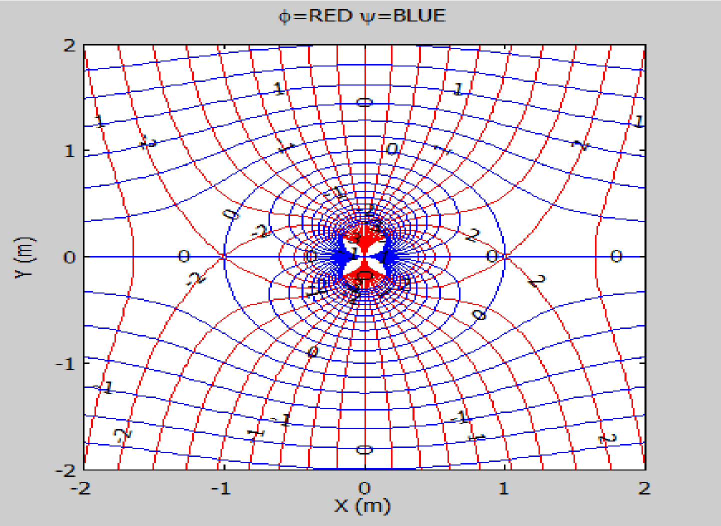

Fig. (5) Velocity potential (red contours) and stream function (blue contours) of flow past a circular cylinder.

the study of this problem. The four basic two-dimensional flow fields, from which all other steady-flow conditions may be modeled, are described and visualized. The classical assumption of irrotational flow is made, meaning that the vorticity is everywhere zero. This also implies inviscid flow. Irrotational flows are potential fields. A potential function, known as the velocity potential, for fluid flow around the cylinder is introduced. It is shown how the velocity components can be determined from the velocity potential. The equation of motion for irrotational flow reduces to a single partial differential equation for velocity potential known as the Lapalce equation. The technique of mapping the complex potential is used for obtaining two-dimensional solutions to the Laplace equation of fluid flow around a cylinder. The calculation and visualization of the flow around a circular cylinder has shown that it is a relatively accurate method of predicting flows over objects and is a widely used today especially in the field of aerodynamics.

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