SEASONAL MODELS FOR MONTHLY TRANSPORT TIME SERIES OF MYANMAR

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by
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ABSTRACT

With the objective to contribute to the application of model building procedure to an economic time series of Myanmar, stochastic models for some monthly transport time series of Myanmar are found by using Box-Jenkins model building approach. Basic statistical characteristics for our transport time series are first investigated and statistical test for seasonality is applied to each series to confirm the existence of seasonality. The traditional methods of seasonal measurement and stochastic seasonal models with their characteristics are also investigated. Suitable stochastic models for our monthly transport time series of Myanmar are found by following the three stages of model building, namely, identification, estimation and diagnostic checking. Whenever needed, computer programs for the systematic development of the model building procedure are developed. It is found that ARI, IMA and SMA models are suitable for our series. They are fitted to the series differenced by year or differenced by month as well as year.

INTRODUCTION

One of the essential things in the development of a country is an excellent treanportation system. The State Law and Order Restoration Council (SLORC) endeavoured to lift the standard of social, economic and cultural conditions of Myanmar by improving the transportation system.

In fact, transport plays a vital role in a nation 's economic, social and curtural development as well as national defense and national unity. That is why, SLORC is giving priority to the safty and smoothness of transportation as one of its four main taskss.

It is necessary to study the situation of transportation in the country which is fundamental to the social, economic and cultural development. Airways, Railways, Inland Water Transport and Road Transport are important components of the state 's transportation. In this thesis, some monthly time series of Airways, Railways and Inland Water Transport are investigated.

The monthly transport data are normally recorded in two ways, freight lb or freight ton and number of passengers. For road transport, we get yearly data of the states and divisions and monthly data of only Yangon city. We cannot get monthly data for the whole country. So, we omit the road transport data to investigate in this thesis. The monthly transport time series of Airways, Railways and Inland Water Transport are collected for 7 years (from 1989 to 1995). The data are obtained from Central Statistical Organization (C. S. O.). We choose some monthly transport time series such as Airways(freight lb), Airways (number of passengers), Railways (freight ton) and Inland Water Trasnsport (freight ton) series to study in this thesis.

The transportation system of a country is usually comprised of transportation by airways, railways, water transport and road transport. Passengers and cargoes are carried from place to place by such transport. In Myanmar, the Ministry of Railway Transportation is in charge of Railways, and the Ministry of Transport is in charge of Airways and Inland Water Transport.

The number of passengers, freight lb and freight ton are shown in the following table.

(In Million)

Series	1987-88	1994-95
Airways (Freight lb)	2.97	3
Airways (Number of passengers)	441	626
Railways (Freight ton)	1764	3297
Railways (Number of passengers)	60859	53362
Inland Water Transport (Freight ton)	2368	3194
Inland Water Transport (Number of passengers)	17892	26582

Source: Review of the Financial, Economic and Social Conditions (1991-92, 1996-97)

The above table shows considerable increase except in Airways (freight lb) and decrease in Railways (number of passengers) between 1987-88 and 1994-95. That is, there is only 1% increase in Airways (freight lb), 41.95 % increase in Airways (number of passenger), 86.90 % increase in Railways (freight ton), 12.32% decrease in Railways (number of passengers), 34.88 %

increase in Inland Water Transport (freight ton) and 48.57% increase in Inland Water Transport (number of passengers).

The objective of this thesis is to find suitable stochastic models for the monthly transport time series consisting of Airways, Railways, Inland Water Transport series and in so doing to investigate the model building procedure suggested by Box and Jenkins (1976). There exists traditional methods of analysing a monthly time series and these will also be discussed in this thesis. Smoothing method and use of dummy variable method will also be presented in order to have a more or less complete coverage of all possible methods of analysis.

In chapter I, basic statistical characteristics of the time series and seasonality of these series are investigated. In chapter II, traditional methods of seasonal measurement are discussed. Stochastic seasonal models for monthly time series are presented in chapter III. Chapter IV gives a description of the stochastic model building procedure. Model building procedures and accepted stochastic models for the monthly transport time series under investigation are discussed in chapter V. Model building procedures of the analyzed transport time series, their limitations, results, comments and suggestions for further research are summarized in the Conclusion.

CHAPTER I

MONTHLY TRANSPORT TIME SERIES OF MYANMAR

1.1 Introduction

Monthly time series over the years display variations over the months as well as variations over the years. Graph of a monthly time series shows these variations in detail. Monthly transport time series of Myanmar. for the years 1989 to 1995 are shown in Tables A1.1 to A1.4 of Appendix A and Graphs B1.1 to B1.4 of Appendix B. Distinct seasonal variations of Airways (freight lb), Airways (number of passengers), Railways (freight ton) and Inland Water Transport (freight ton) monthly time series can be discerned in the graphs. Seasonality in a time series can best be discerned in a tier chart. Graphs B1.5 to B1.8 show tier charts for each of transport time series of Myanmar. They show that the patterns of variation over the months are similar for most of the seven years.

In this chapter, basic statistical characteristics of some of transport time series of Myanmar will first be investigated. Statistical tests for seasonality will also be applied to each series to confirm the existence of seasonality.

1.2 Basic Statistical Characteristics

In this section, some basic statistics of some transport time series are presented in order to be able to see their significant variations in a summarized form. The statistical measures used are the mean, the variance, coefficient of variation, maximumand minimum. These values are calculated from the monthly series for each month (January to December) over a number of years and for each year over a number of months.

To calculate these values, we define y_{ij} as the value of the random variable y during j^{th} month of i^{th} year and compute,

$$\bar{y}_{i.} = \frac{1}{k}\sum_{j=1}^{k} y_{ij} \qquad ; i = 1, 2, \ldots, 7$$

= the mean value for i^{th} year

$$\bar{y}_{.j} = \frac{1}{n}\sum_{i=1}^{n} y_{ij} \qquad ; j = 1, 2, \ldots, 12$$

= the mean value for j^{th} month

$$V_i = \frac{1}{k-1}\sum_{j=1}^{k} (y_{ij} - \bar{y}_{i.})^2 \qquad ; i = 1, 2, \ldots, 7$$

= the variance for i^{th} year

$$V_j = \frac{1}{n-1}\sum_{i=1}^{n} (y_{ij} - \bar{y}_{.j})^2 \qquad ; j = 1, 2, \ldots, 12$$

= the variance for j^{th} month

$$C.V(i) = \frac{\sqrt{V_i}}{\bar{y}_{i.}} x100$$

= the coefficient of variation for i^{th} year

$$C.V(j) = \frac{\sqrt{V_j}}{\bar{y}_{\cdot j}} x100$$

= the coefficient of variation for j^{th} month

These values enable us to compare the statistical characteristics from month to month and from year to year.

1.2.1 Airways (Freight Lb) Series

The monthly data of Airways (freight lb) series are collected for 7 years, from 1989 to 1995 and presented in Table A1.1 of Appendix A. Basic statistical characteristics of this series are investigated from two aspects. Firstly, the basic statistics for each month over a number of years (7 years) are computed. This enable us to see the pattern clearly from January to December throughout the year of the means and variances. Secondly, the basic statistics for each year over a number of months (12 months) are computed. The pattern over the years of the means and variances can be seen clearly from these.

Some basic statistics of the Airways (freight lb) series are computed for each month and presented in Table 1.1(a).

Table 1.1(a)

Basic Statistical Characteristics For Each Month : Airways (Freight Lb)

Month	Mean (Thousand lb)	Variance (Thousand lb)^2	C.V.	Maximum (Thousand)	Minimum (Thousand)
January	255.86	7236.41	33.25	426	175
February	377.00	49656.57	59.11	744	171
March	566.00	69926.57	45.02	984	200
April	519.00	50431.43	43.27	791	185
May	276.86	13882.98	42.56	528	155
June	300.14	22296.12	49.75	568	118
July	262.86	12184.98	41.99	523	166
August	205.29	1607.06	19.53	260	135
September	255.14	6092.98	30.59	428	157
October	245.57	1821.96	17.38	321	192
November	425.14	82464.40	67.55	899	225
December	443.71	43025.06	46.75	885	248

From Table 1.1(a), it can be seen that the monthly mean values vary from month to month for this series. For instance, January and May to October have the means which are less than the overall mean 334.38 (thousand lb). The monthly mean is highest in March with 566.00 (thousand lb) and the lowest in August with 205.29 (thousand lb). Therefore, the highest which occurs in March is about 2.5 times the lowest mean which occurs in August. The variances for each of the months vary from 1607.66 (thousand lb)^2 to 82464.40 (thousand lb)^2 and the coefficient of variations vary from 17.38 percent to 67.55 percent. The coefficient of variations for November is found to be largest (67.55%), a fact which indicates that the monthly data over the years for November differs to a certain extent. The maximum value for each month is the lowest in August and the highest in March. The minimum value for each month is the lowest in June and the highest in December. When the mean value for the month is large, the maximum value and the minimum value of the series are also large. For the whole observed records, the minimum value of Airways (freight lb) series is 118 (thousand lb), which occurs in June, 1989. Similarly, the maximum value is 899 (thousand lb) which occurs in March, 1993. Therefore, during the observed period, the maximum value of this series is about 8 time that of the minimun value.

The yearly mean value, the variance, the coefficient of variations, maximum and minimum over the twelve months for each year from 1989 to 1995 of Airways (freight lb) series are presented in Table 1.1(b).

Table 1.1(b)

Basic Statistical Characteristics For Each Year : Airways (Freight Lb)

Year	Mean (Thousand)	Variance $(\text{Thousand})^2$	C. V.	Maximum (Thousand)	Minimum (Thousand)
1989	312.08	53157.08	73.88	858	118
1990	412.83	59806.14	59.24	899	171
1991	264.75	15488.52	47.01	568	162
1992	331.25	39406.85	59.93	796	157
1993	415.92	61120.24	59.44	984	212
1994	366.50	29930.08	47.20	791	196
1995	307.33	19813.06	45.80	655	182

From Table 1.1(b), it can be seen that the yearly means vary from 312.08 (thousand lb) in 1989 to 415.92 (thousand lb) in 1993. The variance is large in each year and the yearly coefficient of variations fluctuate around 55 percent. The maximum value is about 3.5 to 7 times of minimum value for each year.

1.2.2 Airways (Number of Passengers) Series

The monthly data of Airways (number of passengers) series are collected for 7 years, from 1989 to 1995 and presented in Table A1.2 of Appendix A. Basic statistical characteristics of this series are computed in the same way as in Airways (freight lb) series.

For each month these statistical characteristics are computed and presented in Table 1.2(a).

Table 1.2(a)

Basic Statistical Characteristics For Each Month : Airways (Number of Passenger)

Month	Mean (Thousand)	Variance (Thousand)^2	C.V.	Maximum (Thousand)	Minimum (Thousand)
January	41.14	109.55	16.32	62	27
February	39.43	88.82	15.02	56	24
March	46.14	104.69	15.06	61	28
April	42.71	170.20	19.96	59	22
May	41.86	89.84	14.65	56	28
June	31.57	62.24	14.04	46	22
July	32.29	77.06	15.45	48	21
August	33.14	60.98	13.56	47	21
September	34.29	62.20	13.67	47	24
October	38.57	87.96	15.10	53	26
November	43.00	114.28	16.30	56	28
December	41.71	149.35	18.92	59	31

From Table 1.2(a), it can be seen that the monthly mean values vary from 31.57 (thousand) to 46.14 (thousand). The months June to October. have a smaller mean value than the overall mean value of 38.82 (thousand). June has the lowest mean value and March has the highest. The variances for each of the months vary from 60.98 (thousand)^2 to 170.20 (thousand)^2. Variance for August is lowest whereas variance for April is the highest. The coefficient of variations lie between 13.56 percent and 19.96 percent. The largest coefficient of variations is found to be in April and the smallest in August. In this series, low values of the monthly mean from June to October is due to the rainy season since these months are very wet and rainy in Myanmar. For the whole series, the minimum value is 21 in July and August, 1989 and the maximum value is 62 in January, 1995. Therefore, the maximum value of the Airways (number of passengers) series is about 3 times that of the minimum value during the period of January, 1989 to December, 1995.

Some basic statistics for each year from 1989 to 1995 of Airways (number of passengers) series are also computed and presented in Table 1.2(b).

Table 1.2(b)

Basic Statistical Characteristics For Each Year : Airways (Number of Passenger)

Year	Mean (Thousand)	Variance (Thousand)^2	C. V.	Maximum (Thousand)	Minimum (Thousand)
1989	25.75	10.02	12.29	31	21
1990	32.75	28.19	16.21	39	23
1991	34.25	64.02	23.36	48	22
1992	35.00	22.00	13.40	43	30
1993	41.25	43.19	15.93	51	30
1994	48.67	60.22	15.95	59	38
1995	54.08	30.41	10.20	62	46

From Table 1.2(b), it can be seen that the yearly means vary from 25.75 (thousand) in 1989 to 54.08 (thousand) in 1995. The variance of each year varies from 10.02 $(\text{thousand})^2$ to 64.02 $(\text{thousand})^2$ and coefficient of variation for each year lies between 10.20 percent and 23.36 percent.

1.2.3 Railways (Freight Ton) Series

The monthly data of Railways (freigh ton) series are collected for 7 years, from 1989 to 1995 and presented in Table A1.3 of Appendix A. Basic statistical characteristics of this series are computed in the same way as in Airways series.

For each month these basic statistical characteristics are computed and presented in Table 1.3(a).

Table 1.3(a)

Basic Statistical Characteristics For Each Month : Railways (Freight Ton)

Month	Mean (Thousand Ton)	Variance $(\text{ThousandTon})^2$	C.V.	Maximum (Thousand Ton)	Minimum (Thousand Ton)
January	212.86	609.55	11.60	249	165
February	201.57	406.24	10.00	231	169
March	212.29	655.35	12.06	260	175
April	175.43	578.53	13.71	209	131
May	179.86	564.98	13.22	212	146
June	171.00	446.57	12.36	196	131
July	168.71	540.49	13.78	199	125
August	167.43	833.96	17.25	210	121
September	160.43	1169.10	21.31	207	107
October	175.29	1101.35	18.93	214	123
November	194.86	1967.27	22.85	276	127
December	209.86	1487.55	18.38	285	171

From Table 1.3(a), it can be seen that the monthly mean values vary from 160.43 (thousand ton) to 212.86 (thousand ton). The months, April to October, have a smaller mean value than the overall mean value of 185.74 (thousand ton). September has the lowest mean value and January has the highest. The variance of each month varies from 446.57 (thousand ton)^2 to 1967.27 (thousand ton)^2. Variance for February is the lowest whereas variance for November is the highest. The coefficient of variations for each month lies between 10.00 percent and 22.85 percent. The largest coefficient of variations is found to be in November and the smallest in February. In this series, the monthly mean decreases from June to October, then increases from November to highest in January. The low mean values from June to October is due to the rainy season since these months are very wet and rainy in Myanmar. The maximum value for each month is the smallest in June and the highest in December. Also, the minimum value for each month is the smallest in September and the largest in March. For the whole series, the minimum value is 107 in September, 1989 and the maximum values is 285 in December, 1994. Therefore, the maximum value of the Railways (freight ton) series is about 2.7 times that of the minimum value during the period of January, 1989 to December, 1995.

Some basic statistics for each year from 1989 to 1995 of Railways (freight ton) series are also computed and presented in Table 1.3(b).

Table 1.3(b)

Basic Statistical Characteristics For Each Year : Railways (Freight Ton)

Year	Mean (Thousand Ton)	Variance (Thousand Ton)^2	C. V.	Maximum (Thousand Ton)	Minimum (Thousand Ton)
1989	141.42	530.58	16.29	177	107
1990	162.33	634.06	15.51	208	124
1991	179.00	440.67	11.73	204	141
1992	185.42	368.74	10.36	220	147
1993	204.58	517.74	11.14	260	175
1994	224.00	864.33	13.12	285	181
1995	203.42	374.24	9.51	249	182

From Table 1.3(b), it can be seen that the yearly mean value of each year varies from 141.42 (thousand ton) to 224.00 (thousand ton). The variance for each year varies from 368.74 (thousand ton)^2 to 864.33 (thousand ton)^2 and coefficient of variation lies between 9.51 percent and 16.29 percent.

1.2.4 Inland Water Transport (Freight Ton) Series

The monthly data of Inland Water Transport (freight ton) series are collected for 7 years, from 1989 to 1995 and presented in Table A1.4 of Appendix A. Basic statistical characteristics of this series are computed in the same way as in Airways and Railways series.

For each month, these basic statistical characteristics are computed and presented in Table 1.4(a).

Table 1.4(a)

Basic Statistical Characteristics For Each Month : Inland Water Transport (Freight Ton)

Month	Mean (Thousand Ton)	Variance (Thousand Ton)^2	C.V.	Maximum (Thousand Ton)	Minimum (Thousand Ton)
January	245.57	1012.82	12.96	289	194
February	230.57	1068.53	14.18	270	170
March	265.29	1494.78	14.57	308	198
April	243.57	1153.67	13.94	277	195
May	245.00	1161.71	13.91	287	190
June	233.43	1065.10	13.98	280	180
July	234.43	723.10	11.47	275	200
August	228.29	179.63	5.87	250	210
September	215.86	280.69	7.76	236	190
October	228.29	817.63	12.53	260	181
November	229.43	696.82	11.51	264	187
December	239.00	691.71	11.00	269	191

From Table 1.4(a), it can be seen that the monthly mean values vary from 215.86 (thousand ton) to 265.29 (thousand ton). The monthly mean values vary do not much from month to month in this series. The variance for each month varies from 179.63 (thousand ton)^2 to 1494.78 thousand ton)^2. The coefficient of variations for each month lies between 5.87 percent and 14.57 percent. The largest coefficient of variations is found to be in March and the smallest in August. The minimum value is the smallest in February and the largest in August. For the whole series, the minimum value is 170 (thousand ton) in February, 1989 and the maximum is 308 (thousand ton) in March, 1993. Therefore, the maximum value of the Inland Water transport (freight ton) series is about 1.8 times that of the minimum value during the period of January, 1989 to December, 1995.

Some basic statistics for each year from 1989 to 1995 of Inland Water Transport(freight ton) series are also computed and presented in Table 1.4(b).

Table 1.4(b)

Basic Statistical Characteristics For Each Year : Inland Water Transport (Freight Ton)

Year	Mean (Thousand Ton)	Variance $(\text{Thousand Ton})^2$	C. V.	Maximum (Thousand Ton)	Minimum (Thousand Ton)
1989	199.50	284.75	8.46	233	170
1990	213.00	487.33	10.36	261	181
1991	213.68	74.41	4.04	228	201
1992	237.50	177.42	5.61	267	217
1993	263.83	477.64	8.02	308	226
1994	263.92	368.58	7.27	292	232
1995	264.58	451.58	8.03	306	236

From Table 1.4(b), it can be seen that the yearly mean value varies from 199.50 (thousand ton) to 264.58 (thousand ton). The variance for each year varies from 74.41 $(\text{thousand ton})^2$ to 487.33 $(\text{thousand ton})^2$ and coefficient of variation lies between 4.04 percent and 10.36 percent.

1.3 Test of Seasonality

The following model for the randomized complete block design (Daniel, W. W. and Terrell, T. C., 1992) will be used in testing seasonality in our monthly transport time series.

$$y_{ij} = \mu + \beta_i + \gamma_j + e_{ij} \qquad ; 1 \le i \le n \;, \quad 1 \le j \le k$$

where y_{ij} is a typical value from the overall population,

μ is an known constant,

β_i represents a yearly effect, reflecting the fact that the experimental unit fell in the i^{th} year,

γ_j represents a monthly effect, reflecting the fact that the experimental unit received the j^{th} month and

e_{ij} is a residual component representing all sources of variation other than months and years.

We make three assumptions when we use the randomized complete block design. (a) Each observed y_{ij} constitutes an independent random variable of size 1 from one of the kn populations represented. (b) Each of these kn populations is normally distributed with mean μ_{ij} and the same variance σ^2 . The e_{ij} are independently and normally distributed with mean 0 and variance σ^2 . (c) The block and treatment effects are additive. To state this assumption another way, we say that there is no interaction between months and years.

In general, we test

H_o: $\mu_1 = \mu_2 = \mu_3 = \ . \ . \ . = \mu_k$ versus

H_1: at least one equality does not hold.

In other words, we test the null hypothesis that the monthly means are all equal or equivalently, that there are no differences in monthly effects.

To analyze the data the needed quantities are the total sum of squares SST, the sum of squares for months SSM, the sum of squares for years SSY and the error sum of squares SSE. When these sum of squares are divided by the appropriate degree of freedom, we have the mean squares necessary for computing the F statistic. For our monthly transport data k=12 months and n=7 years, the degrees of freedom are computed as follows:

Total		Months		Years		Error
(kn-1)	=	(k-1)	+	(n-1)	+	(n-1)(k-1)

where k=12, n=7.

We find the error degrees of freedom, like the error sum of squres, by subtraction:

$$\begin{aligned}(kn-1) - (k-1) - (n-1) &= kn-1-k+1-n+1 \\ &= kn-k-n+1 \\ &= k(n-1) - (n-1) \\ &= (k-1)(n-1)\end{aligned}$$

Short-cut formulas for computing the required sum of squares are as follows:

$$SSM = \sum_{j=1}^{k} \frac{y_{.j}^2}{n} - C \quad ; \quad y_{.j} = \sum_{i=1}^{n} y_{ij}$$

$$SSY = \sum_{i=1}^{n} \frac{y_{i.}^2}{k} - C \quad ; \quad y_{i.} = \sum_{j=1}^{k} y_{ij}$$

$$SST = \sum_{i=1}^{n} \sum_{j=1}^{k} y_{ij}^2 - C$$

$$SSE = SST - (SSM + SSB)$$

where $C = \frac{y_{..}^2}{nk}$; $y_{..} = \sum_{i=1}^{n} \sum_{j=1}^{k} y_{ij}$.

We can display the results of the calculations for the randomized complete block design in an analysis of variance (ANOVA) Table.

ANOVA Table for a Two-Way Analysis of Variance

Source	S. S.	D. F.	M. S.	F-Ratio
Between Months	SSM	k-1	MSM=SSM/k-1	F_1=MSM/MSE
Between Years	SSY	n-1	MSY=SSY/n-1	F_2=MSY/MSE
Error	SSE	(n-1)(k-1)	MSE=SSE/(n-1)(k-1)	
Total	SST	kn-1		

We compare the computed ratios F_1 and F_2 with the critical values $K_1 = F_{\alpha,\,(k-1),\,(n-1)(k-1)}$ and $K_2 = F_{\alpha,\,(n-1),\,(n-1)(k-1)}$, respectively. If this ratios are equal to or exceed the critical values, we reject the null hypothesis.

The ANOVA Tables for our four monthly transport time series are shown in Table A1.5 to A1.8 of Appendix A.

Table 1.5

Mean Sum of Squares, F-Ratio, Critical Value and Decision For Each Series

S.N.	Series	MSM	F-Ratio	$K_1=F_{0.05,11,66}$	Decision
1.	Airways (freight lb)	180989.3	5.4	1.95	reject H_0
2.	Airways (number of passengers)	277.5	13.3	1.95	reject H_0
3.	Railways (freight ton)	4755.6	19.3	1.95	reject H_0
4.	Inland Water Transport (freight ton)	1982.5	8.4	1.95	reject H_0

Table 1.5 shows the Mean Sum of Squares for months (MSM), F-Ratio, Critical value K_1 and the Decision for each series. All the series are

found to be subject to variations from month to month. The ANOVA tables shown in Appendix A also show that yearly means are different in the population except for Airways (freight lb) series. Thus, the series are found to be subject to variations from year to year except for Airways (freight lb) series.

CHAPTER II

SEASONALITY IN A TIME SERIES

2.1 Introduction

A time series is a series of values of a variable at successive points in time or for successive intervals of time. The reason for analysing a time series is that the past behaviour of a time series may tell the investigator something about the future and hence forecasts can be made.

In classical approach, a time series is assumed to be composed of four components. These components are trend, cyclical, seasonal and random components. A trend refers to a smooth upward or downward movement of a time series over a long period of time. A periodic movement with a period of not more than a year is known as seasonal variation. Seasonal variations are cycles that complete themselves within the period of a calender year and then continue to repeat the same basic pattern. A periodic movement which has a longer duration than one year is known as cyclical variation. It refers to the long - term osscillations about a trend line or curve. A movement which is caused by unforeseeable and uncontrollable forces is known as a random variation.

Many time series, such as the monthly or quarterly series have an important seasonal component. Thus, it is important to separate the seasonal and trend components so that the overall deseasonalized series can be discerned and the magnitude and pattern of seasonal variation better be understood.

In this chapter, basic types of time series and kinds of seasonal pattern will be presented. Some methods of seasonal measurement based on the two basic models will be investigated. The methods discussed will include the well known methods of Ratio to Moving Averages, Link Relatives, the widely used B. L. S. and Census Mark II methods. The Regression methods, although it is not used much will also be presented. All these methods have been developed by various authors for the purpose of finding the seasonal indexes to represent seasonality in a time series.

2.2 Basic Types of Time Series Model

According to the way a time series is constructed from the four compoments, time series models are traditionaly differentiated. Two basic time series models are additive and multiplicative models. The time series is constructed by adding the components (additive model) or by multiplying them (multiplicative model). That is, in additive model, the value of the trend; the cycle, the seasonal and random or irregular components are added. In the multiplicative model, they are multiplied and are traditionaly expressed as percentages. In a mixed model with a multiplicative seasonal but an additive random component, the seasonal component is usually expressed as a percentage of the trend. Thus, the two basic models used for analyzing a time series are

$$Y_t = T_t + C_t + S_t + R_t$$

which is called the additive model and

$$Y_t = T_t\ C_t\ S_t\ R_t$$

which is called the multiplicative model. In these models, Y_t is the time series, T_t is the trend component, C_t is the cyclical component, S_t is the seasonal component and R_t is the random or irregular component.

Sometimes the trend and cyclical components are considered together (Kendall, 1973) and then the two models are

$$y_t = m_t + s_t + e_t$$

which is the additive model and

$$y_t = m_t\ s_t\ e_t$$

which is the multiplicative model. Other possible model is

$$y_t = m_t\ s_{1t} + s_{2t} + e_t$$

which is the mixed model. In these models, y_t is the time series, m_t is the smooth component (trend and short term oscillation) of the time series, s_t, s_{1t} and s_{2t} are the seasonal components and e_t is the error term. The mixed model is of additive multiplicative type in which the components add and multiply.

2.3 Nature of Seasonal Time Series

A time series is periodic with period s when similarity in the series occurs after s basic time intervals. Basic time interval means one month in monthly time series and the period is s = 12. It means one quarter in quarterly time series and the period is s = 4. In some series, there can be more than one period. Thus, in a weekly time series, there can be a period of s = 4 and a period of s = 12.

Monthly or quarterly time series may show seasonal effects within years. Seasonality means a tendency to repeat a pattern of behaviour over a seasonal period of one year. Seasonal series are characterized by a display of strong seasonal correlation at the seasonal lag, that is, the lag corresponding to the number of observations per seasonal period and usually at multiples of that lag.

Seasonal time series usually display time to time (for example, month to month) changes over the years, showing also within year variations. It is useful to understand the actural situation and is used for short term planning. For example, if the data represent the daily sales of a large restaurant, a considerable variation may be noticed depending on the day of the week. Another example, the publisher of a monthly magazine, may be concerned with monthly variations in sales throughout the year. The restaurant owner is concerned with the day of the week that is involved. The publisher of a monthly magazine is concerned with which month of the year is involved. The restaurant' s week-end sales are usually considerly higher than the other sales. The seasonal cycle for the restaurant is a week and the days of the week are seasons. For a monthly magazine publisher, months are the seasons and a year is a seasonal cycle.

The seasonally adjusted time series shows the tendency or trend of the time series and can be used for long term planning. Some seasonal adjustment procedures use seasonal indices to find seasonally adjusted time series. For short term planning, unadjusted time series is better to be used than the adjusted time series. Seasonal models can then be used to describe a seasonal times series and hence for forecasting.

2.4 Kinds of Seasonal Patterns

Steiner (1956) defined the following kinds of seasonal pattern usually found in a time series.

(a) Constant Seasonal

It is the simple kind of seasonal pattern. In this kind of seasonal variation, the seasonal factor operate in precisely the same fashion year after year. With a constant seasonal the generalized relationship is simply stated, with one figure for each segment (month or quarter) of the year. That is the simple average statement of each section, usually an index number is used to represent a constant seasonal. Also it assumes that such portion of within year variation as can be generalized does not systematically change. Thus, a constant seasonal pattern means that insofar as a recurrent systematic pattern can be generalized it is unchanging in form.

(b) Gradually Changing Seasonal

It is also called as progressive seasonal. Many series have changing systematic seasonal pattern and these patterns may be of several types. In one such types, the variations of the quarters or months get larger and larger as time progresses. Within the year the same ranking of quarters or months occurs, but the pattern is no longer constant among years. The amplitude of seasonal fluctuations steadily increases. In this case, the change in amplitude is itself systematic and it can not be represented by a single set of numbers, as was done with the constant seasonal.

A different kind of gradually changing (or progressive) seasonal also exists. In this type, the same segments of the year no longer bear the same relationship to each other over the period. The pattern of a year shifts in terms of timing in later years. There is a gradual progression in the patterns of successive years and such a seasonal is said to be progressive with respect to timing.

A seasonal pattern might systematically vary in amplitude and timing. There are a multitude of forms which progressive seasonal variation might take including cases in which both timing and amplitude change progressively over time.

(c) Osscillating Seasonal

This kind of seasonal is occasionally found in economic time series. In this kind, amplitudes of seasonal variations showed marked variation in amplitude at different levels of activity. For example, the percentage seasonal variation might appear to be more marked at high levels of activity than low ones. A series affected by cyclical variations might have such kind of variation. If shown by a seasonal index, it would be alternately rising and falling.

(d) Abruptly Changing Seasonal

In some economic time series a seaaonal pattern may abrutply altered. Over some period there is one seasonal pattern for part of the period, and a second different seasonal pattern thereafter. Causes for such abrupt changes are usually identifiable. Treatment of such cases involves separate determination of the kind and form of seasonal for each of the sub-periods involved.

2.5 Traditional Methods of Measurement

Many time series have an important seasonal component and for such series it is important to be able to separate out the seasonal and trend components so that the overall deseasonalized trend can be discerned and the magnitude and pattern of seasonal variation better understood. It is important to know whether a falling value or rising value of a time series indicates the real change in the long term movement or whether it is just a temporary change due to seasons. In other words, the seasonal effects in a time series need to be examined. There are different reasons for wanting to examine seasonal effects. Some of the reasons given by Kendall (1973) are

(a) to compare the variable under study at different points of the year in order to know the inter-year variation and to take action correspondingly,

(b) to remove seasonal effects from the series in order to study its other components uncontaminated by the seasonal element and

(c) to adjust the time series for seasonal effects.

Because these objectives are not the same, different methods of seasonal determination may be needed to be used. There are many different methods for computation of seasonal index, some of which are quite accurate and some of which are only appropriate. The following methods will be discussed in this section.

(1) Ratio to moving average method

(2) Link relatives method

(3) Census Mark II method

(4) B. L. S. seasonal factor method

(5) Regression method.

These methods have been developed to meet different objectives of estimating seasonals and under the assumed models of the time series. The seasonal pattern itself is important in the application of these methods since most of the methods assume that the seasonal pattern is constant or stable. Of these methods, the Ratio to Moving Average method and the Link Relatives method are simple and which are the most widely used.

Ratio to Moving Average Method

Under the assumptions of multiplicative model and constant seasonal pattern, the followings are the steps for the computation of the seasonal index by the Ratio to Moving Average method. (Steiner, 1956)

(1)Find the twelve months centered moving averages. This is equivalent to a moving average of thirteen months with weights $\frac{1}{24}(1,2,2,\ldots,2,2,1)$.

By finding twelve months centered moving averages, we eliminate the seasonality, since the seasonal pattern is periodic with a period of twelve months. Also it will eliminate the random component or irregular movements. Therefore, the centered twelve month moving averages are the approximates of trend and cyclical components.

(2) Compute the ratio to moving average values, that is, the original data is divided by its appropriate moving average value. There, the first and last six months may not be obtained.

By this step, the trend and cyclical components are removed from the original data and the ratios are the values due to seasonal and random components. They are called specific seasonals. (Steiner, 1956)

(3) Compute the averages of these ratios referring to the same months. These averages are the crude seasonal index values.

This step involves two different purposes : the elimination of the random components and averaging the seasonal relatives referring to the same months.

(4) Adjust the crude seasonal index.

In multiplicative model, the total seasonal index values have to be equal to twelve (or 1200 percent) for monthly series. Therefore, the crude seasonal index is adjusted to get a total of twelve (or 1200 percent).

(5) In order to get the deseasonalized series, the observed records have to be divided by the appropriate seasonal index value.

Link Relatives Method

The same assumptions as in ratio to moving average method have to be made to compute the seasonal index by the link relatives method. The followings are the steps for the computation of the seasonal index by the link relatives method.(Kendall, 1973)

(1) Find the link relatives values.

This is to divide the current value by the previous values. Then, the first one may not be obtained. These values show the relative changes of the consecutive values.

(2) Find the averages of the link relatives values referring to the same months.

These averages show the average changes in consecutive months within the whole period of twelve months.

(3) Compute the chain relative values by assumimg that the chain relative value of the first month is unity.

The chain relative value for the current month is the product of the chain relative value of the previous month and the average of link relatives for the current month. These chain relative values constitute seasonal pattern and the trend within a year.

(4) Determine the trend component within the year and adjust for the trend.

To determine the trend component within a year, the chain relative value of the first month is computed, that is, the product of the chain value of the last month and the average of the link relatives for the first month is computed and the difference between the chain relative value and the setting value unity is found. This difference is regarded as the trend for twelve months. By dividing this value by twelve, the difference for a month is obtained, which is assumed to be the coefficient of linear trend and denoted by Δ (delta). If the delta is positive, there exists an upward trend and the respective trend values $(i-1)\Delta$, $i = 1, 2, \ldots, 12$ are subtracted from the corresponding chain relative values. Similarly, if the delta is negative, there exists a downward trend and the respective trend values $(i-1)|\Delta|$, $i = 1, 2, \ldots, 12$ are added to the corresponding chain relative values. After the adjustment, the adjusted chain relative values are regarded as the crude seasonal index.

(5) Adjust the crude seasonal index.

The crude seasonal index is adjusted to get a total of twelve (or 1200 percent) and the seasonal index is obtained.

(6) In order to get the deseasonalized series, the observed records have to be divided by the appropriate seasonal index value.

The Census Mark II Method

For a great many practical purpose where monthly or quarterly data are involved, use may be made of a powerful program known as Census Mark II devised by Shiskin for the U. S. Bureau of Census. It is widely used and notwithstanding its vulnerability on a few theoretical points seem to work very well in practice. Its purporse is to separate off the seasonal and residual variation, but it does not dissect the smooth component into trend and short term oscillation. There are several versions which differ in minor particulars, but basically the procedure for monthly data proposed by Kendall (1973) is as follows.

(1) An option is offered whether to adjust the series for number of trading or working days. If it is adopted, all subsequent operations are on the adjusted series.

(2)A moving average is taken. There are a number of options in the extent and weighting of the average.

(3) This is divided into the series to give a first estimate of the seasonal-plus-irregular component. End values are estimated, usually as the nearest value for the same month. Extreme values are replaced by the mean of the two values for that month lying on either side of it.

(4) To decide the relative importance of seasonal and irregular components, an analysis of variance is carried out between years, between months and residual. If the variance between months is significant, on an F-test as compared with the residual variance, there is evidence of genuine seasonal effects.

(5) For any month the ratio of within-month to residual variance for that month is allowed to decide among number of options what

moving average shall be used to smooth the random-residual term. A different average may be used for different months.

(6) In some cases the seasonal so determined is divided into the primary series to get a preliminary deseasonalized series, and another moving average taken to get a second estimate of the trend. The seasonal factors are adjusted so as to sum to 12.

(7) These results provide estimates of the smooth component and of a moving seasonal component. The residual is obtained by subtraction(or sometimes by dividing the primary series by smooth-plus-seasonal component if the error is regarded as multiplicative).

(8) Various subsidiary statistics such as the error variance are computed.

The B. L. S. Seasonal Factor Method

The B. L. S. method for developing seasonal factors for economic time series was introduced in 1960 by the staff of the Bureau of Labour Statistics, Department of Labour, United States. Since then, continued research in seasonal factor methodology by the Bureau's staff has resulted in some changes in the method, the latest version was introduced early in 1963.

The B. L. S. method attempts to separate an economic time series into three constituent parts. Underlying movement or trend-cycle (T), annual repetitive or seasonal (S) and random or irregular (I). The three components, when multiply together, completely and exactly exhaust the original observations. The process used to develop the consitituent parts is an iterative one, each successive iteration (cycle) resulting in an improved estimate for each of the components of the original series. The method involves four or seven iterations, depending on whether any original observations are identified or introduced as extreme for a monthly time series.

If the test (fourth) iteration reveals no extreme value, and none are introduced from earlier runs of the same series, then the trend-cycle, seasonal and irregular component of the third iteration are considered to be the final components of the original series. Seasonally adjusted values are computed accordingly.

If the test (fourth) iteration reveals one or more extreme original observations, each such observation is replaced by a substitute value which is the product of its trend-cycle and seasonal components as developed in the test iteration. The modified original series in which extreme observations have been replaced is then processed in the regular maner, starting with a centered 12-month moving average and continuing through three complete iterations (fifth, sixth and seventh). These last three iterations are designed to remove the effects of the extreme original observations from the trend-cycle, seasonal and irregular components being developed. The trend-cycle and seasonal components resulting from the seventh iteration are final.

To summarize, the seven stages of the B. L. S. procedure are:

Iteration 1. Startingtrend-cycle is a centered 12-month moving average of the original observations.

Iteration 2. First modified trend-cycle is the 12-month average modified by a set of 7-term parabolic weights applied to the irregulars of the preceding cycle.

Iteration 3. Second modified trend-cycle is the preceding iteration trend-cycle modified by a set of 7-term flat (uniform) weights applied to irregulars of preceding cycle.

Iteration 4. Test iteration trend-cycle is the preceding iteration trend-cycle modified by central-zero, 7-term flat weights applied to irregulars of preceding cycle. Seasonal factors are secured by applying a 5-term central-zero weight pattern to seasonal-irregulars. Extreme observations are identified and replaced by values developed in the test iteration.

Iteration 5, 6, 7. Same as 1, 2, 3 except that any replacement value developed in the fourth iteration, or introduced from prior runs, is used instead of an original values identified as "extreme". The original extreme values are restored for calculating the final irregular components and the seasonally adjusted values.

Because the method requires a large volume of calculations, an electronic computer has to be used. The modifications introduced in 1963 required changes in the computer program which reduced the length of the series that could be handled. But, the Census Mark II method and the B. L. S. seasonal factor method are quite lengthy and complicated.

The Regression Method

Let y_{ij} be the values of the original series for the i^{th} year and j^{th} month. Let x_{ij} be the estimate of trend based on a centered 12-point moving average. Then the regressions supposed by Kendall (1973)

$$y_{ij} = a_j + b_j x_{ij}$$

are calculated for each of the 12 months. If $a_j = 0$, a multiplicative model is obtained. If $b_j = 1$, an additive model resulted.

Adjust the constants a and b to

$$a_j' = a_j - \bar{a}, \qquad \text{where} \qquad \bar{a} = \frac{1}{12}\sum_{j=1}^{12} a_j$$

$$b_j' = b_j - \bar{b}, \qquad \text{where} \qquad \bar{b} = \frac{1}{12}\sum_{j=1}^{12} b_j$$

The seasonally adjusted values are then given by

$$y_{ij}(\text{adjusted}) = \frac{y_{ij} - a_j'}{b_j'} .$$

This model has the advantage that it copes with both additive and multiplicative effects. However, there are serious disadvantages which would outweight the advantage for many purposes. It requires the estimation of 24 constants, and unless the number of years is substantial those estimates are rather unreliable. The constants a' and b' are estimated for the whole series and make no allowance for rapid recent changes in seasonal pattern.

And consequently the estimators are troublesome to update. The method has not come into general use.

The seasonal indexes were calculated by the appropriate method and these may be used to deseasonalize the original time series data. Removing the seasonal effect provides a series that may prove useful in analyzing longer term movement in the series. The resulting series is sometimes used instead of the smooth data to identify cyclical activity. Deseasonalized data may be used for instance, to compare suuccessive time periods to aid in determining whether a turning point in the current cycle has been reached.

Removal of seasonal fluctuation is accomplished by dividing the original time series by the corresponding seasonal indexes. Under the classical model this is expressed symbolically as,

$$\text{Deseasonalized value} = \frac{\text{Actual (monthly or quarterly)data}}{\text{Seasonal Index}}$$

$$= \frac{T_t x C_t x S_t x I_t}{S_t}$$

$$= T_t x C_t x I_t$$

Dividing actual data (Y_t) by seasonal index removes the seasonal component (S_t) from the series, leaving only the trend (T_t), cyclical (C_t) and irregular component (I_t). Eliminating seasonal movement makes it simpler to identify the longer term oscillations.

Besides these traditional methods, the seasonal can be measured by the Box-Jenkins (1976) seasonal models which will be discussed in chapter III. Our monthly transport time series are fitted by using Box-Jenkins models and will be presented in chapter V.

2.6 Use of Dummy Variables for Seasonal Adjustment

Many of the models that have been suggested incorporate seasonal variation in a deterministic way. For example, a seasonal time series might be modeled as a periodic function of a time plus a random component. A seasonal series modeled as a periodic function of time y_t, plus a random component e_t, is

$$y_t = \sum_{j=1}^{6} (\alpha_j \cos \lambda_j t + \beta_j \sin \lambda_j t)$$

and $\lambda_j = \frac{2\pi j}{12}$, $\beta_6 = 0$.

This formulation was used by Hannan (1960). Alternatively, dummy variables could be introduced to reflect additive effects associated with particular months or quarters. Economic time series rarely exhibit these kinds of deterministic seasonality. Rather, the pattern and intensity of seasonal variation change as the time changes.

Johnston (1972) proposed that dummy variables also play an important role in problems of seasonal adjustment. There is the conventional and long - standing problem of deseasonalizing a given quarterly or monthly time series. Suppose we have 4n quarterly observations on z so that z_{ij} in is the value of z in the j^{th} quarter of the i^{th} year (i = 1, . . ., n; j=1, 2, 3, 4).

Let us define 4n x 4 matrix $\underset{\sim}{D}$.

$$\underset{\sim}{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is a sample matrix for four dummy variables, defined by

D_i = 1 ; if the observation relates to quarter i; i = 1, 2, 3, 4.

= 0 ; otherwise.

The regression of $\underline{z}$ on $\underline{D}$ is

$$\underline{z} = \underline{D}\,\underline{b} + \underline{z}^{\alpha} \qquad (2.6.1)$$

where $\underline{b}$ is the vector of least squares coefficients and $\underline{z}^{\alpha}$ is the vector of residuals.

By ordinary least squares method, we have

$$\underline{b} = (\underline{D}'\underline{D})^{-1}\underline{D}'\underline{z}$$

and from equation (2.6.1), we get

$$\begin{aligned}\underline{z}^{\alpha} &= \underline{z} - \underline{D}\underline{b}\\ &= \underline{z} - \underline{D}\,(\underline{D}'\underline{D})^{-1}\underline{D}'\underline{z}\\ &= [\underline{I} - \underline{D}(\underline{D}'\underline{D})^{-1}\underline{D}']\underline{z} \qquad (2.6.2)\\ &= \underline{M}\underline{z} \qquad (2.6.3)\end{aligned}$$

where $\underline{M} = [\underline{I} - \underline{D}(\underline{D}'\underline{D})^{-1}\underline{D}']$ giving $\underline{z}^{\alpha}$ as a linear transformation of $\underline{z}$. It is seen that $\underline{M}$ is symmetric idempotent (that is. $\underline{M}' = \underline{M}$, $\underline{M} = \underline{M}^2 = \underline{M}^3 = \ldots$) with the property $\underline{M}\underline{D} = \underline{0}$.

The series $\underline{z}^{\alpha}$ cannot serve directly as a deseasonalized series for two reasons. First of all, it sums to zero, whereas it would seem plausible to require a deseasonalized series to have the same sum as the uncorrected. original series. Secondly, it can be seen from the nature of $\underline{D}$ that

$$\underline{b} = \begin{bmatrix}\bar{z}_1\\ \bar{z}_2\\ \bar{z}_3\\ \bar{z}_4\end{bmatrix}$$

where $\bar{z}_j = \frac{1}{n} \sum_{j=1}^{n} z_{ij}$; j = 1, 2, 3, 4 is the mean of j^{th} quarter z value. This $\underline{z}^{\alpha}$ merely consists of deviations of the z values from the quarterly means. If the original series contain trend and cyclical elements these will effect the quarterly means and thus in turn the deseasonalized series.

Since $\underline{z}$ is composed of trend, cyclical component, seasonal component and a disturbance. Thus, $\underline{z}$ is now regressed on an expanded matrix [$\underline{P}$ $\underline{D}$] where $\underline{P}$ is an appropriate set of powers of time, that is,

$$[\underline{P} \;\; \underline{D}] = \begin{bmatrix} 1 & 1^2 & 1^3 & \cdots & 1^p & 1\;0\;0\;0 \\ 2 & 2^2 & 2^3 & \cdots & 2^p & 0\;1\;0\;0 \\ 3 & 3^2 & 3^3 & \cdots & 3^p & 0\;0\;1\;0 \\ 4 & 4^2 & 4^3 & \cdots & 4^p & 0\;0\;0\;1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots\;\vdots\;\vdots\;\vdots \\ 4n & (4n)^2 & (4n)^3 & \cdots & (4n)^p & 0\;0\;0\;1 \end{bmatrix}$$

Thus, the regression may be written as

$$\underline{z} = \underline{P}\,\underline{a} + \underline{D}\,\underline{b} + \underline{e} \qquad (2.6.4)$$

and the deseasonalized series would now be defined as

$$\underline{z}^{s} = \underline{z} - \underline{D}\,\underline{b} \qquad (2.6.5)$$

Jorgenson (1964) has argued that if the $\underline{P}$ and $\underline{D}$ matrices are properly specified then $\underline{a}$ and $\underline{b}$ will be best linear unbiased estimates of the systematic and seasonal components. Then, equation (2.6.4) can be written as

$$\underline{z} = [\underline{P} \;\; \underline{D}] \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} + \underline{e}$$

By ordinary least squares, the estimates of $\underline{a}$ and $\underline{b}$ are given as

$$\begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} = \begin{bmatrix} \underline{P}'\underline{P} & \underline{P}'\underline{D} \\ \underline{D}'\underline{P} & \underline{D}'\underline{D} \end{bmatrix} \begin{bmatrix} \underline{P}'\underline{z} \\ \underline{D}'\underline{z} \end{bmatrix}$$

Applying the results for the inverse of a partitioned matrix, we get

$$\underline{b} = (\underline{D}'\underline{N}\underline{D})^{-1}\underline{D}\underline{N}\underline{z}$$

where $\underline{N} = \underline{I} - \underline{P}(\underline{P}'\underline{P})^{-1}\underline{P}'$.

By substituting the value of $\underline{b}$ in equation (2.6.5) gives

$$\underline{z}^{g} = \underline{z} - \underline{D}(\underline{D}'\underline{N}\underline{D})^{-1}\underline{D}'\underline{N}\underline{z}$$

$$= [\, \underline{I} - \underline{D}(\underline{D}'\underline{N}\underline{D})^{-1}\underline{D}'\underline{N}\,]\,\underline{z}$$

$$= \underline{T}\,\underline{z}$$

where $\underline{T} = \underline{I} - \underline{D}(\underline{D}'\underline{N}\underline{D})^{-1}\underline{D}'\underline{N}$.

Thus, the deseasonalized series can be expressed as a linear transformation of $\underline{z}$. However, in contrast with the matrix $\underline{M}$ defined in equation (2.6.3), the $\underline{T}$ matrix is not symmetric (that is, $\underline{T}' \neq \underline{T}$), but it is idempotent (that is, $\underline{T} = \underline{T}^2 = \underline{T}^3 = \ldots$) and does satisfy the condition $\underline{T}\,\underline{D} = \underline{0}$.

2.7 Use of Smoothing Methods

Winters (1960) has suggested an adaptive scheme for forecasting seasonal time series in which both the level and the seasonal factor in each time period are revised in accordance with a smoothing formula. It is conceptually an extension of simple exponential smoothing having the advantages of adaption but still lacking a basis in statistical theorey.

Consider a seasonal time series z_t with period s (so that $s = 4$ for quarterly series and $s = 12$ for monthly series). The most commonly employed variant of the Holt (1957) - Winters (1960) method regards the seasonal factor F_t as multiplicative (while trend remains additive). Then F_t is estimated as

$$F_t = D\,(z_t / \bar{z}_t) + (1 - D)\,F_{t-s}\,; \qquad 0 < D < 1 \qquad (2.7.1)$$

It is assumed that $z_t > 0$ for all t. The $\bar{z}_t$ is estimated by

$$\bar{z}_t = A(z_t / F_{t-s}) + (1 - A)(\bar{z}_{t-1} + T_{t-1}) \quad ; 0 < A < 1 \qquad (2.7.2)$$

The trend component is again estimated as

$$T_t = C(\bar{z}_t - \bar{z}_{t-1}) + (1 - C)T_{t-1} \quad ; 0 < C < 1 \qquad (2.7.3)$$

In order to employ equations (2.7.1), (2.7.2) and (2.7.3) it is necessary to specify starting values. A very simple way to accomplish this is to take

$$F_j = z_j / (\tfrac{1}{s} \sum_{k=1}^{s} z_k) \quad ; j = 1, 2, 3, \ldots, s, \text{ and } T_s = 0.$$

The three updating equations are used recursively for t = s+1, s+2, . . . , n. Since trend is additive and seasonality is multiplicative, forecasts of future values are given by

$$\begin{aligned} \hat{z}_t(L) &= (\bar{z}_t + LT_t)F_{t+L-s} \quad ; \quad L = 1, 2, \ldots, s \\ &= (\bar{z}_t + LT_t)F_{t+L-2s} \quad ; \quad L = 1, 2, \ldots, 2s \\ &\;\vdots \end{aligned} \qquad (2.7.4)$$

The Holt - Winters approach can easily be modified to deal with the seasonal factor which is additive. In this case, equations (2.7.1) and (2.7.2) are replaced by

$$F_t = D(z_t - \bar{z}_t) + (1 - D)F_{t-s} \quad ; \quad 0 < D < 1$$

and

$$\bar{z}_t = A(z_t - F_{t-s}) + (1 - A)(\bar{z}_{t-1} + T_{t-1}) \quad ; \quad 0 < A < 1.$$

The forecasting equation (2.7.4) is now replaced by

$$\begin{aligned} \hat{z}_t(L) &= \bar{z}_t + LT_t + F_{t+L-s} \quad ; L = 1, 2, \ldots, s \\ &= \bar{z}_t + LT_t + F_{t+L-2s} \quad ; L = 1, 2, \ldots, 2s \\ &\;\vdots \end{aligned}$$

It remains only to choose the smoothing constants A, C and D employed in the Holt - Winters algorithms. If the value of the smoothing constant are low, it is the same as giving more weights to the past observations. One possibility is to choose the smoothing constants according to one' s assessment of the characteristic of the particular series under consideration. A more objective approach proposed by Holt and Winters is to select those values that would have best forecast the given observations.

The most common procedure is to seek the smoothing constants that provide the best one - step ahead forecasts. The procedure is to choose a grid of possible values of A, C and D and to calculate the one -step ahead forecasts (that is, $\hat{z}_t(L)$, $t = m, m+1, \ldots, n-1$) for each set of the values of the smoothing constants. That particular set for which the sum of squared errors

$$s = \sum_{t=m+1}^{n} [z_t - \hat{z}_t(1)]$$

is smallest is then used to calculate actual forecasts of all future values of the series. The starting points m for this procedure is taken to be an integer sufficiently large as to allow the effects of the choice of initial starting up values to have died down. Box and Jenkins (1976) considered instead a particular class of linear stochastic processes that display seasonal behaviour. These models can be used as the basis for models of seasonal time series. These models will be discussed in chapter III.

2.8 Comments

Many time series have important seasonal component and it is necessary to measure it and adjust the time series for seasonal variation. Considerable efforts have been made for the development of seasonal adjustment procedures which produces deseasonalized series. These efforts led to various methods of finding measures of seasonal variation.

Each method has its own logic, advantages and disadvantages. Under specific conditions, each is likely to be more efficient than others. Some methods are valid under the additive model and some under multiplicative model. It will be found that some methods are quite simple and easy to perform and others like B. L. S. and Census Mark II are quite lengthy and complicated. The investigator has to choose a suitable method to fulfil his own purposes for seasonal adjustment.

These methods of seasonal measurement are based on the two basic models of time series and the trend component of the smooth component represented by the polynomial trend or the moving averages. The methods covered can be used to find the constant seasonals. For other types of seasonals single set of seasonal measures may not be enough to represent the changing seasonal patterns. Besides these methods, the seasonal can be measured by using the models such as the exponential smoothing, the harmonic representation and the Box - Jenkins (1976) seasonal models. Spectral methods can also be used to assess the seasonal adjustment procedures.

CHAPTER III

SEASONAL TIME SERIES MODELS

3.1 Introduction

In chapter II, traditional methods of measuring seasonality in a time series were investigated. In these methods a time series is assumed to be composed of four components if they exist and decomposition of a time series into its components plays an essential role. Instead of this, variations in a time series can be explained by a suitable model. Many models have been suggested, including deterministic and stochastic models. It is common to use deterministic functions of time to represent the systematic variation of a time series, the random part being explained in probabilistic terms. Alternatively random functions of time can be used to explain certain variations in a time series. In this chapter, stochastic time series models which can be used to represent a seasonal time series will be discussed. These models were due to Box and Jenkins (1976) and have been successfully applied to many time series.

3.2 Stochastic Seasonal Models

Linear stochastic process can display seasonal behaviour of a time series. This class can be considered as a basis for models of seasonal time series.

Consider the stochastic model for non - seasonal series.

$$\varphi(B)\tilde{z}_t = \theta(B)a_t$$

where $\varphi(B) = \phi(B)(1-B)^d$ and $\tilde{z}_t = z_t - \mu_z$.

$\theta(B)$ is a moving average operator and $\phi(B)$ is a stationary autoregressive operator.

The generalized autoregressive operator $\varphi(B)$ determines the eventual forecast function, which is the solution of the difference equation

$$\varphi(B)\hat{z}_t(L) = 0$$

where B operates on L.

To represent the seasonal behaviour, the forecast function should trace out a periodic pattern. Then, φ (B) should produce a forecast function consisting of a mixture of sines, cosines and possibly mixed with polynomial terms, to allow for changes in the level of the series and changes in the seasonal pattern.

For example, with monthly data, a forecast function which is a sine wave with a twelve month period adaptive in phase and amplitude, will satisfy the difference equation

$$(1 - \sqrt{3}\,B + B^2)\,\hat{z}_t(L) = 0 \qquad (3.2.1)$$

This equation can be solved as

$$\hat{z}_t(L) - \sqrt{3}\,\hat{z}_t(L-1) + \hat{z}_t(L-2) = 0$$

Let $\hat{z}_t(L) = \lambda^L$.

Therefore,

$$\lambda^L - \lambda^{L-1} + \lambda^{L-2} = 0 \qquad ;\lambda \neq 0 \qquad (3.2.2)$$

$$\lambda = \frac{\sqrt{3} \pm \sqrt{3-4}}{2}$$

$$= \frac{\sqrt{3}}{2} \pm i(\frac{1}{2})$$

$$= \cos\frac{2\pi}{12} \pm i\sin\frac{2\pi}{12} \qquad (3.2.3)$$

Then,

$$
\begin{aligned}
\hat{z}_t(L) &= A\ \lambda_1{}^L + B\ \lambda_2{}^L \\
&= A\,(\cos \frac{2\pi}{12} + i\ \sin\ \frac{2\pi}{12})^L + B\,(\cos \frac{2\pi}{12} - i\ \frac{2\pi}{12}\sin\ )^L \\
&= A\,(\cos \frac{2\pi L}{12} + i\ \sin\ \frac{2\pi L}{12}) + B\,(\cos \frac{2\pi L}{12} - i\ \sin\ \frac{2\pi L}{12}) \\
&= (A + B)\cos\ \frac{2\pi L}{12} + i\,(A - B)\ \sin\ \frac{2\pi L}{12}
\end{aligned}
$$

where A and B have to be determined from the initial conditions.

$\hat{z}_t(L)$ will then exhibit periodic behaviour. However, it is not true the periodic behavior is necessarily represent economically by mixture of sines and cosines because many sine - cosine components would need to represent 'single - spike' of some sales data.

3.3 Seasonal ARMA Models

A particular class of linear stochastic processes that display seasonal behaviour can be considered as a basis for models of seasonal time series.

3.3.1 Seasonal AR Model

This model has the following specifications:

(1) if s is the number of observations per seasonal period, the order of AR process is an integer multiple of s.

(2) the only non - zero coefficients are those with subscripts that are an integer multiple of s.

The model is

$$z_t = \phi_s\ z_{t-s} + \phi_{2s}\ z_{t-2s} + \ldots + \phi_{Ps}\ z_{t-Ps} + a_t$$

where P is the largest multiple of s represented in the model and a's are random shocks with mean zero. constant variance σ_a^2 and uncorrelated.

Let $\phi_{js} = \phi_j$, then the model becomes

$$z_t = \phi_1 z_{t-s} + \phi_2 z_{t-2s} + \ldots + \phi_P z_{t-Ps} + a_t \qquad (3.3.1)$$

which refers to a seasonal AR process of order P. The seasonal autoregressive model (3.3.1) expresses the current value of the process z_t as finite weighted sum of P previous values $z_{t-s}, z_{t-2s}, \ldots, z_{t-Ps}$ of the process plus random shock a_t. Here

$$E[a_t] = 0 \text{ for all } t,$$

$$V[a_t] = E[a_t^2] = \sigma_a^2 \text{ for all } t,$$

and $$Cov[a_t, a_{t'}] = E[a_t a_{t'}] = 0 \text{ for all } t \neq t'.$$

Autocorrelation Structure

The autocovariance function of the seasonal AR (P) model is found by multiplying throughout in (3.3.1) by z_{t-k} and taking the expected values, that is,

$$\begin{aligned}\gamma_k &= Cov[z_t, z_{t-k}] \\ &= E[z_t z_{t-k}] \\ &= E[\phi_1(z_{t-s} z_{t-k}) + \phi_2(z_{t-2s} z_{t-k}) + \ldots + \phi_P(z_{t-Ps} z_{t-k}) + (a_t z_{t-k})] \\ &= \phi_1 \gamma_{k-s} + \phi_2 \gamma_{k-2s} + \ldots + \phi_P \gamma_{k-Ps} \quad ; k = 1, 2, \ldots Ps\end{aligned}$$

(3.3.2)

where $E[z_{t-j} z_{t-k}] = \gamma_{k-j}$ and the last term $E[a_t z_{t-k}]$ vanishes for $k > 0$ since z_{t-k} can only involve the random shock term a_j up to time t–k which are uncorrelated with a_t for $k \neq 0$. The autocorariances are non - zero at lags s, 2s, . . . , Ps.

For k = 0, the variance of the seasonal AR process is obtained as

$$\gamma_0 = \phi_1 \gamma_s + \phi_2 \gamma_{2s} + \ldots + \phi_P \gamma_{Ps} + \sigma_a^2 \qquad (3.3.3)$$

since $\gamma_{-j} = \gamma_j$ and $E[z_t a_t] = E[a_t^2] = \sigma_a^2$.

Dividing γ_k by γ_0, the autocorrelation function satisfies the difference equation

$$\rho_k = \phi_1 \rho_{k-s} + \phi_2 \rho_{k-2s} + \ldots + \phi_P \rho_{k-Ps} \quad ; k = 1, 2, \ldots, Ps \qquad (3.3.4)$$

The autocorrelation function will be non - zero at only lags that are integer multiples of s . The autocorrelation at seasonal lags persists indefinitely, although with declining intensity.

SAR (1) Model

Consider the SAR (1) model (P = 1),

$$z_t = \phi_1 z_{t-s} + a_t$$

where a_t's are random shocks with usual assumptions.

The autocovariance function of the SAR (1) model is obtained by substituting P = 1 in equation (3.3.2)·

We get

$$\gamma_k = \phi_1 \gamma_{k-s} \quad ; k = 1, 2, \ldots, Ps$$

For k = 0 , the variance of the SAR (1) model is obtained as

$$\gamma_0 = \text{Cov}[z_t, z_t] = E[z_t \; z_t]$$
$$= E[(\phi_1 z_{t-s} + a_t) z_t]$$
$$= \phi_1 E[z_{t-s} \; z_t] + E[a_t \; z_t]$$
$$= \phi_1 E[z_{t-s}(\phi_1 z_{t-s} + a_t)] + E[a_t(\phi_1 z_{t-s} + a_t)]$$
$$= \phi_1^2 E[z_{t-s}^2] + \phi_1 E[a_t z_{t-s}] + \phi_1 E[a_t z_{t-s}] + E[a_t^2]$$
$$= \phi_1^2 \gamma_0 + \sigma_a^2$$

where $E[z_{t-s}^2] = \gamma_0$, $E[a_t^2] = \sigma_a^2$ and $E[a_t \; z_{t-s}] = 0$, $s \neq 0$. Then, we get

$$\gamma_0 = \frac{\sigma_a^2}{1 - \phi_1^2} \; .$$

When k=1, $\gamma_1 = \text{Cov}[z_t, z_{t-1}] = E[z_t z_{t-1}]$

$$= E[(\phi_1 \; z_{t-s} + a_t) z_{t-1}]$$
$$= \phi_1 E[z_{t-s} \; z_{t-1}]$$
$$= \phi_1 E[(\phi_1 \; z_{t-s-1} + a_{t-1}) z_{t-s}]$$
$$= \phi_1^2 \; E[z_{t-s-1} \; z_{t-s}]$$
$$= 0$$

where $E[a_t \; z_{t-1}] = 0$, $E[a_{t-1} z_{t-s}] = 0$, $s \neq 1$ and

$E[z_{t-s-1} \; z_{t-s}] = E[z_{t-s-1}(a_{t-s} + \phi_1 a_{t-2s} + \phi_1^2 \; a_{t-3s} + \ldots)] = 0$.

Similary, $\gamma_k = 0$ for $k = 2, 3, \ldots, s-1$.

When $k = s$, $\gamma_s = \text{Cov}[z_t, z_{t-s}] = E[z_t \; z_{t-s}]$

$$= E[(\phi_1 z_{t-s} + a_t) z_{t-s}]$$
$$= \phi_1 E[z_{t-s}^2] + E[a_t \; z_{t-s}]$$
$$= \phi_1 \gamma_0$$

where $E[z^2_{t-s}] = \gamma_0$ and $E[a_t z_{t-s}] = 0$.

Then, $\gamma_k = 0$ for $k = s+1, s+2, \ldots, 2s-1$.

When $k = 2s$, $\gamma_{2s} = \text{Ccov}[z_t, z_{t-2s}] = E[z_t z_{t-2s}]$

$$\begin{aligned}
&= E[(\phi_1 z_{t-s} + a_t) z_{t-2s}] \\
&= \phi_1 E[z_{t-s} z_{t-2s}] + E[a_t z_{t-2s}] \\
&= \phi_1 E[(\phi_1 z_{t-2s} + a_{t-s}) z_{t-2s}] \\
&= \phi_1^2 E[z^2_{t-2s}] + E[a_{t-s} z_{t-2s}] \\
&= \phi_1^2 \gamma_0
\end{aligned}$$

where $E[z^2_{t-2s}] = \gamma_0$, $E[a_t z_{t-2s}] = 0$ and $E[a_{t-s} z_{t-2s}] = 0$.

Then, $\gamma_k = 0$ for $k = 2s+1, 2s+2, \ldots, 3s-1$.

When $k = 3s$, $\gamma_{3s} = \phi_1^3 \gamma_0$ and

$\gamma_k = 0$ for $k = 3s+1, 3s+2, \ldots, 4s-1$ and so on.

The autocovariances will be zero at lags that are not integer multiple of s.

Therefore, the autocovariance function of the SAR (1) model is

$$\gamma_k = \begin{cases} \dfrac{\sigma_a^2}{1-\phi_1^2} & ; k = 0 \\ \phi_1^k \gamma_0 & ; k = s, 2s, 3s, \ldots \\ 0 & ; k \neq 0, s, 2s, 3s, \ldots \end{cases}$$

and the autocorrelation function of the model is

$$\rho_k = \begin{cases} 1 & ; k = 0 \\ \phi_1^k & ; k = s, 2s, 3s, \ldots \\ 0 & ; k \neq 0, s, 2s, 3s, \ldots \end{cases}$$

Therefore, the autocovariance and the autocorrelations are non - zero at lags that are integer multiples of s .

SAR (2) Model

For SAR (2) model (P = 2),

$$z_t = \phi_1 z_{t-s} + \phi_2 z_{t-2s} + a_t$$

where a_t's are random shocks satistiying usual assumptions.

The autocovariance function of the SAR (2) model is obtained by substituting P = 2 in equation (3.3.2)

We get

$$\gamma_k = \phi_1 \gamma_{k-s} + \phi_2 \gamma_{k-2s} \qquad ; k = 1, 2, \ldots, Ps.$$

When k = 1 , 2 ,. . . , s −1 ,

$$\gamma_k = 0$$

and for k = s,

$$\gamma_s = \phi_1 \gamma_0 + \phi_2 \gamma_s$$

$$= \left(\frac{\phi_1}{1-\phi_2}\right) \gamma_0 .$$

When k = s + 1 , s + 2, . . . , 2s − 1 ,

$$\gamma_k = 0$$

and for k = 2s

$$\gamma_{2s} = \phi_1 \gamma_s + \phi_2 \gamma_0$$

$$= \left(\frac{\phi_1^2}{1-\phi_2} + \phi_2\right) \gamma_0 .$$

When k = 2s + 1 , 2s + 2 , . . . , 3s − 1 ,

$$\gamma_k = 0$$

and $\gamma_k = \phi_1 \gamma_{k-s} + \phi_2 \gamma_{k-2s}$ for k = 3s , 4s, ...

Similarly, the autocovariances will be zero at lags that are not integer multiples of s .

For k = 0, the variance of the SAR (2) model is obtained by substitutating P = 2 in equation (3.3.3).

$$\gamma_0 = \phi_1 \gamma_s + \phi_2 \gamma_{2s} + \sigma_a^2$$

$$= \phi_1 \left(\frac{\phi_1}{1-\phi_2}\right) \gamma_0 + \phi_2 \left(\frac{\phi_1^2}{1-\phi_2} + \phi_2\right) \gamma_0 + \sigma_a^2$$

$$= \left(\frac{\phi_1^2}{1-\phi_2} + \frac{\phi_1^2\phi_2}{1-\phi_2} + \phi_2^2\right) \gamma_0 + \sigma_a^2$$

$$= \left(\frac{1-\phi_2}{1+\phi_2}\right)\left(\frac{\sigma_a^2}{(1-\phi_2)^2 - \phi_1^2}\right).$$

Therefore, the autocovariance function of the SAR (2) model is

$$\gamma_k = \left(\frac{1-\phi_2}{1+\phi_2}\right)\left(\frac{\sigma_a^2}{(1-\phi_2)^2 - \phi_1^2}\right) \quad ; k = 0$$

$$= \left(\frac{\phi_1}{1-\phi_2}\right) \gamma_0 \quad ; k = s$$

$$= \left(\frac{\phi_1^2}{1-\phi_2} + \phi_2\right) \gamma_0 \quad ; k = 2s$$

$$= \phi_1 \gamma_{k-s} + \phi_2 \gamma_{k-2s} \quad ; k = 3s, 4s, \ldots$$

$$= 0 \quad ; k \neq 0, s, 2s, 3s, 4s, \ldots .$$

Then, the autocorrelation function of the model is

$$\rho_k = 1 \quad ; k = 0$$

$$= \frac{\phi_1}{1-\phi_2} \quad ; k = s$$

$$= \frac{\phi_1^2}{1-\phi_2} + \phi_2 \quad ; k = 2s$$

$$= \phi_1 \rho_{k-s} + \phi_2 \rho_{k-2s} \quad ; k = 3s, 4s, \ldots$$

$$= 0 \quad ; k \neq 0, s, 2s, 3s, 4s, \ldots .$$

Therefore, the autocovariance and the autocorrelation function are non - zero at lags that are integer multiples of s .

3.3.2 Seasonal MA Model

This model has the following specifications:

(1) if s is the number of observations per seasonal period, the order of the MA process is an integer multiple of s.

(2) the only non - zero coefficients are those with subscripts that are integer multiples of s.

The model is

$$z_t = a_t - \theta_s a_{t-s} - \theta_{2s} a_{t-2s} - \ldots - \theta_{Qs} a_{t-Qs}$$

where Q is the largest multiple of s represented in the model and a's are random shocks with mean zero, constant variance σ_a^2 and are uncorrelated.

Let $\theta_j = \theta_{js}$, then the model becomes

$$z_t = a_t - \theta_1 a_{t-s} - \theta_2 a_{t-2s} - \ldots - \theta_Q a_{t-Qs} \qquad (3.3.5)$$

which refers to a seasonal MA process of order Q. The seasonal moving average model (3.3.5) expresses the current value of the process z_t as a finite weighted sum of Q previous random shocks ($a_{t-s}, a_{t-2s}, \ldots, a_{t-Qs}$) plus current random shock a_t . Here

$$E[a_t] = 0 \quad \text{for all } t,$$

$$V[a_t] = \sigma_a^2 \quad \text{for all } t,$$

and

$$Cov[a_t, a_{t'}] = E[a_t a_{t'}] = 0 \quad \text{for all } t \neq t'.$$

Autocorrelation Structure

The autocovariance function of the seasonal model is found by multiplying throughout in (3.3.5) by z_{t-k} and taking expected values, that is,

$$\gamma_k = Cov[z_t, z_{t-k}] = E[z_t \, z_{t-k}]$$

$$= E[(a_t - \theta_1 a_{t-s} - \theta_2 a_{t-2s} - \ldots - \theta_Q a_{t-Qs})(a_{t-k} - \theta_1 a_{t-k-s}$$

$$- \theta_2 a_{t-k-2s} - \ldots - \theta_Q a_{t-k-Qs})] \qquad (3.3.6)$$

The autocovariances are non - zero at lags s, 2s, . . . , Qs.

For k = 0 , the variance of the seasonal MA (Q) process is obtained as

$$\gamma_0 = E[(a_t - \theta_1 a_{t-s} - \theta_2 a_{t-2s} - \ldots - \theta_Q a_{t-Qs})(a_t - \theta_1 a_{t-s} - \ldots - \theta_Q a_{t-Qs})]$$

$$= (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_Q^2)\,\sigma_a^2$$

where $E[a_t^2] = \sigma_a^2$ for all t, and $E[a_t \, a_{t'}] = 0$ for all $t \neq t'$.

For k = 1 , 2 , . . . , s − 1,

$$\gamma_k = 0$$

and when k = s,

$$\gamma_s = (-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{Q-1}\theta_Q)\,\sigma_a^2 \; .$$

For k = s + 1, s + 2, . . . , 2s − 1 ,

$$\gamma_k = 0$$

and when k = 2s,

$$\gamma_{2s} = (-\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \ldots + \theta_{Q-2}\theta_Q)\,\sigma_a^2 \; .$$

For $k = 2s + 1,\ 2s + 2,\ \ldots,\ 3s - 1,$

$$\gamma_k = 0$$

and when $k = Qs$,

$$\gamma_{Qs} = -\theta_Q\ \sigma_a^2.$$

Then, $\gamma_k = 0$ for $k = Qs+1, Qs+2$ and so on.

The autocovariances will be zero at lags that are not integer multiples of s .

Therefore, the autocovariance function of the seasonal MA (Q) process is

$$\begin{aligned}
\gamma_k &= (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_Q^2)\,\sigma_a^2 && ; k = 0 \\
&= (-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{Q-1}\theta_Q)\,\sigma_a^2 && ; k = s \\
&= (-\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \ldots + \theta_{Q-2}\theta_Q)\,\sigma_a^2 && ; k = 2s \\
&\ \vdots \\
&= -\theta_Q\ \sigma_a^2 && ; k = Qs \\
&= 0 && ; k \neq 0, s, 2s, \ldots, Qs.
\end{aligned}$$

Then, the autocorrelation function of the process is

$$\begin{aligned}
\rho_k &= 1 && ; k = 0 \\
&= \frac{-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{Q-1}\theta_Q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_Q^2} && ; k = s \\
&= \frac{-\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \ldots + \theta_{Q-2}\theta_Q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_Q^2} && ; k = 2s \\
&\ \vdots \\
&= \frac{-\theta_Q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_Q^2} && ; k = Qs \\
&= 0 && ; k \neq 0, s, 2s, \ldots, Qs.
\end{aligned}$$

Therefore, the autocovariance and the autocorrelation function are non-zero at lags that are integer multiples of s and are less than or equal to Qs. For a seasonal MA process of order Q the correlation persists only for Q seasonal periods.

SMA (1) Model

Consider SMA (1) model $(Q = 1)$,

$$z_t = a_t - \theta_1 a_{t-s}$$

where a_t's are random shocks with usual assumptions.

The autocovariance function of SMA (1) model is obtained by substituting $Q = 1$ in equation (3.3.6), we get

$$\gamma_k = E\,[(a_t - \theta_1 a_{t-s})(a_t - \theta_1 a_{t-k-s})] \qquad ; k = 1, 2, \ldots, s$$

For $k = 0$, the variance of the SMA (1) process is

$$\gamma_0 = (1 + \theta_1^2)\,\sigma_a^2 .$$

For $k = 1, 2, \ldots, s-1$,

$$\gamma_k = 0$$

and when $k = s$,

$$\gamma_s = -\theta_1\,\sigma_a^2 .$$

For $k = s+1, s+2, \ldots,$

$$\gamma_k = 0 .$$

Therefore, the autocovariance function of the SMA (1) process is

$$\begin{aligned} \gamma_k &= (1 + \theta_1^2)\,\sigma_a^2 && ; k = 0 \\ &= -\theta_1\,\sigma_a^2 && ; k = s \\ &= 0 && ; k \neq 0, s . \end{aligned}$$

Then, the autocorrelation function of the process is

$$\rho_k = 1 \qquad ; k = 0$$

$$= \frac{-\theta_1}{1+\theta_1^2} \qquad ; k = s$$

$$= 0 \qquad ; k \neq 0, s \; .$$

Therefore, the autocovariance and the autocorrelation are non - zero at lags 0 and s .

SMA (2) Model

For SMA (2) model $(Q = 2)$,

$$z_t = a_t - \theta_1 a_{t-s} - \theta_2 a_{t-2s}$$

where a_t's are random shocks with usual assumptions.

The autocovariance function of SMA (2) model is obtained by substituting $Q = 2$ in equation (3.3.6), we get

$$\gamma_k = E\,[(a_t - \theta_1 a_{t-s} - \theta_2 a_{t-2s})\,(a_{t-k} - \theta_1 a_{t-k-s} - \theta_2 a_{t-k-2s})]$$

$; k = 1,2,\ldots 2s$

For $k = 0$, the variance of the SMA (2) process is

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2)\,\sigma_a^2 \; .$$

For $k = 1, 2, \ldots, s-1$,

$$\gamma_k = 0$$

and when $k = s$,

$$\gamma_s = (-\theta_1 + \theta_1\theta_2)\,\sigma_a^2 \; .$$

For $k = s+1, s+2, \ldots, 2s-1$,

$$\gamma_k = 0$$

and when $k = 2s$,

$$\gamma_{2s} = -\theta_2 \sigma_a^2 .$$

For $k = 2s+1, 2s+2, \ldots,$

$$\gamma_k = 0 .$$

Therefore, the autocovariance function of the SMA (2) process is

$$\gamma_k = \begin{cases} (1+\theta_1^2+\theta_2^2)\,\sigma_a^2 & ; k = 0 \\ (-\theta_1+\theta_1\theta_2)\,\sigma_a^2 & ; k = s \\ -\theta_2\,\sigma_a^2 & ; k = 2s \\ 0 & ; k \neq 0, s, 2s . \end{cases}$$

Then, the autocorrelation function of the process is

$$\rho_k = \begin{cases} 1 & ; k = 0 \\ \dfrac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & ; k = s \\ \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2} & ; k = 2s \\ 0 & ; k \neq 0, s, 2s . \end{cases}$$

Therefore, the autocovariances and the autocorrelations are non - zero at lags 0, s and 2s .

3.3.3 Seasonal ARMA Model

The logical generalization of the seasonal AR and MA processes is a mixed model incorporating both. Such a model would be of the form

$$z_t = \phi_1 z_{t-s} + \phi_2 z_{t-2s} + \ldots + \phi_P z_{t-Ps} + a_t - \theta_1 a_{t-s} - \theta_2 a_{t-2s} - \ldots - \theta_Q a_{t-Qs} \qquad (3.3.7)$$

This model is called seasonal ARMA process and denoted by SARMA (P,Q).

The seasonal autoregressive-moving average model (3.3.7) expresses the current value of the process z_t as a finite weighted sum of the P previous values ($z_{t-s}, z_{t-2s}, \ldots, z_{t-Ps}$) plus current random shock a_t and a finite weighted sum of Q previous random shocks ($a_{t-s}, a_{t-2s}, \ldots, a_{t-Qs}$). The a's are random shocks satisfying the usual assumptions. That is ,

$$E[a_t] = 0 \text{ for all } t,$$

$$V[a_t] = \sigma_a^2 \text{ for all } t,$$

and $\quad Cov[a_t, a_{t'}] = E[a_t a_{t'}]$ for all $t \neq t'$.

Autocorrelation Structure

The autocovariance of the SARMA (P, Q) process is

$$\gamma_K = Cov[z_t, z_{t-k}] = E[z_t z_{t-k}]$$
$$= E[\phi_1 z_{t-s} z_{t-k} + \phi_2 z_{t-2s} z_{t-k} + \ldots + \phi_P z_{t-Ps} z_{t-k} + a_t z_{t-k}$$
$$- \theta_1 a_{t-s} z_{t-k} - \theta_2 a_{t-2s} z_{t-k} - \ldots - \theta_Q a_{t-Qs} z_{t-k}]$$
$$= \phi_1 \gamma_{k-s} + \phi_2 \gamma_{k-2s} + \ldots + \phi_P \gamma_{k-Ps} + \gamma_{za}(k) - \theta_1 \gamma_{za}(k-s)$$
$$- \theta_2 \gamma_{za}(k-2s) \ldots - \theta_Q \gamma_{za}(k-Qs) \qquad (3.3.8)$$

where $E[z_{t-j} z_{t-k}] = \gamma_{k-j}$, $E[a_{t-j} z_{t-k}] = \gamma_{za}(k-j)$ and $E[a_t z_{t-k}] = \gamma_{za}(k)$.

For $k = 0$ the variance of the process is obtained as

$$\gamma_0 = \phi_1 \gamma_s + \phi_2 \gamma_{2s} + \ldots + \phi_P \gamma_{Ps} + \sigma_a^2 - \theta_1 \gamma_{za}(-s)$$
$$- \theta_2 \gamma_{za}(-2s) - \ldots - \theta_Q \gamma_{za}(-Qs) \qquad (3.3.9)$$

since $\gamma_{-j} = \gamma_j$ and $\gamma_{za}(0) = \sigma_a^2$.

The autocovariance of the process satisfies the difference equation

$$\gamma_k = \phi_1 \gamma_{k-s} + \phi_2 \gamma_{k-2s} + \ldots + \phi_P \gamma_{k-Ps} \quad ; k \geq Qs \qquad (3.3.10)$$

where $\gamma_{za}(k) = 0$ for $k = s, 2s, \ldots, Qs$.

Dividing γ_k by γ_0 , the autocorrelation function of the process satisfies the difference equation

$$\rho_k = \phi_1 \rho_{k-s} + \phi_2 \rho_{k-2s} + \ldots + \phi_P \rho_{k-Ps} \quad ; k \geq Qs \qquad (3.3.11)$$

The autocorrelation structure of the sesonal ARMA process is precisely analogous to that of nonseasonal ARMA process with nonzero correlations occuring only at the lags s, 2s, 3s and so forth.

SARMA (1, 1) Model

Consider SARMA (1, 1) model (P = 1, Q = 1) ,

$$z_t = \phi_1 z_{t-s} + a_t - \theta_1 a_{t-s} \qquad (3.3.12)$$

where a 's are random shocks with usual assumptions.

The autocovariance function of the SARMA (1, 1) process is

$$\begin{aligned} \gamma_k &= \text{Cov}\,[z_t , z_{t-k}] = E\,[z_t\, z_{t-k}] \\ &= E\,[\theta_1 z_{t-s}\, z_{t-k} + a_t\, z_{t-k} - \theta_1 a_{t-s}\, z_{t-k}] \\ &= \phi_1 \gamma_{k-s} + \gamma_{za}(k) - \theta_1 \gamma_{za}(k-s) \end{aligned} \qquad (3.3.13)$$

where $E\,[z_{t-s}\, z_{t-k}] = \gamma_{k-s}$, $E\,[a_t\, z_{t-k}] = \gamma_{za}(k)$

and $E\,[a_{t-s}\, z_{t-k}] = \gamma_{za}(k-s)$.

For k = 0, the variance of the SARMA (1, 1) model is obtained as

$$\begin{aligned} \gamma_0 &= \phi_1 \gamma_{-s} + \gamma_{za}(0) - \theta_1 \gamma_{za}(-s) \\ &= \phi_1 \gamma_s + \sigma_a^2 - \theta_1 \gamma_{za}(-s) \end{aligned} \qquad (3.3.14)$$

where $\gamma_{-s} = \gamma_s$ and $\gamma_{za}(0) = \sigma_a^2$.

For $k = s$,

$$\gamma_s = \phi_1 \gamma_0 + \gamma_{za}(s) - \theta_1 \gamma_{za}(0)$$
$$= \phi_1 \gamma_0 - \theta_1 \sigma_a^2$$

where $\gamma_{za}(s) = 0$ and $\gamma_{za}(0) = \sigma_a^2$.

For $k = 2s$,

$$\gamma_{2s} = \phi_1 \gamma_s + \gamma_{za}(2s) - \theta_1 \gamma_{za}(s)$$
$$= \phi_1 \gamma_s$$

where $\gamma_{za}(k) = 0$ for $k = s, 2s, \ldots$.

Similarly, for $k = 3s$,

$$\gamma_{3s} = \phi_1 \gamma_{2s} + \gamma_{za}(3s) - \theta_1 \gamma_{za}(2s)$$
$$= \phi_1 \gamma_{2s} .$$

In general, we get

$$\gamma_k = \phi_1 \gamma_{(k-1)s} \quad ; \; k \geq 2s \qquad (3.3.15)$$

But, in the equation (3.3.14)

$$\gamma_0 = \phi_1 \gamma_s + \sigma_a^2 - \theta_1 \gamma_{za}(-s)$$
$$\gamma_{za}(-s) = E[a_t z_{t+s}]$$
$$= E[a_t(\phi_1 z_t + a_{t-s} - \theta_1 a_t)]$$
$$= \phi_1 E[a_t z_t] + E[a_t a_{t-s}] - \theta_1 E[a_t^2]$$
$$= (\phi_1 - \theta_1)\sigma_a^2$$

where $E[a_t z_t] = \gamma_{za}(0) = \sigma_a^2$, $E[a_t a_{t-s}] = 0$, $s \neq 0$ and $E[a_t^2] = \sigma_a^2$.

Therefore, the equation (3.3.14) becomes

$$\gamma_0 = \phi_1 \gamma_s + \sigma_a^2 - \theta_1 (\phi_1 - \theta_1)\sigma_a^2 \qquad (3.3.16)$$

By substituting $\gamma_s = \phi_1\gamma_0 - \theta_1\sigma_a^2$ in equation (3.3.16),the variance of the process is obtained as

$$\gamma_0 = \phi_1 (\phi_1 \gamma_0 - \theta_1\sigma_a^2) + \sigma_a^2 - \theta_1 (\phi_1 - \theta_1) \sigma_a^2$$

$$= \left(\frac{1 - 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}\right) \sigma_a^2$$

By substituting the value of γ_0 in $\gamma_s = \phi_1 \gamma_0 - \theta_1\sigma_a^2$ we get,

$$\gamma_s = \phi_1 \left(\frac{1 - 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}\right) \sigma_a^2 - \theta_1\sigma_a^2$$

$$= \left(\frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 - \phi_1^2)}\right) \sigma_a^2 .$$

Therefore, the autocorariance function of the SARMA (1,1)process is

$$\gamma_k = \left(\frac{1 - 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}\right) \sigma_a^2 \qquad ; k = 0$$

$$= \left(\frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 - \phi_1^2)}\right) \sigma_a^2 \qquad ; k = s$$

$$= \phi_1 \gamma_{k-s} \qquad ; k = 2s , 3s , \ldots$$

$$= 0 \qquad ; k \neq 0 , s , 2s , \ldots$$

Dividing γ_k by γ_0, the autocorrelation function of the process is obtained as

$$\rho_k = 1 \qquad ; k = 0$$

$$= \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 - 2\phi_1\theta_1 + \theta_1^2)} \qquad ; k = s$$

$$= \phi_1 \rho_{k-s} \qquad ; k = 2s , 3s, \ldots$$

$$= 0 \qquad ; k \neq 0 , s , 2s , \ldots$$

3.4 The General Multiplicative Seasonal Model

Box and Jenkins (1976) proposed that correlation between observations withinseasonalperiods may be introduced by supposing that the noise input to the seasonal ARIMA is serially correlated rather than independent. In particular, they suggested that z_t be generated by the seasonal model of the form

$$\Phi(B^s)\nabla_s^D z_t = \Theta(B^s)\alpha_t \quad (3.4.1)$$

where $\nabla_s = 1 - B^s$ and $\Phi(B^s)$, $\Theta(B^s)$ are polynomials in B^s of degrees P and Q, respectively, and satisfying stationarity and invertibility conditions. Similarly, a model

$$\Phi(B^s)\nabla_s^D z_t = \Theta(B^s)\alpha_{t-1} \quad (3.4.2)$$

might be used to link the current behavior of a month (e.g. March) with previous March observations, and so on, for each of the twelve months. Moreover, it would usually be resonable to assume that the parameters Φ and Θ contained in these monthly models would be approximately the same for each month.

Now the error components $\alpha_t, \alpha_{t-1}, \ldots,$ in these models would not in general be uncorrelated. For example, the data for April, 1960, while related to previous April, would also be related to March of 1960, February of 1960, and January of 1960, etc. Thus it would be expected that α_t in (3.4.1) would be related to α_{t-1} in (3.4.2) and to α_{t-2} etc. Therefore, to take care of such relationships, a seasonal model

$$\phi(B)\nabla^d \alpha_t = \theta(B)\, a_t \quad (3.4.3)$$

is introduced. Where now a_t is a white noise process and $\phi(B)$ and $\theta(B)$ are polynomials in B of degrees p and q respectively, and satisfying stationarity and invertibility conditions, and $\nabla = \nabla_1 = 1 - B$.

Substituting (3.4.3) in (3.4.1) a general multiplicative model is finally obtained as

$$\phi_p(B)\,\Phi_P(B^s)\nabla^d\nabla_s^D z_t = \theta_q(B)\;\Theta_Q(B^s)\,a_t \qquad (3.4.4)$$

In (3.4.4), the subscripts p, P, q, Q have been added to remind the orders of the various operators. The resulting multiplicative process will be said to be of order (p,d,q) x $(P,D,Q)_s$. A similar argument can be used to obtain models with three or more periodic components to take care of multiple seasonalities.

It has been assumed that stationary series $\nabla^d\nabla_s^D z_t$ has a zero mean. The degree of seasonal differencing D and that of consecutive differencing d will in economic contexts usually be either 0 or 1 as required to achieve stationarity in the differenced series.

The seasonal model is multiplicative in the sense that the observed data result from the successive filtering of the random noise series a_t through the non - seasonal filter (3.4.3) and the seasonal filter (3.4.1).

Multiplicative (0,1,1) x (0,1,1)$_{12}$ Model

Consider the multiplicative (0, 1, 1) x (0, 1, 1)$_{12}$ model. Such a model

$$\nabla_{12} z_t = (1 - \Theta B^{12})\alpha_t$$

is employed for linking z's one year apart. Suppose futher that a similar model is employed

$$\nabla \alpha_t = (1 - \theta B)\, a_t$$

for linking α_t's one month apart, where in general θ and Θ will have different values. Then on combining these expressions, the seasonal multiplicative model

$$\nabla\nabla_{12} z_t = (1 - \theta B)(1 - \Theta B^{12})a_t$$

of order $(0,1,1) \times (0,1,1)_{12}$ is obtained.

Now, $\nabla\nabla_{12} z_t = (1-B)(1-B^{12})z_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$

and

$$(1-\theta B)(1-\Theta B^{12})a_t = a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13} .$$

Then the model written explicitly is

$$z_t = z_{t-1} + z_{t-12} - z_{t-13} + a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta \Theta a_{t-13} .$$

The invertibility region for this model, required by the condition that the roots of $(1-\theta B)(1-\Theta B^{12}) = 0$ lie outside the unit circle, is defined by the inequalities,

$$-1 < \theta < 1 \text{ and } -1 < \Theta < 1.$$

Note that the moving average operator

$$(1-\theta B)(1-\Theta B^{12}) = 1 - \theta B - \Theta B^{12} + \theta \Theta B^{13}$$

is of order $q + sQ = 1 + (12)\,1 = 13$.

3.4.1 Autocorrelation Structure

The autocorrelation structure of the stationary process $(1-B^s)^D(1-B)^d z_t$ is generally very complex. Nevertheless, it is worthwhile to make some general observations and work out an example.

First of all if we combine the polynomials in B on both sides of the model, it is seen to be essentially an ARMA process of order Ps + p and Qs + q. Many of the coefficients appearing in the expanded model will, however, be zero, resulting in certain simplifications in the autocorrelation structure.

Multiplicative (0,1,1) x (0,1,1)$_{12}$ Model

Consider the multiplicative (0,1,1) x (0,1,1)$_{12}$ process

$$(1 - B^{12})(1 - B)\ z_t = (1 - \Theta_1\ B^{12})(1 - \theta_1 B)\ a_t$$

which in expanded form is

$$y_t = (1 - \theta_1 B - \Theta_1 B^{12} + \theta_1 \Theta_1 B^{13})\ a_t$$

where y_t denotes the stationary series, $(1- B^{12})(1- B)\ z_t$.

Allowing the backshift operators to act on a_t, we have

$$y_t = a_t - \theta_1\ a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1\ a_{t-13} \qquad (3.4.5)$$

which for analytical purposes is simply a MA process where only the first, twelfth and thirteeth coefficients are non - zero. The a's are random shocks with mean zero, cosntant variance σ_a^2 and are uncorrelated. That is,

$$E[a_t] = 0 \text{ for all } t,$$

$$V[a_t] = E[a_t^2] = \sigma_a^2 \text{ for all } t,$$

and $$E[a_t\ a_{t'}] = 0 \text{ for all } t \neq t'.$$

The autocovariance of y_t is found by multiplying equation (3.4.5) with y_{t-k} and taking expected value. That is,

$$\gamma_k = Cov[\ y_t, y_{t-k}] = E[\ y_t\, y_{t-k}]$$

$$= E[(a_t - \theta_1\ a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1\ a_{t-13})$$

$$(a_{t-k} - \theta_1\ a_{t-k-1} - \Theta_1 a_{t-k-12} + \theta_1 \Theta_1\ a_{t-k-13})]$$

where $E[a_t^2] = \sigma_a^2$ for all t and $E[a_t\ a_{t'}] = 0$ for all $t \neq t'$.

For k = 0 , the variance of the process is obtained as

$$\gamma_0 = Cov[\ y_t, y_t] = E[\ y_t\, y_t] = E[\ y_t^2]$$

$$= E[(a_t - \theta_1\ a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1\ a_{t-13})^2]$$

$$= (1 + \theta_1^2)(1 + \Theta_1^2)\ \sigma_a^2$$

The autocovariances of y_t are

$$\gamma_1 = Cov[\, y_t, y_{t-1} \,] = E[\, y_t y_{t-1} \,]$$

$$= E[(a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13})$$

$$(a_{t-1} - \theta_1 a_{t-2} - \Theta_1 a_{t-13} + \theta_1 \Theta_1 a_{t-14})]$$

$$= -\theta_1 \, (1 + \Theta_1^2)\, \sigma_a^2$$

$$\gamma_2 = Cov[\, y_t, y_{t-2} \,] = E[\, y_t y_{t-2} \,]$$

$$= E[(a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13})$$

$$(a_{t-2} - \theta_1 a_{t-3} - \Theta_1 a_{t-14} + \theta_1 \Theta_1 a_{t-15})]$$

$$= 0$$

Similarly,

$$\gamma_3 = 0 \;,\; \gamma_4 = 0 \;, \;. \;. \;. \;, \; \gamma_{10} = 0$$

$$\gamma_{11} = Cov[\, y_t, y_{t-11} \,] = E[\, y_t y_{t-11} \,]$$

$$= E[(a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13})$$

$$(a_{t-11} - \theta_1 a_{t-12} - \Theta_1 a_{t-23} + \theta_1 \Theta_1 a_{t-24})]$$

$$= \theta_1 \Theta_1 \sigma_a^2$$

$$\gamma_{12} = Cov[\, y_t, y_{t-12} \,] = E[\, y_t y_{t-12} \,]$$

$$= E[(a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13})$$

$$(a_{t-12} - \theta_1 a_{t-13} - \Theta_1 a_{t-24} + \theta_1 \Theta_1 a_{t-25})]$$

$$= -\Theta_1 \,(\, 1 + \theta_1^2 \,) \sigma_a^2$$

$$\gamma_{13} = Cov[\, y_t, y_{t-13} \,] = E[\, y_t y_{t-13} \,]$$

$$= E[(a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13})$$

$$(a_{t-13} - \theta_1 a_{t-14} - \Theta_1 a_{t-25} + \theta_1 \Theta_1 a_{t-26})]$$

$$= \theta_1 \Theta_1 \, \sigma_a^2$$

$$\gamma_{14} = \text{Cov}[\, y_t, y_{t-14}] = E[\, y_t\, y_{t-14}]$$

$$= E[(a_t - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13})$$

$$(a_{t-14} - \theta_1 a_{t-15} - \Theta_1 a_{t-26} + \theta_1 \Theta_1 a_{t-27})]$$

$$= 0$$

Similary,

$\gamma_{15} = 0$, $\gamma_{16} = 0$, . . . , and all other γ_k are equal to zero.

Therefore, the autocovariance function of y_t is

$$\gamma_k = (1 + \theta_1^{\,2})(1 + \Theta_1^{\,2})\,\sigma_a^2 \qquad ; \; k = 0$$

$$= -\theta_1 (1 + \Theta_1^{\,2})\,\sigma_a^2 \qquad ; \; k = 1$$

$$= \theta_1 \Theta_1\, \sigma_a^2 \qquad ; \; k = 11$$

$$= -\Theta_1 (1 + \Theta_1^{\,2})\,\sigma_a^2 \qquad ; \; k = 12$$

$$= \theta_1 \Theta_1\, \sigma_a^2 \qquad ; \; k = 13$$

$$= 0 \qquad ; \; k \neq 0,1,11,12,13.$$

Dividing γ_k by γ_0, the autocorrelation function of y_t is

$$\rho_k = 1 \qquad ; \; k = 0$$

$$= \frac{-\theta_1}{(1 + \theta_1^2)} \qquad ; \; k = 1$$

$$= \frac{\theta_1 \Theta_1}{(1 + \theta_1^2)(1 + \Theta_1^2)} \qquad ; \; k = 11$$

$$= \frac{-\Theta_1}{(1 + \Theta_1^2)} \qquad ; \; k = 12$$

$$= \frac{\theta_1 \Theta_1}{(1 + \theta_1^2)(1 + \Theta_1^2)} \qquad ; \; k = 13$$

$$= 0 \qquad ; \; k \neq 0,1,11,12,13.$$

Therefore, the only non - zero autocorrelations of y_t are those at lags 1, 11, 12 and 13. Thus, the correlogram of this process will display spikes at lags 1, 11, 12 and 13 with the later 3 spikes being symmetric around ρ_{12} .

CHAPTER IV

STOCHASTIC MODEL BUILDING FOR A TIME SERIES

4.1 Introduction

From the general statistical point of view, an important step in the analysis of time series is to construct a model for the underlying stochastic process. The model can be used for prediction, system design, simulation of a system and other purposes. Box and Jenkins (1976) proposed stochastic models for prediction, system design, simulation of a system and for the representation of a time series and the model building procedure. The procedure consists of a three steps iterative cycle of

(a) Identification of the form of the model,

(b) Estimation of the model parameters,

and (c) Diagnostic checking of the model.

This procedure is quite general and can be used in building any statistical model. In building a stochastic model of the ARIMA type, the three steps consists of the following tasks. The ARIMA type of model is

$$\phi(B)\nabla^d z_t = \theta(B)a_t$$

where $\nabla^d = (1-B)^d$, $\phi(B) = 1-\phi_1 B - \ldots -\phi_p B^p$ and $\theta(B) = 1-\theta_1 B - \ldots -\theta_q B^q$. For seasonal time series, seasonal models are fitted to a given time series by employing essentially the same principles as for the nonseasonal time series.

In the identification step, the type of model and order of that model, that is, p, d and q of the model have to be determined.

Procedures employed in this step are of necessarity inexact and required a good deal of judgement. The chosen model is not final and is to be discarded if subsequent analysis suggests that some alternative form might provide more adequate representation of the data. Nature of the estimation techniques applied in this step is such that, initial rough estimates of the coefficients ϕ's and θ's of the identified models are only obtained.

In the estimation step, coefficients of the identified model are estimated using efficient statistical techniques. To obtain the estimates of the parameters ϕ's and θ's, at least one of the methods such as maximum likelihood, least squares or Bayesian has to be used.

In the diagnostic checking step, checks are employed to determine whether or not the tentatively chosen model adequately represents the given set of data. Any inadequacy revealed may suggest an alternative model specification.

If this is the case, the whole iterative cycle of identification, estimation and diagnostic checking is repeated until a statisfactory model is obtained.

In the following section, the steps in model building procedure, namely, identification, estimation and diagnostic checking will be classified. The statistical tools used in identification step such as autocorrelation function and partial autocorrelation function will be discussed in detail. The estimation of parameter by using maximum likelihood estimates will also be described and the tests associated with model diagnostic checking will also be presented. The models used to illustrate the tools and techniques are those which are usually found to represent a time series in practice.

4.2 Statistical Tools Used in Identification

Identification of the model is the most difficult step in the model building procedure. In this step only a number of general principles can be laid down. No sure - fire deterministic approach to the problem exist. It is necessary to exert a degree of judgement which is greatly improved by experience.

In selecting a model at this stage, one is committed to no more than an assessment of its validity. The initially chosen model can be discarded at a later stage of the analysis if it appears to be not suitable. It is also possible that one may wish to carry forward from this stage not one, but two or more possible models.

The two most useful tools to be used in this stage are the sample autocorrelation function and the sample partial autocorrelation function. That is, r_k, $k = 0, 1, 2, \ldots$ as an estimate of the theoretical autocorrelation function ρ_k, $k = 0, 1, 2, \ldots$ and $\hat{\phi}_{kk}$, $k = 0, 1, 2, \ldots$ as an estimate of the theoretical partial autocorrelation function ϕ_{kk}, $k = 0, 1, 2, \ldots$. They have to be used in order to judge the values of p, d and q. Similarly, in the estimation and diagnostic checking of a chosen model these functions have to be used.

4.2.1 Autocorrelation Function of A Time Series

Let us define

$$z_t = z_{ij} \; ; t = 12(i-1)+j \; ; \; i = 1,2,\ldots,n \; ; \; j = 1,2,\ldots,12$$

where z_{ij} is the value of the variable z at the j^{th} month in the i^{th} year. Then, the autocovariance between z_t and z_{t+k} is denoted by $\gamma_{t,t+k}$ and is defined as

$$\gamma_{(t,t+k)} = \text{Cov}(z_t, z_{t+k}) = E[z_t - E(z_t)]\,[z_{t+k} - E(z_{t+k})] .$$

Similarly, the autcorrelation between z_t and z_{t+k} is denoted by $\rho_{(t,t+k)}$ and is defined as

$$\rho_{(t,t+k)} = \frac{\text{Cov}(z_t, z_{t+k})}{\sqrt{V(z_t)\,V(z_{t+k})}} \qquad ; k = 1, 2, \ldots$$

After imposing the stationary assumptions, the autocovariance and autocorrelations do not depend on the actual time, but on the time lag (time difference) and they can be written as γ_k and ρ_k. Then

$$\gamma_k = E[(z_t - \mu)(z_{t+k} - \mu)] \qquad ; k = 1, 2, \ldots$$

where μ is the common mean of z and similarly

$$\rho_k = \frac{\gamma_k}{\gamma_0} \qquad ; k = 1, 2, \ldots$$

where

$$\gamma_0 = E(z_t - \mu)^2 \text{ is the variance of } z .$$

Note that $\gamma_k = \gamma_{-k}$ and $\rho_k = \rho_{-k}$.

A set of autocorrelations $\{\rho_k\}$, $k = 1, 2, \ldots$ is known as autocorrelation function (acf) and the plot of ρ_k against the lag k is called the correlogram.

The autocorrelation is the dimensionless measurement and it lies between -1 and $+1$. The most important use of autocorrelation function is in the determination of the model for the underlying process and in the estimation of its parameters.

The above definition of acf are for the theoretical values. In practice, we have a finite time series $z_1, z_1, \ldots, z_N$ of N observations.

Thus, the theoretical autocorrelations are to be estimated from the observed series.

As an estimte of the k^{th} lag autocorrelation, ρ_k,

$$\hat{\rho}_k = r_k = \frac{\sum_{t=1}^{N-k} (z_t - \bar{z})(z_{t+k} - \bar{z})}{\sum_{t=1}^{N} (z_t - \bar{z})^2} \qquad (4.2.1)$$

where $\bar{z} = \frac{1}{N} \sum_{t=1}^{N} z_t$ is to be used.

4.2.2 Autocorrelation Function of An Autoregressive Process

The autoregressive model is an extremely useful model in the representation of certain practically occuring time series. In this model, the current value of the process is expressed as finite, linear aggregate (weighted sum) of previous values of the process and a random shock term a_t.

Let us denote the values of a process at time t, $t - 1$, $t - 2$, . . . by z_t, z_{t-1}, z_{t-2}, . . . and also let μ be the mean value of z, then

$$z_t - \mu = \phi_1 (z_{t-1} - \mu) + \phi_2 (z_{t-2} - \mu) + \ldots + \phi_p (z_{t-p} - \mu) + a_t$$

is called an autoregressive process of order p and denoted by AR (p).

$(z_{t-1} - \mu)$ is linearly regressed on $(z_{t-1} - \mu)$, $(z_{t-2} - \mu)$, . . . , $(z_{t-p} - \mu)$. But as a regession equation, the independent variables $(z_{t-1} - \mu)$, $(z_{t-2} - \mu)$, . . . , $(z_{t-p} - \mu)$ are the previous values of the dependent variable $(z_t - \mu)$. Thus, the variable z is regressed on previous values of itself and hence the model is called as autoregressive.

In this model, ϕ_1, ϕ_2, . . . , ϕ_p are the autoregressive coefficients and a_t is assumed to be independently and identically distributed random error term with mean zero and variance σ_a^2 . That is

$$E[a_t] = 0$$

$$E[a_t a_{t+k}] = 0 \qquad ; k \neq 0$$

$$= \sigma_a^2 \qquad ; k = 0 \cdot$$

The most widely used AR models are only the lower order autoregressive models, especially of order 1 and 2. For p = 1 , the AR(1) model is obtained as

$$z_t - \mu = \phi_1 (z_{t-1} - \mu) + a_t$$

and there are only one coefficient ϕ_1 to be estimated.

Similarly, the AR(2) model is

$$z_t - \mu = \phi_1 (z_{t-1} - \mu) + \phi_2 (z_{t-2} - \mu) + a_t$$

and there are only two coefficients ϕ_1 and ϕ_2 to be estimated.

The AR (1) model is also known as the Markov model and the AR (2) model is also known as the Yule model.

The autocorrelation function of a stationary AR process is found by multiplying throughout in

$$z_t - \mu = \phi_1 (z_{t-1} - \mu) + \phi_2 (z_{t-2} - \mu) + \ldots + \phi_p (z_{t-p} - \mu) + a_t$$

by $(z_{t-k} - \mu)$ and taking the expected value, that is,

$$\gamma_k = E[(z_t - \mu)(z_{t-k} - \mu)]$$

$$= E[\phi_1 (z_{t-1} - \mu)(z_{t-k} - \mu) + \ldots + \phi_p (z_{t-p} - \mu)(z_{t-k} - \mu) + (z_t - \mu) a_t]$$

$$= \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \ldots + \phi_p \gamma_{k-p} \quad ; k = 1, 2, 3, \ldots$$

where $E[(z_{t-j} - \mu)(z_{t-k} - \mu)] = \gamma_{k-j}$ and the last term $E[(z_{t-k} - \mu) a_t]$ vanishes for $k > 0$ since $(z_{t-k} - \mu)$ can only involve the random shock term a_j up to time t–k , which are uncorrelated with a_t for $k \neq 0$.

For k = 0 the variance of the process is obtained as

$$\gamma_0 = \sigma^2 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \ldots + \phi_p \gamma_p + \sigma_a^2$$

since $\gamma_{-j} = \gamma_j$ and $E[(z_t - \mu) a_t] = E(a_t^2) = \sigma_a^2$.

Dividing γ_k by γ_0, the autocorrelation function satisfies the same form of difference equation as the model and

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \ldots + \phi_p \rho_{k-p} \; ; k = 1, 2, 3, \ldots, p \tag{4.2.2}$$

which is known as Yule - Walker equations.

For AR (1) process (p = 1), the autocorrelation function satisfies the first - order difference equation

$$\rho_k = \phi_1 \rho_{k-1} \qquad ; k = 1, 2, 3, \ldots$$

which with $\rho_0 = 1$ and replacing k = 1, 2, 3, . . . , we have

$$\rho_1 = \phi_1$$

$$\rho_2 = \phi_1 \rho_1 = \phi_1^2$$

$$\vdots$$

$$\rho_k = \phi_1^k \qquad ; k = 1, 2, \ldots .$$

In this process, the autocorrelation function decays exponentially to zero when ϕ_1 is positive, but decays exponentially to zero and oscillates in sign when ϕ_1 is negative.

For AR (2) process (p = 2), the autocorrelation function satisfies the second - order difference equation

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \qquad ; k = 1, 2, \ldots$$

which with starting value $\rho_0 = 1$ and replacing k = 1, 2, 3, . . . we have

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$$\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 \qquad \text{and etc.}$$

In this process, if $\phi_1^2 + 4\phi_2 \geq 0$, the autocorrelation function consists of a mixture of damped exponentials and if $\phi_1^2 + 4\phi_2 < 0$, the autocorrelation function consists of a damped since wave.

By substituting p = 2 in the Yule - Walker equations, we get

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_2 = \rho_1 \phi_1 + \phi_2$$

and by solving these two equations,

$$\phi_1 = \frac{\rho_1 (1 - \rho_2)}{1 - \rho_1^2}$$

$$\phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} .$$

When ϕ_1 and ϕ_2 are given, the expressions for ρ_1 and ρ_2 in terms of ϕ_1 and ϕ_2 are

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

$$\rho_2 = \phi_2 + \frac{\phi_1^2}{1 - \phi_2} .$$

In general, the autocorrelation function of an AR process will consist of a mixture of damped exponentials and damped sine waves.

Stationarity Condition for An Autoregressive Process

The parameters $\phi_1, \phi_2, \ldots, \phi_p$ of an AR (p) process must satisfy certain conditions for the process to be stationary. The AR(p) can be expressed as

$$z_t - \mu = \phi_1(z_{t-1} - \mu) + \phi_2(z_{t-2} - \mu) + \ldots + \phi_p(z_{t-p} - \mu) + a_t$$

By using backward shift operator B , it acn be written as,

$$z_t - \mu = \phi_1 B(z_{t-1} - \mu) + \phi_2 B^2(z_{t-2} - \mu) + \ldots + \phi_p B^p(z_{t-p} - \mu) + a_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p)(z_t - \mu) = a_t$$

$$\phi(B)(z_t - \mu) = a_t$$

$$(z_t - \mu) = \phi^{-1}(B)\, a_t$$

where $\phi(B) = (1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p)$.

Since $\phi(B)$ is the p^{th} order polinomial in B, it can be written as

$$\phi(B) = (1 - G_1 B)(1 - G_2 B)\ldots(1 - G_p B^{p})$$

where Gi^{-1} are the roots of the equation $\phi(B) = 0$. Then, equation

$$(z_t - \mu) = \phi^{-1}(B)\, a_t$$

can be expressed in the partial fractions,

$$(z_t - \mu) = \sum_{i=1}^{p} \frac{K_i}{(1 - G_i B)} a_t$$

where K_i are arbitrary constants.

If $\phi^{-1}(B)$ is to be a convergent series for $|B| \leq 1$, then we must have $|G_i| < 1$, where $i = 1,2,\ldots,p$. Since G_i^{-1} are the roots of the equation $\phi(B) = 0$, it can be said that the roots of the equation $\phi(B) = 0$ must lie outside the unit circle. The equation $\phi(B) = 0$ is called the characteristic equation and the conditions $|G_i| < 1$; $i = 1, 2, \ldots, p$ are called the stationarity conditions of the AR (p) process.

For $p = 1$, the AR (1) process

$$z_t - \mu = \phi_1 (z_{t-1} - \mu) + a_t$$

can be expressed as

$$(1 - \phi_1 B)(z_t - \mu) = a_t$$

$$(z_t - \mu) = \phi^{-1}(B)\, a_t$$

where $\phi(B) = (1 - \phi_1 B)$.

Since the root of the characteristic equation $(1 - \phi_1 B) = 0$ is $B = \phi_1^{-1}$, this condition is equivalent to saying that the root of $(1 - \phi_1 B) = 0$ must lie outside the unit circle. This implies that the parameter ϕ_1 of an AR (1) process, must satisfy the condition $|\phi_1| < 1$ to ensure stationarity.

For $p = 2$, the AR (2) process

$$z_t - \mu = \phi_1 (z_{t-1} - \mu) + \phi_2 (z_{t-2} - \mu) + a_t$$

can be expressed as

$$z_t - \mu = \phi_1 B(z_t - \mu) + \phi_2 B^2 (z_t - \mu) + a_t$$

$$(1 - \phi_1 B - \phi_2 B^2)(z_t - \mu) = a_t$$

$$(z_t - \mu) = \phi^{-1}(B)\ a_t$$

where $\phi(B) = (1 - \phi_1 B - \phi_2 B^2)$.

For stationarity, the roots of the characteristic equation $(1-\phi_1 B-\phi_2 B^2) = 0$ must lie outside the unit circle. The roots of $(1-\phi_1 B-\phi_2 B^2) = 0$ are

$$B_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad \text{and} \quad B_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

Then, $\phi_2 B_1^2 + 4\,\phi_2 B_1^2 = 4 B_1 - 4 B_1 \phi_1 + \phi_1^2$

and for $|B_1| < 1$,

$$4\phi_2 < 4 - 4\phi_1$$

or $\phi_1 + \phi_2 < 1$.

Similarly, for $|B_2| < 1$, the condition $\phi_2 - \phi_1 < 1$ must be satisfied. By multiplying both roots, $B_1 B_2 = -\phi_2$ and since $|B_1| < 1$ and $|B_2| < 1$, $|\phi_2|$ must be less than one or $-1 < \phi_2 < 1$.

Therefore, the parameters ϕ_1 and ϕ_2 of an AR (2) process, must lie in the triangular region

$$\phi_1 + \phi_2 < 1$$

$$\phi_2 - \phi_1 < 1$$

and $$-1 < \phi_2 < 1 .$$

4.2.3 Autocorrelation Function of A Moving Average Process

The moving average model is an extremely useful model in the representation of certain practically occuring time series. In this model, the current value of the process is expressed as a random error term a_t and finite weighted sum of previous values of a 's .

Let us denote the value of a process at time t by z_t and let μ be the mean value of z, then

$$z_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}$$

is called a moving average process of order q and denoted by MA (q).

$z_t - \mu$ is linearly regressed on a finite number q of previous a's. and current value of a.

In this model $\theta_1, \theta_2, \ldots, \theta_q$ are the coefficients of the moving average terms and a_t is assumed to be independently and identically distributed random error term with mean zero and variance σ_a^2 . That is ,

$$E[a_t] = 0$$

$$E[a_t a_{t+k}] = 0 \qquad ; k \neq 0$$

$$= \sigma_a^2 \qquad ; k = 0 .$$

The most widely used MA models are only the lower order moving average models, especially of order 1 and 2. For $q = 1$, the MA (1) model is obtained as

$$z_t - \mu = a_t - \theta_1 a_{t-1}$$

and there are only one coefficient θ_1 to be estimated.

Similarly, for $q = 2$, the MA (2) model is

$$z_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

and there are two coefficients θ_1 and θ_2 to be estimated.

The autocorrelation function of a MA process is found by multiplying throughout in

$$z_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}$$

by $(z_{t-k} - \mu)$ and taking expected values, that is

$$\gamma_k = E[(z_t - \mu)(z_{t-k} - \mu)]$$

$$= E[(a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q})$$

$$(a_{t-k} - \theta_1 a_{t-k-1} - \ldots - \theta_q a_{t-k-q})]$$

$$= E[(-\theta_k a^2_{t-k} + \theta_1\theta_{k+1} a^2_{t-k-1} + \ldots + \theta_{q-k}\theta_q a^2_{t-k-q} + \text{CPT's})]$$

where E (CPT's) are expectations of cross product terms and are equal to zeros.

Therefore,

$$\gamma_k = (-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q)\sigma_a^2 \quad ; k = 1, 2, \ldots, q$$

$$= 0 \quad ; k > q.$$

For $k = 0$, the variance of the process is obtained as

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)\sigma_a^2.$$

Dividing γ_k by γ_0, the autocorrelation function is

$$\rho_k = \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2} \quad ; k = 1, 2, \ldots, q$$

$$= 0 \quad ; k > q.$$

We can see that the autocorrelation function of MA (q) process is zero, beyond the order q, of the process. In other words, the autocorrelation function of a MA (q) process has a cut - off after lag q.

For MA (1) process (q = 1) the autocorrelation function is

$$\rho_k = \frac{-\theta_1}{1 + \theta_1^2} \quad ; k = 1$$

$$= 0 \quad ; k > 1.$$

Thus, the autocorrelation function of MA (1) process has a cut - off after lag 1.

For MA (2) process (q = 2), the autocorrelation function is

$$\rho_k = \frac{-\theta_1 (1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2} \quad ; k = 1$$

$$= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \quad ; k = 2$$

$$= 0 \quad ; k > 2.$$

Thus, the autocorrelation function of MA (2) process has a cut - off after lag 2.

Invertibility Conditions for A Moving Average Process

The parameters $\theta_1, \theta_2, \ldots, \theta_q$ of a MA(q) process must satisfy certain conditions for the process to be invertible. The MA (q) process can be expressed as

$$z_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}.$$

By using backward shift operator B, it can be written as,

$$z_t - \mu = a_t - \theta_1 B a_t - \theta_2 B^2 a_t - \ldots - \theta_q B^q a_t$$

$$= (1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q)\, a_t$$

$$= \theta(B)\, a_t$$

where $\theta(B) = (1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q)$.

Since $\theta(B)$ is the q^{th} order polinomial in B, it can be written as

$$\theta(B) = (1 - H_1 B)\,(1 - H_2 B) \ldots (1 - H_q B)$$

where H_i^{-1} are the roots of the equation $\theta(B) = 0$. Then, equation

$$(z_t - \mu) = \theta(B)\, a_t$$

or

$$a_t = \theta^{-1}(B)\,(z_t - \mu)$$

can be expressed in the partial fractions,

$$a_t = \sum_{j=1}^{q} \frac{M_j}{(1 - H_j B)}\,(z_t - \mu)$$

where M_j are arbitrary constants.

If $\theta^{-1}(B)$ is to be a convergent series for $|B| \leq 1$, we must have $|H_j| < 1$, where $j = 1, 2, \ldots, q$. Since H_j^{-1} are the roots of the equation $\theta(B) = 0$, it can be said that the roots of the equation $\theta(B) = 0$ must lie outside the unit circle. The equation $\theta(B) = 0$ is called the characteristic equation and the conditions $|H_j| < 1;\ j = 1, 2, \ldots, q$ are called the invertibility conditions of the MA (q) process.

Note, since the series

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$$

is finite, no restrictions are needed on the parameters of moving average process to ensure stationarity.

For $q = 1$, the MA (1) process

$$z_t - \mu = a_t - \theta_1 a_{t-1}$$

can be expressed as

$$z_t - \mu = a_t - \theta_1 B a_t$$

$$= (1 - \theta_1 B) a_t$$

$$= \theta(B) a_t$$

where $\theta(B) = (1 - \theta_1 B)$.

Since the root of the characteristic equation $(1 - \theta_1 B) = 0$ is $B = \theta_1^{-1}$, this condition is equivalent to saying that the root of $(1 - \theta_1 B) = 0$ must lie outside the unit circle. This implies that the parameter of a MA (1) process, must satisfy the condition $|\theta_1| < 1$ to ensure invertibility

For q = 2 the MA(2) process,

$$z_t - \mu = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

can be expressed as

$$z_t - \mu = a_t - \theta_1 B a_t - \theta_2 B^2 a_t$$

$$= (1 - \theta_1 B - \theta_2 B^2) a_t$$

$$= \theta(B) a_t$$

where $\theta(B) = 1 - \theta_1 B - \theta_2 B^2$.

For invertibility, the roots of the characteristic equation $(1 - \theta_1 B - \theta_2 B^2) = 0$ must lie outside the unit circle. The roots of $(1 - \theta_1 B - \theta_2 B^2) = 0$ are

$$B_1 = \frac{\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}}{2} \quad \text{and} \quad B_2 = \frac{\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}}{2}$$

Then, $\theta_1^2 B_1 + 4\,\theta_2 B_1^2 = 4 B_1 - 4 B_1 \theta_1 + \theta_1^2$

and for $|B_1| < 1$,

$$4\,\theta_2 < 4 - 4\,\theta_1$$

or $$\theta_1 + \theta_2 < 1 \ .$$

Similarly, for $|B_2| < 1$, the condition $\theta_2 - \theta_1 < 1$ must be satisfied. By multiplying both roots, $B_1 B_2 = -\theta_2$ and since $|B_1| < 1$ and $|B_2| < 1$, $|\theta_2|$ must be less than one or $-1 < \theta_2 < 1$.

Therefore, the parameters and of a MA (2) process, must lie in the triangular region

$$\theta_1 + \theta_2 < 1$$

$$\theta_2 - \theta_1 < 1$$

and $$-1 < \theta_2 < 1 \ .$$

These are parallel to the conditions required for the stationarity of an AR (2) process.

4.2.4 Partial Autocorrelation Function of A Time Series

The partial autocorrelation function is another way of describing the dependent structure of a time series. It is useful for the identification of the type and order of the model to represent a sample time series.

The autocorrection ρ_k measures the correlation of terms of the series separated by k terms or k lags apart. The partial autocorrelation ϕ_{kk} measures the linear dependence between ρ_j and ρ_{j-k} for $j \le k$. In other words, it measures the correlation of the terms of the series k lags apart irrespective of the other terms of the series.

A set of partial autocorrelations $\{\phi_{kk}\}$, k = 1, 2, . . . is known as partial autocorrelation function (pacf) and the plot of ϕ_{kk} against the lag value k, k = 1, 2, . . . is called the partial correlogram.

The theoretical values ϕ_{kk}, k = 1, 2, ... are estimated from the sample time series and in practice, the following recursive formula, due to Durbin, can be used to find the estimator $\hat{\phi}_{kk}$

$$\hat{\phi}_{k+1,k+1} = \frac{r_{k+1} - \sum_{j=1}^{k} \hat{\phi}_{kj}\, r_{k+1-j}}{1 - \sum_{j=1}^{k} \hat{\phi}_{kj} \cdot r_j} \quad ; k=1,2,\ldots \qquad (4.2.3)$$

and

$$\hat{\phi}_{k+1,j} = \hat{\phi}_{kj} - \hat{\phi}_{k+1,k+1}\, \hat{\phi}_{k,k-j+1} \quad ; j=1,2,\ldots,k \qquad (4.2.4)$$

where r_j is the j th sample autocorrelation which estimates ρ_j.

By solving these equations the set of partial autocorrelation estimates { $\hat{\phi}_{kk}$ }, k = 1 , 2, . . . can be obtained. In these equations $\hat{\phi}_{11}$ is equal to r_1.

4.2.5 Partial Autocorrelation Function of An Autoregressive Process

Deciding the order of the autoregressive process to be fitted to an observed time series is similar to the process of deciding on the number of independent variables to be included in a multiple regression. Denoting ϕ_{kj}, as the j th coefficient in an autoregressive process of order k, the partial autocorrelation is ϕ_{kk} which is the last coefficient of the k order AR process.

From the Yule - Walker equations, ϕ_{kj} satisfy the set of equations

$$\rho_j = \phi_{k1}\rho_{j-1} + \ldots + \phi_{k(k-1)}\rho_{j-k+1} + \phi_{kk}\rho_{j-k} \quad ; j = 1, 2, \ldots, k$$

and it may be written as

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix}$$

Solving these equations for k = 1 , 2, 3, . . . , successively, gives

$$\phi_{11} = \rho_1$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

$$\phi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}}$$ and so on.

In general , for ϕ_{kk}, the determinant in the numerator has the same elements as that in the denominator but with the last column replaced by $[\rho_1 \ \rho_2 \ \ldots \rho_k]'$.

On the other hand the partial autocorrelation function can be obtained by solving Durbin's recursive formula,

$$\phi_{k+1,k+1} = \frac{\rho_{k+1} - \sum_{j=1}^{k} \phi_{kj}\rho_{k+1-j}}{1 - \sum_{j=1}^{k} \phi_{kj}\rho_j} \quad ; k=1, 2, \ldots \qquad (4.2.5)$$

and $\phi_{k+1,j} = \phi_{kj} - \phi_{k+1,k+1}\,\phi_{k,k-j+1}$; j = 1,2, . . . (4.2.6) with starting value $\phi_{11} = \rho_1$.

The estimated partial autocorrelation can be obtained by replacing the estimate r_j in place of ρ_j.

For the AR (1) process,

$$\phi_{11} = \phi_1$$

$$= \rho_1$$

and the partial autocorrelation of order higher than one are

$$\phi_{22} = \frac{\rho_2 - \phi_{11}\rho_1}{1 - \phi_{11}\rho_1}$$

since $\rho_k = \phi_1^k = \rho_1^k$.

Now, from (4.2.6)

$$\phi_{21} = \phi_{11} - \phi_{22}\phi_{11}$$

$$= \phi_{11}$$

$$= \rho_1$$

and from (4.2.5)

$$\phi_{33} = \frac{\rho_3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1 - \phi_{21}\rho_1 - \phi_{22}\rho_2}$$

since $\rho_3 = \rho_1^3$, $\phi_{21}\rho_2 = \rho_1^3$ and $\phi_{22}\rho_1 = 0$ we get

$$\phi_{33} = 0$$

In general,

$$\phi_{kk} = 0 , \; k = 2, 3, 4, \ldots .$$

It can be concluded that, the pacf of AR (1) process has a cut - off after lag 1.

For AR (2) process, the partial autocorrelations are obatined as,

$$\phi_{11} = \phi_1$$

$$= \rho_1$$

$$\phi_{22} = \phi_2$$

$$= \frac{\rho_2 - \phi_{11}\rho_1}{1 - \phi_{11}\rho_1}$$

$$= \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

from (4.2.6)

$$\phi_{21} = \phi_{11} - \phi_{22}\phi_{11}$$

$$= \rho_1 - \left(\frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}\right)\rho_1$$

$$= \frac{\rho_1(1 - \rho_2)}{1 - \rho_1^2}$$

and from (4.2.5)

$$\phi_{33} = \frac{\rho_3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1 - \phi_{21}\rho_1 - \phi_{22}\rho_2} \qquad (4.2.7)$$

where ρ_3 can be obtained by solving the Yule - Walker equation (4.2.2) for p =2 that is,

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \qquad ; k = 1, 2, 3, \ldots$$

with $\phi_1 = \frac{\rho_1(1 - \rho_2)}{1 - \rho_1^2}$ and $\phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$ we have

$$\rho_3 = \frac{\rho_1}{1 - \rho_1^2} [2 \rho_2 - \rho_1^2 - \rho_2^2] \cdot$$

By substituting ρ_3 value in (4.2.7), we get

$$\phi_{33} = 0$$

since $\phi_{21} = \frac{\rho_1 - (1 - \rho_2)}{1 - \rho_1^2}$ and $\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$.

It can be shown that

$$\phi_{kk} = 0 \qquad ; k = 3, 4, 5, \ldots$$

by the same way.

It can be concluded that the pacf of AR(2) process has a cut off after lag 2. Generally, for AR(p) process,the partial autocorrelation function ϕ_{kk} will be nonzero for k = 1, 2, . . . , p and zero for k = p+1, p+2, ... In other words, the partial autocorrelation function of AR (p) process has a cut off after lag p. This characteristic of the partial autocorrelatoin function can be used in the determination of the order of an AR process.

4.2.6 Partial Autocorrelation Function of A Moving Average Process

For MA(1) process, using Yule - Walker equations,

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix}$$

with $\rho_1 = \dfrac{-\theta_1}{1+\theta_1^2}$ and $\rho_k = 0$, for k>1, we obtain

$$\phi_{22} = \frac{-\rho_1^2}{1-\rho_1^2}$$

$$= \frac{-(-\theta_1 / 1+\theta_1^2)}{1-(-\theta_1 / 1+\theta_1^2)^2}$$

$$= \frac{-\theta_1^2}{(1+\theta_1^2)-\theta_1^4}$$

$$= \frac{-\theta_1^2}{1+\theta_1^2+\theta_1^4}$$

$$= \frac{-\theta_1^2(1-\theta_1^2)}{(1+\theta_1^2+\theta_1^4)\,(1-\theta_1^2)}$$

$$= \frac{-\theta_1^2(1-\theta_1^2)}{1-\theta_1^6}$$

$$\phi_{33} = \frac{\rho_1^3}{1-2\rho_1^2}$$

$$= \frac{(-\theta_1/1+\theta_1^2)^3}{1-2(--\theta_1/1+\theta_1^2)^2}$$

$$= \frac{-\theta_1^3}{(1+\theta_1^4)(1+\theta_1^2)}$$

$$= \frac{-\theta_1^3(1-\theta_1^2)}{1-\theta_1^8} \qquad \text{and so on.}$$

In general , we get

$$\phi_{kk} = \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}} \; .$$

It ρ_1 is positive so that θ_1 is negative, the partial autocorrelation alteranate in sign. It ρ_1 is negative, so that ϕ_1 is positive, the partial autocorrelation is negrative. Thus, $|\phi_{kk}| < \theta_1^k$, and the partial autocorrelation function is dominated by a damped exponential and tails off.

The exact expression for the partial autocorrelation function of MA(2) process is complicated. but if the roots of characteristic equation $(1- \theta_1 B - \theta_2 B^2) = 0$ are real, it is dominated by the sum of two exponentials and if the roots of characteristic equation are complex it has a damped sine wave.

Therefore, the partial autocorrelation function of MA process tails off and is dominated by the damped exponentials and / or damped sine waves.

The characteristic behavior of autocorrelations, partial autocorrelations for the three classes of processes, namely autoregressive process, moving average process and the mixed autoregressive moving average process is shown in the following table.

Characteristic Behavior of ACF, PACF of AR, MA and AR MA Processes

Class of Process	Autocorrelation	Partial Autocorrelation
AR(p)	Infinite (damped exponentials and / or damped sine waves). Tail off according to $\rho_j = \phi_1\rho_{j-1}+\phi_2\rho_{j-2}+ \ldots +\phi_p\rho_{j-p}$.	Finite. Spikes at lag 1 though p, then cut off.
MA(q)	Finite. Spike at lag 1 through q, then cut off.	Infinite(dominated by damped exponentials and/or damped sine waves). Tail off.
ARMA (p,q)	Infinite (damped exponentials and /or damped sine waves after first q-p lags) . Irregular pattern at lag 1 through q, then tail off according to $\rho_j = \phi_1\rho_{j-1}+\phi_2\rho_{j-2}+ \ldots +\phi_p\rho_{j-p}$	Infinite (dominated by damped exponential and /or damped sine waves after first q-p lags). Tail off.

Source : Box, G.E.P and Jekins, G.M (1976) " Time Series Analysis, Forecasting and Control"

Remarks : ϕ_j is the j[th] autoregressive parameter and

ρ_j is the j[th] autocorrelation coefficient.

4.2.7 Stndard Error of Autocorrelation and Partial Autocorrelation Estimates

Since we do not know the theoretical correlations and since the estimates which we compute from the sample time series will differ somewhat from their theoretical counterparts, it is important to have some indication of how far an estimated value may differ from the corresponding theoretical value.

In particular, we need some means for judging whether the autocorrelations and partial autocorrelations are effectively zero after some specific lag q. For larger lags, we can compute the variance of r_j from Bartlett's formula. That is,

$$V(r_j) \approx \frac{1}{N}\left(1 + 2\sum_{i=1}^{q} \rho_i^2\right) \qquad ; j > q \qquad (4.2.8)$$

where N is number of observations.

In practice, the autocorrelations ρ_i are unknown and must be replaced by sample estimates r_i. Thus, a standard error for estimated autocorrelation r_j is given by

$$S.E(r_j) \approx \frac{1}{\sqrt{N}}\left(1 + 2\sum_{i=1}^{q} r_i^2\right)^{\frac{1}{2}} \qquad ; j > q \qquad (4.2.9)$$

For partial autocorrelations, it was shown by Quenouille that if the process is autoregressive of order p, the estimated partial autocorrelations of order p+1 and higher are approximately independently distributed.

Also if N is the number of observations, the variance of the estimated partial autocorrelation $\hat{\phi}_{kk}$ is

$$V(\hat{\phi}_{kk}) \approx \frac{1}{N} \qquad ; k \geq p+1 \qquad (4.2.10)$$

Thus the standard error of the estimated partial autocorrelation $\hat{\phi}_{kk}$ is

$$S.E(\hat{\phi}_{kk}) \approx \frac{1}{\sqrt{N}} \qquad ; k \geq p+1 \ . \qquad (4.2.11)$$

4.2.8 Identification Techniques

The autocorrelations of a nonstationary process z_t will not die out rapidly and this failure to die out of the autocorrelation function at high lags indicates that differencing is required. This is so even though the first two autocorrelations are not large enough. Assume that the process z_t has been differenced a sufficient number of times as to produce the stationary process $w_t = (1 - B)^d z_t$.

(a) If w_t is an AR (p), its autocorrelations will die out according to the difference equations

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \ldots + \phi_p \rho_{k-p} \quad ; k = 1, 2, \ldots$$

that is, according to a mixture of damped exponentials and / or sine waves and its partial autocorrelations will obey

$$\phi_{kk} = 0 \quad \text{for } k = p+1, p+2, \ldots$$

(b) If w_t is MA (q), its autocorrelations will obey

$$\rho_k = 0 \ , k = q + 1, q + 2, \ldots$$

and its partial autocorrelations will die out, though not according to any clearly recognizable pattern.

(c) If w_t is an ARMA (p,q), its autocorrelations will die out according to the difference equations

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \ldots + \phi_p \rho_{k-p} \quad ; k > q$$

and its partial autocorrelations will also die out, though not according to any clearly recognizable pattern.

These three characteristics of the class of models can be employed as the basis of an attempt to identify an appropriate model for suitably differenced series.

In practice, these ρ 's and ϕ 's are unknown and have to be estimated from the given time series realization. Therefore, in identifying an appropriate model, we have to rely on the sample autocorrelation function and partial autocorrelation function which imitate the behaviour of the corresponding quantities. The imitation will be better if the sample is large. Therefore, in order to have any reasonable success in model identification, one requires a moderately long series of observations. In fact, we cannot be confident of the success in identification with short series. Not less than 45 or 50 observations will be needed to have reasonable success in model identification.

Some of the steps in model identification are

(1) Calculate the sample autocorrelations and partial autocorrelations of the time series.

(2) Calculate them from the first or second differences of the time series.

(3) Failure of the sample autocorrelations to die out quickly at higher lags is an indication that further differencing is needed.

(4) Having achieved stationarity by suitable differencing, attempt to identify the ARMA or AR, MA patterns as indicated in (a), (b) and (c).

(5) As a rough guide for determining whether the autocorrelations are in fact zero after lag q, use Bartletts (1946) formula, that is, for a sample of size N, the standard error of the sample autocorrelation r_k is approximately,

$$\frac{1}{\sqrt{N}} [\, 1 + 2 \sum_{j=1}^{q} r_j \,]^{\frac{1}{2}} \qquad \text{for} \quad k > q .$$

Quenouille (1949) result gives the standard error of the partial autocorrelations $\hat{\phi}_{kk}$ for AR(p) process. The standard error is approximately,

$$\frac{1}{\sqrt{N}} \qquad \text{for } k > p.$$

Anderson's (1942) result is that in moderately large sample, we can assume normality of the sample estimates.

This implies that limits plus or minus two standard error about zero can be used as a reasonable guide in assessing whether or not the quantities are in fact zero or not.

In the identification stage of model building, we have to find the sample autocorrelations. These can be used to find the initial estimates of the parameters in the model.

4.2.9 Initial Estimates of the Parameters

For an assumed AR process of order p, initial estimates for ϕ_j, j= 1, 2, . . . p. can be calculated by first substituting r_j for the theoretial autocorrelations ρ_j in the Yule - Walker equations,

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} + \ldots + \phi_p \rho_{j-p} \qquad ; j = 1, 2, \ldots p$$

and solving them.

In particular, for an AR(1) process,

$$\hat{\phi}_1 = r_1 \qquad (4.2.12)$$

and for an AR (2) process,

$$\hat{\phi}_1 = \frac{r_1(1 - r_2)}{1 - r_1^2}$$

and
$$\hat{\phi}_2 = \frac{r_2 - r_1^2}{1 - r_1^2}$$

where ϕ_j denotes the j^{th} autoregressive parameter in a process of order p.

For MA (q) process, it has been shown that the first q autocorrelations are nonzero and can be written in terms of parameters θ_j, j = 1, 2, . . . , q of the model as

$$\rho_k = \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2} \quad ; k = 1, 2, \ldots, q.$$

Expressing θ's in terms of ρ's will result in q nonlinear equations in q unknowns, Preliminary estimates of θ's can be obtained by substituting the estimates r_k for ρ_k in this equation and solving the resulting nonlinear equations which can be done only by numerical methods, except for MA (1) model.

For MA(1) process, we have

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2} \quad (4.2.13)$$

So an estimate of θ_1 may be obtained from

$$r_1 = \frac{-\hat{\theta}_1}{1+\hat{\theta}_1^2}$$

Now this is a quadratic equation in $\hat{\theta}_1$ and has two solutions, namely,

$$\hat{\theta}_1 = \frac{-1+\sqrt{1-4r_1^2}}{2r_1}$$

$$= -\frac{1}{2r_1} + \frac{1}{2r_1}(1-4r_1^2)^{\frac{1}{2}}$$

$$= -\frac{1}{2r_1} + \left[\frac{1-4r_1^2}{(2r_1)^2}\right]^{\frac{1}{2}}$$

$$= -\frac{1}{2r_1} + \left[\frac{1}{(2r_1)^2} - 1\right]^{\frac{1}{2}}$$

and $$\hat{\theta}_1 = \frac{-1 - \sqrt{1 - 4r_1^2}}{2r_1}$$

$$= -\frac{1}{2r_1} - \frac{1}{2r_1}(1 - 4r_1^2)^{\frac{1}{2}}$$

$$= -\frac{1}{2r_1} - \left[\frac{1 - 4r_1^2}{(2r_1)^2}\right]^{\frac{1}{2}}$$

$$= -\frac{1}{2r_1} - \left[\frac{1}{(2r_1)^2} - 1\right]^{\frac{1}{2}}$$

The only value of θ_1 that satisfies the invertibility condition, $-1 < \theta_1 < 1$, is taken as the estimate of θ_1. In fact, it is always true that only one of the multiple solutions can satisfy the invertiblity condition, $|\theta_1| < 1$.

4.3 Estimation of the Parameters

After a model is identified for a given time series it is important to obtain efficient estimates of the parameters. Then the fitted model will be subject to diagnostic checks and tests of goodness of fit.

For testing of goodness of fit to be relevant, it is necessary that efficient use of data should have been made in the fitting process. If this is not so, inadequacy of fit may simply arise because of the inefficient fitting and not because the form of the model is inadequate. Therefore, in the fitting process, the estimation of the parameters should be made by an efficient method.

Suppose the tentatively accepted model is

$$\phi(B)\, w_t = \theta(B)\, a_t$$

where $w_t = (1 - B)^d z_t$ and p, d, q being fixed in the identification stage.

To obtain the estimates of the parameters $\phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q$ we may use the least squares method since it can be proved that the least squares estimates are approximately maximum likelihood estimates in ARIMA models. If the least squares, method is used, we have to choose those values of ϕ 's and θ 's of the parameter set which minimize the sum of squared errors Σa_t^2 obtained from the observed time series.

There arises two difficulties in estimation stage.

(i) The equations involve unknown starting values,

$(w_0, w_{-1}, \ldots, w_{1-p}, a_0, a_{-1}, \ldots, a_{1-q})$.

(ii) The sum of squared errors function is in general nonlinear in the coefficients to be estimated.

There are two approaches to (i).

(a) The unknown starting values are simply replaced by some appropriately assumed values and estimation is conditional on these assumed starting values.

(b) The estimation is based on estimated starting values from the sample data. This unconditional approach is more efficient than the conditional approach. For long series, the difference between the results obtained by the two approaches is negligible.

4.3.1 Conditional Approach

The model $\phi(B) w_t = \theta(B) a_t$ can be written as

$$w_t - \phi_1 w_{t-1} - \ldots - \phi_p w_{t-p} = a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q}$$

where $w_t = (1-B)^d z_t$. Then

$$a_t = w_t - \phi_1 w_{t-1} - \ldots - \phi_p w_{t-p} + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q} \, .$$

Given the set of differenced observations $w_1, w_2, \ldots, w_n$, a_t can be calculated from this particular set of observations using the tentatively identified model subject to the specified values $w_0, w_{-1}, \ldots, w_{1-p}$ and $a_0, a_{-1}, \ldots, a_{1-q}$.

The most obvious values to choose as specified values of w's and a 's are their unconditional expectations namely zero. This allow the calculation of a_t, $t = 1, 2, \ldots, n$ given the coefficients $\phi_1, \phi_2, \ldots, \phi_p$ and $\theta_1, \theta_2, \ldots, \theta_q$. That is,

$$a_1 = w_1 - \phi_1 w_0 - \phi_2 w_{-1} - \ldots - \phi_p w_{1-p} + \theta_1 a_0 + \theta_2 a_{-1} + \ldots + \theta_q a_{1-q}$$

Therefore, $a_1 = w_1$

$$a_2 = w_2 - \phi_1 w_1 - \phi_2 w_0 - \ldots - \phi_p w_{2-p} + \theta_1 a_1 + \theta_2 a_0 + \ldots + \theta_q a_{2-q}$$

Therefore, $a_2 = w_2 - \phi_1 w_1 + \theta_1 a_1$

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$$a_n = w_n - \phi_1 w_{n-1} - \phi_2 w_{n-2} - \ldots - \phi_p w_{n-p} + \theta_1 a_{n-1} + \theta_2 a_{n-2} + \ldots + \theta_q a_{n-q}$$

Then, the conditional sum of squares to be minimized is

$$S_*(\underline{\phi}, \underline{\theta}) = \sum_{t=1}^{n} a_t^2$$

where $\underline{\phi} = (\phi_1, \phi_2, \ldots, \phi_p)$ and $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_q)$.

4.3.2 Unconditional Approach

In order to avoid the starting values problem, we may use the unconditiional approach to the estimation of the parameters of the temporarily selected ARMA model. In this approach, unknown values are replaced by their expectations conditional on the given data set and the sum of squares, $S(\underline{\phi}, \underline{\theta}) = \sum_{t=-\alpha}^{n} \{E_t[a_t]\}^2$, where E_t denotes the conditional expectation, is minimized with respect to ϕ's and θ's . That is, given an observed time series, the value of a_t that generates the observations is estimated by the conditional expectations given the data. For the stationary process, these conditional expectations become very small for $t < 1 - Q$, where Q is some sufficiently large positive integer and so the sum of squares to be minimized becomes, $S(\underline{\phi}, \underline{\theta}) = \sum_{t=1-Q}^{n} \{ E_t [a_t] \}^2$. Thus, increased efficiency is obtained by estimating the initial values from observed time series. This is justified by noting that the density function of observations given the parameters under assumption of normality is

$$\Pr(a_1, a_2, \ldots, a_n) \propto \sigma_a^{-n} \, g(\underline{\phi}, \underline{\theta}) \exp\left[-\frac{1}{2\sigma_a^2} \sum_{t=-\infty}^{n} \{ E_t[a_t] \}^2 \right]$$

where g is a function of the coefficients ϕ's and θ's but does not involve the observations.

4.3.3 Computation of the Conditional Expectations

For any given set of parameter values, the conditional expectation can be computed by using the fact that the two processes,

$$\phi(B)\, w_t = \theta(B)\, a_t \quad (4.3.1)$$

and

$$\phi(F)\, w_t = \theta(F)\, e_t \quad (4.3.2)$$

where a_t and e_t are white noise processes describe the same stochastic process in the sense that e_t have the same autocovariance structure as a_t. By expanding them, we have

$$w_t - \phi_1 w_{t-1} - \phi_2 w_{t-2} - \ldots - \phi_p w_{t-p} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}$$

and

$$w_t - \phi_1 w_{t+1} - \phi_2 w_{t+2} - \ldots - \phi_p w_{t+p} = a_t - \theta_1 a_{t+1} - \theta_2 a_{t+2} - \ldots - \theta_q a_{t+q}$$

since $F = B^{-1}$.

If the conditional expectations given the observations are taken,

$$E_t[w_t] - \phi_1 E_t[w_{t-1}] - \ldots - \phi_p E_t[w_{t-p}] = E_t[a_t] - \theta_1 E_t[a_{t-1}] - \ldots - \theta_q E_t[a_{t-q}]$$

(4.3.3)

and

$$E_t[w_t] - \phi_1 E_t[w_{t+1}] - \ldots - \phi_p E_t[w_{t+p}] = E_t[a_t] - \theta_1 E_t[e_{t+1}] - \ldots - \theta_q E_t[e_{t+q}]$$

(4.3.4)

Given a set of observations $w_1, w_2, \ldots, w_n$ have

$E_t[w_t] = w_t$; $t = 1, 2, \ldots, n$

$E_t[a_{t+j}] = 0$; $j = 1, 2, \ldots$

and $E_t[e_{t-j}] = 0$; $j = 1, 2, \ldots$.

Using these properties, the conditional expectation can be computed iteratively. They are started by setting,

$E_t[e_{n-p+j}] = 0$; $j = 1, 2, \ldots, q$.

For moderately long series, this approximation has no effect. The values of $E_t[e_t]$; $t = n - p, n - p-1, \ldots, 1$ are then obtained from (4.3.4). After that the values of $E_t[w_t]$; $t = 0, -1, -2, \ldots, 1\text{-}Q\text{-}p$ can be obtained. The integer Q is chosen so that the conditional expectations will be negligibly small for $t < 1 - Q$. A reasonable rule is to stop when three successive values of $E_t[w_t]$ less than 1% of the standard deviation of $w_1, w_2, \ldots, w_n$ in absolute magnitude occur. Finally, the values of $E_t[a_t]$; $t = 1 - Q, 2 - Q, \ldots, -1, 0, 1, \ldots, n$ are obtained from (4.3.3) , after setting $E_t[a_t] = 0$, $t < 1 - Q$.

Usually, single iteration is enough for the computation of the conditional expectations. For shorter time series, values of $E_t[w_t]$; $t = n+1, n+2, \ldots,$ can be obtained from the (4.3.3.) and hence from the (4.3.4) more accurate estimates of the starting up values

$E_t[e_{n-p+j}]$; $j = 1, 2, \ldots, q$ can be obtained. Iteration may then proceed until the sum of squares, $\sum_{t=1-Q}^{n} \{E_t[a_t]\}^2$ converges.

4.3.4 General Procedure for Calculating the Unconditional Sum of Squares

In general, the dual set of equations, for generating the conditional expectations $E_t[a_t]$ is obtained by taking conditional expectation in (4.3.1) and (4.3.2). Then we can write

$$\phi(F) E_t[w_t] = \theta(F) E_t[e_t] \tag{4.3.5}$$

and

$$\phi(B) E_t[w_t] = \theta(B) E_t[a_t] \tag{4.3.6}$$

The first equation is used to generate the backward forecasts $E_t[w_t]$'s and the second equation is used to generate the $E_t[a_t]$'s.

If we find that the forecasts are negligible in magnitude beyond some lead time Q, then the recursive calculation goes forward with

$$E_t[e_t] = 0 \qquad ; j = 0, 1, 2, \ldots$$

$$E_t[a_{-j}] = 0 \qquad ; j > Q - 1$$

In practice, the estimates $E_t[w_t]$ at and beyond some point $t = -Q$ with Q of moderate size, become essentially equal to zero.

Thus, to a sufficient approximation we can write

$$w_t = \phi^{-1}(B)\,\theta(B)\,a_t = \sum_{j=0}^{\alpha} \Psi_j a_{t-j} \approx \sum_{j=0}^{Q} \Psi_j a_{t-j}$$

This means that the original mixed process can be replaced by a moving average process of order Q and the procedure for estimating the parameters of a moving average process may be used .

The backward equation $\phi(F)\, E_t[w_t] = \theta(F)\, E_t[e_t]$ is now started off in precisely the same procedure for the conditional sum of squares and substitute $E_t[e_{n-p+j}] = 0$; $j = 1, 2, \ldots q$.

Then, the values of back forecasts, $E_t[w_t]$ are calculated recursively. The back forecasts, $E_t[w_{-j}]$; $j = 0, 1, 2, \ldots$ die out quickly and are equal to zero for $j > Q - 1$. Using $E[a_{-j}] = 0$; $j > Q - 1$, the forward recursion is now begun with equation (4.3.6) and the $E_t[a_t]$'s are computed.

The unconditional sum of squares $S(\phi, \theta)$ to be minimized is obtained by summing the squares of all the calculated $E_t[a_t]$'s. Thus,

$$S(\phi, \theta) = \sum_{t=1-Q}^{n} \{E_t[a_t]\}^2 .$$

4.3.5 Variances and Covariances of Estimates of Parameters

For the apporpriately parameterized AR (p) model, the log likelihood will be

$$L = L(\underline{\phi}, \sigma_a / w_n) = -\frac{n}{2} \ln \sigma_a^2 + \frac{1}{2} \ln |M_{-p}| - \frac{S(\phi)}{2\sigma_a^2} \qquad (4.3.7)$$

where $\underline{\phi} = (\phi_1, \phi_2, \ldots, \phi_p)$, $\underline{w}_n = (w_1, w_2, \ldots, w_n)$ and

$$S(\underline{\phi}) = \underline{w}'_p \underline{M}_p \underline{w}_p + \sum_{t=p+1}^{n} (w_t - \phi_1 w_{t-1} - \ldots - \phi_p w_{t-p})^2 \qquad (4.3.8)$$

and
$$\underline{M}_{p+1} = \begin{bmatrix} & & 0 \\ & & \cdot \\ & \underline{M}_p & \cdot \\ & & \cdot \\ & & \cdot \\ 0 & \ldots & 0 \end{bmatrix} + \begin{bmatrix} \phi_p^2 & \phi_p \phi_{p-1} & \ldots & -\phi_p \\ \phi_p \phi_{p-1} & \phi_{p-1}^2 & \ldots & -\phi_{p-1} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & & \cdot \\ -\phi_p & -\phi_{p-1} & \ldots & 1 \end{bmatrix}$$

For moderate and large sample (4.3.7) is adequate, useful approximations to the variances and covariances of the estimates.

The information matrix for the parameters is obtained by

$$I(\underline{\phi}) = -E\left[\frac{\partial^2 L}{\partial\phi_i \, \partial\phi_j}\right] \approx n\,(\underline{M}_p)^{-1} \cdot$$

The inverse of the information matrix supplies the asymptotic variance - covariance matrix of the maximum likelihood (ML) estimates. Moreover, if the log likelihood is approximately quadratic and maximum is not close to boundary, even if the sample size is only moderate, the elements of this matrix will normally provide adequate approximations to the variances and covariances of the estimates.

If the sample size is moderate or large, the elements of variance - covariances matrix will provide approximate variances and covariances of the estimates.

In particular, for autoregressive processes of first and second order

$$V(\hat{\phi}_1) \approx n^{-1}\underline{M}_1$$
$$\approx \frac{1}{n}(1 - \phi_1^2)$$

$$V(\hat{\phi}_1, \hat{\phi}_2) \approx n^{-1}\underline{M}_2$$
$$\approx \frac{1}{n}\begin{bmatrix} 1-\phi_2^2 & -\phi_1(1+\phi_2) \\ -\phi_1(1+\phi_2) & 1-\phi_2^2 \end{bmatrix}.$$

For large samples, the variance - covariance matrix for the maximum likelihood estimates from a moving average process is precisely the same as that for an autoregressive process of the same order. Thus, for moving average processes of order 1 and 2 we have

$$V(\hat{\theta}_1) \approx \frac{1}{n}(1 - \theta_1^2)$$

$$V(\hat{\theta}_1, \hat{\theta}_2) \approx \frac{1}{n}\begin{bmatrix} 1-\theta_2^2 & -\theta_1(1+\theta_2) \\ -\theta_1(1+\theta_2) & 1-\theta_2^2 \end{bmatrix}.$$

For ARMA (1,1) process,

$$(1 - \phi_1 B)\, w_t = (1 - \theta_1 B)\, a_t$$

the variances and covariances are

$$V(\hat{\phi}_1, \hat{\theta}_1) \approx \frac{1}{n}\frac{1-\phi_1\theta_1}{(\phi_1-\theta_1)^2}\begin{bmatrix} (1-\phi_1^2)(1-\phi_1\theta_1) & (1-\phi_1^2)(1-\theta_1^2) \\ (1-\phi_1^2)(1-\theta_1^2) & (1-\theta_1^2)(1-\phi_1\theta_1) \end{bmatrix}.$$

The estimates of variances and covariances of the estimated parameters may be obtained by substituting estimates for the parameters. That is, we may substitute $\hat{\phi}_1$ for ϕ_1, $\hat{\theta}_1$ for θ_1 and $\hat{\theta}_1$ and $\hat{\theta}_2$ for θ_1 and θ_2, respectively.

4.3.6 Constant Terms in the Models and Its Estimates

If the mean of the differenced series w_t is not zero, it needs to represent a constant term in the model. Generally, the average of differences of a time series over a long period of time will be approximately zero. If not, a constant term is to be put in the model. Let the constant term is to be put in the model. Let the constant term be δ. Then, the mean of the differences w_t of an AR(p) process is

$$E(w_t) = \frac{\delta}{1-\phi_1 - \ldots - \phi_p} \qquad (4.3.9)$$

indicating that the average over a long period of time will be nonzero.

An estimate of the constant term may be obtained from the relationship between the mean of the process and the other parameters using the sample mean of the series as an estimate of the mean of the process.

For the AR (p) process, we have theoretical relationship (4.3.9), from which we obtain the estimate of δ given by

$$\hat{\delta} = (1 - \hat{\phi}_1 - \ldots - \hat{\phi}_p) \bar{w} \qquad (4.3.10)$$

where $\bar{w}$ is the sample mean of the differences w_t and $\hat{\phi}_j$'s are the estimates of the parameters ϕ_j's.

The complete AR(p) model with estimates is then

$$w_t = \hat{\delta} + \hat{\phi}_1 w_{t-1} + \ldots + \hat{\phi}_p w_{t-p} + a_t \cdot$$

For the AR(1) process,

$$E(w_t) = \frac{\delta}{1-\phi_1} \quad \text{and} \quad \hat{\delta} = (1 - \hat{\phi}) \bar{w}$$

where $\bar{w}$ is the sample mean of w_t and $\hat{\phi}_1$ is the estimate of the parameter ϕ_1.

Then, a complete AR (1) model with estimates $\hat{\phi}_1$ and $\hat{\delta}$ is

$$w_t = \hat{\delta} + \hat{\phi}_1 w_{t-1} + a_t \cdot$$

For the MA(q) process, we have theoretical relationship

$$E(w_t) = \delta \cdot$$

Denoting the sample mean of the differences w_t as $\bar{w}$, we have then

$$\hat{\delta} = \bar{w} \cdot \qquad (4.3.11)$$

The complete MA(q) model with estimates is then

$$w_t = \hat{\delta} + a_t - \hat{\theta}_1 a_{t-1} - \ldots - \hat{\theta}_q a_{t-q}$$

where $\hat{\theta}_j$'s are the estimates of parameters θ_j's.

For the MA(1) process,

$$E(w_t) = \delta \text{ and } \hat{\delta} = \bar{w}$$

where $\bar{w}$ is the sample mean of the differences w_t .

Then, a complete MA(1) model with estimates $\hat{\theta}_1$ and $\hat{\delta}$ is

$$w_t = \hat{\delta} + a_t - \hat{\theta}_1 a_{t-1}.$$

4.3.7 An Approximate Standard Error for $\bar{w}$

In the general ARIMA model, the mean μ_w of $w_t = \nabla^d z_t$ is not zero, it may be written as

$$\phi(B)(w_t - \mu_w) = \theta(B)\, a_t$$

or

$$\phi(B)\, w_t = \delta + \theta(B)\, a_t$$

where

$$\mu_w = \frac{\delta}{1 - \phi_1 - \phi_2 - \dots - \phi_p}.$$

If $1 - \phi_1 - \phi_2 - \dots - \phi_p \neq 0$, μ_w implies that $\delta = 0$. In general, when $d = 0$, μ_z will not be zero.

At the identification stage of model building, an indication of whether or not a nonzero value for μ_w is needed, may be obtained by comparison of $\bar{w} = \Sigma\, w / n$ with its approximate standard error. With $n = N - d$,

$$\sigma^2(\bar{w}) = n^{-1} \gamma_0 \sum_{-\alpha}^{\alpha} \rho_j$$

$$= n^{-1} \sum_{-\alpha}^{\alpha} \gamma_j . \qquad (4.3.12)$$

Since the covariance generating function is

$$\gamma(B) = \sum_{k=-\alpha}^{\alpha} \gamma_k B^k,$$

the equation (4.3.12) becomes

$$\sigma^2(\bar{w}) = n^{-1} \gamma(1) \qquad (4.3.13)$$

with $B = B^{-1} = 1$.

Consider for the (1, d, 0) process

$$(1 - \phi_1 B)(w_t - \mu_w) = a_t$$

with $w_t = \nabla^d z_t$. Since the covariance generating function is

$$\gamma(B) = \frac{\sigma_a^2}{(1-\phi_1 B)(1-\phi_1 F)} \qquad ; F = B^{-1}$$

from the equation (4.3.13) , we get

$$\sigma^2(\bar{w}) = n^{-1} \frac{\sigma_a^2}{(1-\phi_1)(1-\phi_1)}$$

$$= n^{-1}(1-\phi_1)^{-2}\sigma_a^2$$

with $B = B^{-1} = 1$.

But,

$$\sigma_a^2 = (1-\phi_1^2)\sigma_w^2 \quad ,$$

so that

$$\sigma^2(\bar{w}) = n^{-1}\frac{(1-\phi_1^2)}{(1-\phi_1)^2}\sigma_w^2$$

$$= \frac{\sigma_w^2}{n}\left(\frac{1+\phi_1}{1-\phi_1}\right)$$

and we get,

$$\sigma(\bar{w}) = \sigma_w \left\{\frac{1+\phi_1}{n(1-\phi_1)}\right\}^{\frac{1}{2}} .$$

Now σ_w^2 and ϕ_1 are estimated by c_0 and r_1 respectively,

where

$$r_k = \frac{c_k}{c_0} \quad ,$$

$$c_k = \frac{1}{n}\sum_{t=1}^{n-k}(w_t - \bar{w})(w_{t+k} - \bar{w})$$

and
$$c_0 = \frac{1}{n}\sum_{t=1}^{n}(w_t - \bar{w})^2 .$$

The r_k is the estimate of the autocorrelation ρ_k and c_k is the estimate of the autocovariance γ_k.

Thus, for (1 , d , 0) process, the required standard error is

$$\hat{\sigma}(\bar{w}) = \left\{ \frac{c_0 (1 + r_1)}{n (1 - r_1)} \right\}^{\frac{1}{2}}$$

with $r_1 = \hat{\phi}$.

Similarly, for (0, d , 1) process, the expression for $\hat{\sigma}(\bar{w})$ may be obtained as

$$\hat{\sigma}(\bar{w}) = \left\{ \frac{c_0 (1 + 2r_1)}{n} \right\}^{\frac{1}{2}}$$

with $r_1 = \dfrac{-\hat{\theta}_1}{1 + \hat{\theta}_1^2}$.

4.4 Model Diagnostic Checking

After the model has been fitted to the observed time series (that is, a model or models have been identified and parameters have been estimated) it is to decide whether the model is adequate or not. If it is adequate, it can be used for forecasting. If not, the model is to be modified. It , then, is needed to discover in what way the model is inadequate. This will help in finding out the appropriate modification.

One method of checking the adequacy of the model is to overfit. That is, to consider a more general model. This can indicate the direction in which the model is inadequate. This method is to be supplemented by checks applied to the residuals from the fitted model. All these will allow the data to suggest modifications to the model.

4.4.1 Overfitting

Having identified what is believed to be a correct model, fit a more elaborate one. That is, additional terms are taken into the model, especially from the direction in which the model is inadequate. For example, if the identified model is ARIMA (0, 1 , 1), the model is $\nabla z_t = (1 - \theta_1 B) a_t$. Suppose θ_1 is estimated as $\hat{\theta}_1$. The tentatively accepted model is $\nabla z_t = (1 - \hat{\theta}_1 B) a_t$. Possible extension is to consider the model, $\nabla z_t = (1 - \theta_1 B - \theta_2 B^2) a_t$, and fit it to the data. For the extended model, the residual sum of squares is to be found and compared with the residual sum squares of the previous model.

4.4.2 Diagnostic Checks Applied to Residuals

The method of overfitting by extending the model in a particular direction needs to assume what kind of discrepancies are to be feared. Procedures less dependent upon such knowledge are based on the analysis of residuals. Visual inspection of a plot of the residuals themselves is an indispensable first step in the checking process.

Box and Pierce (1970) have suggested the statistic

$$Q = n (n + 2) \sum_{k=1}^{K} r_k^2 / (n - k)$$

where n = N - d is the number of w's used to fit the model and

$$r_k = \frac{\sum_{t=1}^{n-k} \hat{a}_t \hat{a}_{t+k}}{\sum_{t=1}^{n} \hat{a}_t^2}$$

is the lag k sample autocorrelation of the estimated residuals ($\hat{a}_t$'s) and **k** is the maximum lag considerd. If the models were appropriate and the parameter were known, for moderate or large n, Q is distributed as χ_K^2 since the limiting distribution of $r = (r_1, \ldots, r_K)'$ is multivariate normal with mean vector zero (Anderson, 1942; Anderson & Walker , 1964),

$$\mathrm{var}(r_k) = (n-k) / \{n\,(n+2)\} \quad \text{and} \quad \mathrm{cov}(r_k, r_j) = 0 \ (k \neq j).$$

Results by Anderson & Walker (1964) show that the asymptotic normality of the r_k's does not require normality of the a_t's , only that var (a_t) is finite. The overall test might be expected to be insensitive to departures from normality of the a_t's .

Using further approximation, that is, var (r_k) = 1/n, the overall test for lack of fit of the model proposed by Box and Pierce (1970) is the Portmanteau lack of fit test. The test statistic is

$$Q = n \sum_{k=1}^{K} r_k^2 .$$

The test statistic Q is distributed as χ_K^2. Furthermore, they showed that when the p+q parameters of an appropriate model are estimated, Q would for large n be distributed as χ^2_{K-p-q} where p and q are numbers of autoregressive and moving average parameters, respectively.

Ljung and Box (1978) suggested the statistic

$$Q = n\,(n+2) \sum_{k=1}^{K} r_k^2 / \,(n-k)$$

whose distribution appears closer to that of χ^2_{K-p-q} for finite samples. They computed r_k as

$$r_k = \frac{\sum_{t=1}^{n-k} \hat{a}_t \hat{a}_{t+k}}{\sum_{t=1}^{n} \hat{a}_t^2}$$

assuming that $E\{a_t\} = 0$. We use the definition,

$$r_k = \frac{\sum_{t=1}^{n-k} (\hat{a}_t - \bar{\hat{a}})(\hat{a}_{t+k} - \bar{\hat{a}})}{\sum_{t=1}^{n} (\hat{a}_t - \bar{\hat{a}})^2}$$

to calculate Q, since some of the $\hat{a}_t$'s have non - zero means.

If $Q < \chi^2_{K-p-q}$, then, $\hat{a}_t$'s can be taken to be independent and it in turn implies that the selected model is adequate.

4.4.3 Some Other Tests for the Adequacy of the Model

To check whether the order of the fitted model is adequate, the Akaike Information Criterion (AIC) can be used. Suppose that the true process from which the observations z_t $(t = 1, 2, \ldots, n)$ are generated is a k_0^{th} order model with parameters $\underline{\phi}' = (\phi_1, \phi_2, \ldots, \phi_1, 0, \ldots, 0)$ and σ_a^2 . It should be noted that

$$\underline{\hat{\phi}}_k' = (\hat{\phi}_{k1}, \ldots, \hat{\phi}_{kk}, a, \ldots 0) \qquad ; k = 1, 2, \ldots, p$$

and

$$\hat{\sigma}^2_{a(k)} = \frac{1}{n} \sum_{t=p+1}^{n} \{z_t - \hat{\phi}_{k1} z_{t-1} - \ldots - \hat{\phi}_{kk} z_{t-k}\}^2 \qquad ; k = 1, \ldots, p$$

are consistent, asymptotically normal estimates of $\underline{\phi}'$ and σ_a^2 for any $k \geq k_0$ (Anderson, 1971).

With Akaike's method (Akaike, 1969,1970), we select the order for which

$$AIC(m) = n \log \hat{\sigma}_a^2(m) + 2m \qquad ; m = 0, 1, \ldots, p$$

where n = number of observations,

σ_a^2 = estimated variance of the residuals and

m = total number of estimated parameters,

attains its minimum as a function of m.

This method balances the risk due to the bias when a lower order is selected and the risk due to the increase of variance when a higher order is selected. It is an open problem to seek an accurate evaluation of this balancing behaviour in other situations, such as when the sample sizes are small or the order k_0, p and the parameter $\underline{\phi}$ are not fixed as $n \to \infty$.

The FPE criterion suggested by Akaike (1969, 1970) for the estimating order m is

$$FPE(k) = \hat{\sigma}_k^2 (1+k/n)(1-k/n)^{-1} \tag{4.4.1}$$

$$= \hat{\sigma}_k^2 (1+2k/n) + O(n^{-2})$$

where

$$\hat{\sigma}_k^2 = \frac{1}{n} \sum_{t=1}^{n} \{ z_t - \hat{\phi}_{k1} z_{t-1} - \ldots - \hat{\phi}_{kk} z_{t-k} \}^2,$$

and $\hat{\phi}_{ki}$ (i = 1, 2, . . . , k) are the least squares estimates of the autoregressive coefficients of order k . Here, assume that m is finite and bounded by some positive integer p where p is known. If the terms of $O(n^{-2})$ are ignored, then the FPE criterion may be regarded as a special case of a more general criterion

$$FPE(k) = \hat{\sigma}_k^2 (1 + \alpha k/n)$$

where α is a fixed positive constant. The order m may be estimated by $\hat{m}$, where

$$FPE_\alpha(\hat{m}) = \inf_k FPE_\alpha(k) \qquad ; k = 1,2,\ldots, p.$$

The $FPE_\alpha(k)$ criterion is also related to an alternative $FPE^\beta(k)$ criterion suggested by Akaike (1970), since for $0 < \beta < 1$

$$FPE^\beta(k) = \hat{\sigma}_k^2 (1 + k/n^\beta)(1 - k/n)^{-1}$$

$$= FPE_\alpha(k) + O(n^{-1-\beta})$$

for $\alpha = 1 + n^{1-\beta}$.

The main advantage of the FPE_α (k) criterion is that it balances the risk of bias due to selecting a lower order model with the increase in prediction variance due to selecting a higher order autoregression, in a slightly more general manner than (4.4.1).

Letting $n \to \infty$, Bhansali and Downham (1977) showed that the greater the value of α, the greater is the probability of fitting the correct order and the smaller is the probability of overfitting. Moreover for $\alpha > 1$, the probability of fitting the correct order is greater than a positive constant, which is independent of p and which tends to 1 as $\alpha \to \infty$. Their results also showed that for $\alpha < 1$ the probability of selecting an order greater than the true order is nonnegligible and the probability of selecting the true order is close to zero for $p > m$. The latter result is also true when $\alpha = 1$. Therefore, a choice of $\alpha \leq 1$ is unlikely to be satisfactory. They said that in practice n is finite and the finite sample properties of the FPE_α (k) criterion are difficult to derive analytically.

There exists other functions for the purpose of choosing the order of a model. Hannan and Quinn (1979) proposed that

$$HQ = n \log(\hat{\sigma}^2) + 2m \log(\log n),$$

be used as a criterion in the determination of the order of a model.Geweke and Meese (1981) also proposed the Baysian Information Criterion (BIC)

$$BIC = n \log(\hat{\sigma}^2) + m \log(n)$$

and Akaike (1970)'s Final Prediction Error (FPE) is

$$BIC = \left(\frac{m}{n} - 1\right)\hat{\sigma}^2 + \left(\frac{n+m}{n-m}\right)$$

In each case, the model which minimizes the function is chosen in the fitting. In the fitting of a model to a time series, the model which gives the lowest value of each of these criteria is to be chosen as the best fitted model.

CHAPTER V

STOCHASTIC MODELS FOR SOME TRANSPORT TIME SERIES OF MYANMAR

5.1 Introduction

It is usual to find seasonal indices as a measure of seasonality in a time series by various methods. This is done under the assumption that a time series is composed of trend, seasonal, cyclical and random variations if they exist. Trend and cyclical components are represented by deterministic time functions, seasonal component by seasonal indices and random component by its statistical properties. Later developments in Time Series Analysis made it possible to represent a time series by a stochastic model which explains all types of variation in the time series as having been generated by different kinds of random processes.

Box and Jenkins (1976) proposed a class of stochastic models for representing a time series. They also developed the model building procedure for these models. Their model building procedure is closely followed in finding appropriate models for some transport time series of Myanmar in this thesis. The series for which stochastic models will be built up are:

(1) Airways (freight lb) series,

(2) Airways (number of passengers) series,

(3) Railways (freight ton) series and

(4) Inland Water Transport (freight ton) series.

For these series, suitable stochastic models will be found in the following sections. Steps followed in model building of these series will also be explained.

5.2 Steps Used in Model Building

The following steps are used in finding a suitable model for some transport time series of Myanmar.

(*i*) Identification : In the class of ARIMA model, suitable model or models for the transport time series of Myanmar under consideration will be identified.

(*ii*) Estimation : Parameters involved in the identified model or models will be estimated from the respective time series data.

(*iii*) Diagnostic Checking : Adequacy of the fitted model or models will be checked by using appropriate test.

The detail procedures in carrying out these steps are as follows:

(a) First of all, it needs to decide the class of model and the appropriate order of the model to be fitted. This is done by examining the sample autocorrelation and partial autocorrelation functions of the time series.

(b) In fitting the tentatively chosen model or models, the model parameters have to be estimated from the time series data. Maximum likelihood method of estimation is used in finding the estimates.

(c) From the fitted model or models, the residuals are computed. The Portmanteau lack of fit test is used to decide whether the residuals of the model can be taken as a sample from the random series. If so, the model will be accepted to represent the given time series. If not, the model will be modified and fitted again. Procedures (b) and (c) will be followed.

This procedure will be used in building stochastic models for the transport time series under consideration. In section 5.3 to 5.6, the model building proceduresare explained for each of the four transport time series.

5.3 Stochastic Model for Airways (Freight Lb) Series

The monthly data of Airways (freight lb) series cover 7 years, from January, 1989 to December 1995. The series consists of 84 observations. The nature of the series and seasonality were discussed in chapter 1. In this section, choice of a suitable stochastic model for this series using the model building procedure discussed in section (5.2) will be explained.

5.3.1 Identification

For this series, sample autocorrelations are found for

(a) the original series, z_t

(b) the series differenced with respect to months only, ∇z_t

(c) the series differenced with respect to years only, $\nabla_{12} z_t$

(d) the series differenced with respect to months and years, $\nabla\nabla_{12} z_t$

and are shown in Table A 5.1 of Appendix A. They are displayed in Figures B 5.1 to B 5.4 of Appendix B along with the confidence limits calculated as

$$r_k \pm 2\,\hat{S.E}\,(r_k) \qquad (5.3.1)$$

where r_k and $\hat{S.E}\,(r_k)$ were calculated as in equations (4.2.1) and (4.2.9).

The sample autocorrelations of the original series show slow decay and indicate the need to difference the sereies. Sample autocorrelations of ∇z_t show that the lag 12 sample autocorrelation lies outside the confidence limits. Sample autocorrelations of $\nabla_{12} z_t$ are mostly inside the confidence limits except at lag 1. But, sample autocorrelations of $\nabla\nabla_{12} z_t$ show that the series has been over differenced . Thus, $\nabla_{12} z_t$ is taken to be fitted by an ARMA model.

We also find the sample partial autocorrelations $\hat{\phi}_{kk}$ (k=1, 2, . . . ,24) for $\nabla_{12} z_t$ series. $\hat{\phi}_{kk}$ and respective confidence limits are shown in Table A5.5 and displayed in Figure B 5.17. The confidence limits for ϕ_{kk} are calculated as

$$\hat{\phi}_{kk} \pm 2\ \hat{S.E}\ (\hat{\phi}_{kk}) \qquad (5.3.2)$$

where $\hat{\phi}_{kk}$ and $\hat{S.E}(\hat{\phi}_{kk})$ are calculated as in equation (4.2.3) and (4.2.11). From the nature of the sample autocorrelations and partial autocorrelations, it is concluded to take p = 0, q = 1. That is, the two together suggest that the serie $\nabla_{12} z_t$ might be described by MA (1) model,

$$w_t = \delta + a_t - \theta_1 a_{t-1} \qquad (5.3.3)$$

where $w_t = \nabla_{12} z_t = z_t - z_{t-12}$ and δ is a constant term. That is,

$$z_t = \delta + z_{t-12} + a_t - \theta_1 a_{t-1}$$

The initial estimate of parameter θ_1 is obtained by substituting the sample estimate $r_1 = 0.32$ in place of ρ_1 in (4.2.13). The initial estimate of θ_1 so obtained is $\hat{\theta}_1 = -\ 0.36$. The preliminary estimate of the constant term is from (4.2.15), $\hat{\delta} = \bar{w} = -\ 0.7917$. So, the tentatively chosen model using preliminary estimates is

$$w_t = -\ 0.7917 + a_t + 0.36\ a_{t-1} \qquad (5.3.4)$$

5.3.2 Estimation

In estimating θ_1 of the model (5.3.3), the unconditional sum of squares for $S(\theta_1)$ is used. The value of the sum of squares function $S(\theta_1)$ under the model (5.3.3) is calculated from the series for different admissible values of θ_1, with an interval of 0.05 where $-1 \le \theta_1 \le 1$ and are shown in Table A 5.9. Minimum value of $S(\theta_1)$ occurs at $\theta_1 = -0.35$. When $S(\theta_1)$ is calculated for $\theta_1 = -0.35$ to -0.30, with an interval of 0.01 the minimum value of $S(\theta_1)$ is obtained at $\theta_1 = -0.33$. The estimate of θ_1 is taken as $\hat{\theta}_1 = -0.33$ since it yields the lowest $S(\theta_1)$ value for $-1 \le \theta_1 \le 1$.

For large sample, the estimated variance of $\hat{\theta}_1$ is $\hat{V}(\hat{\theta}_1) \approx n^{-1}(1-\hat{\theta}_1^2)$. For $\hat{\theta}_1 = -0.33$ and $n = 72$, we obtain $\hat{V}(\hat{\theta}_1) \approx 0.0124$ and the estimated standard error $\hat{S.E}(\hat{\theta}_1) \approx 0.1114$. The estimated standard error of $\hat{\delta}$ is 23.78 . Then the model (5.3.3) with likelihood estimates is,

$$w_t = -\underset{(23.78)}{0.7917} + a_t + \underset{(0.1114)}{0.33}\, a_{t-1} \qquad (5.3.5)$$

where $w_t = z_t - z_{t-12}$. Thus, the model for z_t is

$$z_t = -0.7917 + z_{t-12} + a_t + 0.33\, a_{t-1}$$

From, the model (5.3.5) the estimated random shock values are calculated as

$$\hat{a}_t = 0.7917 + w_t - 0.33\, a_{t-1} \qquad (5.3.6)$$

where $w_t = z_t - z_{t-12}$ and are shown in Table A 5.13.

5.3.3 Diagnostic Checking

The estimated autocorrelations of the $\hat{a}_t$ are calculated and are shown in Table A 5.17, along with the confidence interval calculated as

$$r_k(\hat{a}_t) \pm 2\, \hat{S.E}\, [\, r_k(\hat{a}_t)\,] \qquad (5.3.7)$$

where $\hat{S.E}\, [r_k(\hat{a}_t)] = \frac{1}{\sqrt{n}}$.

All the sample autocorrelations of $\hat{a}_t$ lie inside the confidence limits and hence the autocorrelations of $\hat{a}_t$ can be taken as not significantly different from zero.

An overall check is performed by using the test statistic,

$$Q = n \sum_{k=1}^{K} r_k^2(\hat{a}_t) \qquad (5.3.8)$$

which is approximately distributed as χ^2 with k-p-q degrees of freedom. $\sum_{k=1}^{48} r_k^2(\hat{a}_t)$ is obtained as 0.451 . The observed value of Q is thus $72 \times 0.451 = 32.472$ and χ^2 with 47 degrees of freedom are 67.5 and 71.42 at 5% and 2.5% significient levels, respectively.

Therfore, $Q < \chi^2_{47}$ at both levels of significance and the model is found to be appropriate. The model (5.3.5) is chosen to represent the series differenced with respect to the years only, that is, $\nabla_{12} Z_t$.

5.4 Stochastic Model for Airways (Number of Passengers) Series

The monthly data of Airways (number of passengers) series covers 7 years, from January, 1989 to December, 1995. The series consists of 84 observations. The nature of the series and seasonality were discussed in chapter 1. In this section, choice of a suitable stochastic model for this series using the model building procedure discussed in section (5.2) will be explained.

5.4.1 Identification

For this series, sample autocorrelations are found for

(a) the original series, z_t;

(b) the series differenced with respect to months only, ∇z_t

(c) the series differenced with respect to years only, $\nabla_{12} z_t$

(d) the series differenced with respect to months and years, $\nabla\nabla_{12} z_t$

and are shown in Table A 5.2 of Appendix A. They are displayed in Figures B 5.5 to B 5.8 of Appendix B along with the confidence limits calculated as in (5.3.1).

The sample autocorrelations of the original series show slow decay and indicate the need to difference the series. Sample autocorrelations of ∇z_t show that sample autocorrelations for lags 12, 24, 36 and 48 are still considerably , relatively high although they lie within the confidence limits. As to the sample autocorrelations of $\nabla_{12} z_t$ lag 12 sample autocorrelation lies outside the confidence limits and at some lags, sample autocorrelations lie outside the confidence limits or relatively high. On the other hand, sample autocorrelations of $\nabla\nabla_{12} z_t$, although some of them lie outside the confidence limits, sample autocorrelations at lag 24, 36 and 48 are very small. Besides, sample autocorrelations have a declining pattern, two possible models are then considered. AR(1) model to be fitted to ∇z_t or AR(1) model to be fitted to $\nabla\nabla_{12} z_t$.

We also find the sample partial autocorrelations $\hat{\phi}_{kk}(k = 1, 2,..., 24)$

for $\nabla\nabla_{12} z_t$ series. $\hat{\phi}_{kk}$ and confidence limits are shown in Table A 5.6 and displayed in Figure B 5.18. The confidence limits for $\hat{\phi}_{kk}$ are calculated as in (5.3.2).

From these values, we fit AR (1) model

$$w_t = \delta + a_t + \phi_1 \, w_{t-1} \qquad (5.4.1)$$

where $w_t = \nabla z_t = z_t - z_{t-1}$

or

$$w_t = \delta + a_t + \phi_1 \, w_{t-1} \qquad (5.4.2)$$

where $w_t = \nabla\nabla_{12} z_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$ and δ is a constant term. That is ,

$$w_t = \delta + z_{t-1} + z_{t-12} - z_{t-13} + a_t + \phi_1 \, w_{t-1}$$

Model (5.4.1) is fitted and its adequacy is checked by Portmanteau test, the model is not accepted. Model (5.4.2) is fitted as follows and when its adequacy is checked, the model is accepted.

The initial estimate of parameter ϕ_1 of model (5.4.2) is obtained by substituting the sample estimate $r_1 = -0.53$ in (4.2.12). The initial estimate of ϕ_1 so obtained is $\hat{\phi}_1 = -0.53$. The preliminary estimate of the constant term is from (4. 2.14), $\hat{\delta} = -0.1509$. So, the tentatively chosen model using preliminary estimates is

$$w_t = -0.1509 + a_t - 0.53 \, w_{t-1} \qquad (5.4.3)$$

5.4.2 Estimation

In estimating ϕ_1 of the model (5.4.2), the sum of squares for $S(\phi_1)$ is used. The value of the sum of squares function under the model (5.4.2) is calculated from the series for different admissible values of ϕ_1, with an interval of 0.05 where $-1 \le \phi_1 \le 1$ and are shown in Table A 5.10. Minimum value of $S(\phi_1)$ occurs at $\phi_1 = -0.55$. When $S(\phi_1)$ is calculated for $\phi_1 = -0.50$ to -0.55, with an interval of 0.01 the minimum value of $S(\phi_1)$ is obtained at $\phi_1 = -0.53$. The estimate of ϕ_1 is taken as $\hat{\phi}_1 = -0.53$ since it yields the lowest $S(\phi_1)$ value for $-1 \le \phi_1 \le 1$.

For large sample, the estimated variance of $\hat{\phi}_1$ is $\hat{V}(\hat{\phi}_1) \approx n^{-1}(1-\hat{\phi}_1^2)$. For $\hat{\phi}_1 = -0.53$ and $n = 71$, we obtain $\hat{V}(\hat{\phi}_1) \approx 0.0101$ and estimated standard error $\hat{S.E}(\hat{\phi}_1) \approx 0.1005$. The estimated standard error of $\hat{\delta}$ is 1.0366 . Therefore, the model (5.4.2) is then

$$w_t = \underset{(1.0366)}{-0.1509} + a_t \underset{(0.1005)}{- 0.53} w_{t-1} \qquad (5.4.4)$$

From the model (5.4.4), the estimated random shock values are calculated as

$$\hat{a}_t = 0.1509 + w_t - 0.53 \quad w_{t-1} \qquad (5.4.5)$$

where $w_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$ and are shown in Table A 5. 14.

5.4.3 Diagnositic Checking

The estimated autocorrelations of the $\hat{a}_t$ are calculated and are shown in Table A 5. 18 , along with the confidence interval calculated as in (5.3.7) .

All the sample autocorrelations of $\hat{a}_t$ lie inside the confidence limits and hence the autocorrelations of $\hat{a}_t$ can be taken as not significantly different from zero.

An overall check is performed by using the test statistic, in (5.3.8) which is approximately distributed as χ^2 with k-p-q degrees of freedom. $\sum_{k=1}^{48} r_k^2(\hat{a}_t)$ is obtained as 0.7732. The observed value of Q is thus $71 \times 0.7732 = 54.8972$ and χ^2 with 47 degrees of freedom are 67.5 and 71.42 at 5% and 2.5% significient levels, respectively.

Therefore, $Q < \chi^2_{47}$ at both levels of significance and the model is found to be appropriate. The model (5.4.4) is chosen to represent the series differenced with respect to the months and years, that is, $\nabla\nabla_{12} z_t$.

5.5 Stochastic Model for Railways (Freight Ton) Series

The monthly data of Railways (freight ton) series covers 7 years, from January, 1989 to December, 1995. The series consists of 84 observations. The nature of the series and seasonality were discussed in chapter 1. In this section, a suitable stochastic model for this series will be found by using the model building procedure explained in section (5.2).

5.5.1 Identification

For this series, sample autocorrelations are found for

(a) the original series, z_t

(b) the series differenced with respect to months only, ∇z_t

(c) the series differenced with respect to years only, $\nabla_{12} z_t$

(d) the series differenced with respect to both months and years, $\nabla\nabla_{12} z_t$

and are shown in Table A 5.3 of Appendix A. They are displayed in Figures B 5.9 to B 5.12 of Appendix B along with the confidence limits calculated as in (5.3.1)

Sample autocorrelation of the original series show slow decay and are oscillating and indicate the need to difference the series. Sample autocorrelations of ∇z_t show that they are still relatively high at lag 12, 24, 36 and 48 . The same is true for $\nabla_{12} z_t$. But they are relatively lower for the $\nabla \nabla_{12} z_t$ series. All of these sample autocorrelations of $\nabla \nabla_{12} z_t$ series lie inside the confidence limits except at lag 12.

We also find the sample partial autocorrelations $\hat{\phi}_{kk}$ (k=1, 2, . . . ,24) for $\nabla \nabla_{12} z_t$ series. $\hat{\phi}_{kk}$ and respective confidence limits are shown in Table A 5.7 and displayed in Figure B. 5.19. The confidence limits for ϕ_{kk} are calculated as in (5.3.2). Most of $\hat{\phi}_{kk}$ lies within the confidence limits except at k = 3 and 24. Besides $\hat{\phi}_{kk}$ for k = 13 to 24 are mostly less than $\hat{\phi}_{kk}$ for k = 1 to 12. The overall pattern of $\hat{\phi}_{kk}$ can be taken as decreasing, over the whole range of k after k = 12.

The two together suggest that the series $\nabla \nabla_{12} z_t$ might be described by seasonal MA(1) model,

$$w_t = \delta + a_t - \Theta_1 a_{t-12} \tag{5.5.1}$$

where $w_t = \nabla\nabla_{12} z_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$ and δ is a contant term. That is ,

$$z_t = \delta + z_{t-1} - z_{t-12} + z_{t-13} + a_t - \Theta_1 a_{t-12}$$

The initial estimate of the parameter Θ_1 is obtaind by substituting the sample estimate $r_{12} = -0.37$ in the place of ρ_{12} in $\rho_{12} = \dfrac{-\Theta_1}{1 + \Theta_1^2}$. The initial estimate of Θ_1 so obtained is $\hat{\Theta}_1 = 0.44$. The preliminary estimate of the constant term is $\hat{\delta} = -1.4225$. So, the tentatively chosen model using preliminary estimates is

$$w_t = -1.4225 + a_t - 0.44\, a_{t-12} \tag{5.5.2}$$

5.5.2 Estimation

In estimating Θ_1 of the model (5.5.1) , the unconditional sum of squares for $S(\Theta_1)$ is used. The value of the sum of squares function $S(\Theta_1)$ under the model (5.5.1) is calculated from the series for different admissible values of Θ_1, with an interval of 0.05 where $-1 \leq \Theta_1 \leq 1$ and are shown in Table A 5.11. Minimum value of $S(\Theta_1)$ occurs at $\Theta_1 = 0.75$. When $S(\Theta_1)$ is calculated for Θ_1 = 0.71 to 0.77 with an interval of 0.01, the minimum value of $S(\Theta_1)$ is obtained at $\Theta_1 = 0.75$. The estimate of Θ_1 is taken as $\Theta_1 = 0.75$ since it yields the lowest $S(\Theta_1)$ value for $-1 \leq \Theta_1 \leq 1$.

For large sample, the estimated variance of $\hat{\Theta}_1$ is $\hat{V}(\hat{\Theta}_1) \approx n^{-1}(1-\hat{\Theta}_1^2)$. For $\hat{\Theta}_1 = 0.75$ and $n = 71$, we obtain $\hat{V}(\hat{\Theta}_1) \approx 0.0062$ and the estimated standard error $\hat{S.E}(\hat{\Theta}_1) \approx 0.0787$. The estimated standard error of $\hat{\delta}$ is 2.6514. Therefore, the model (5.5.1) is then

$$w_t = -1.4225 + a_t - 0.75\, a_{t-12} \qquad (5.5.3)$$

(2.6514) (0.0787)

From the model (5.5.3) the estimated random shock values are calculated as

$$\hat{a}_t = 1.4225 + w_t - 0.75\, a_{t-12} \qquad (5.5.4)$$

where $w_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$ and are shown in Table A 5.15.

5.5.3 Diagnostic Checking

The estimated autocorrelations of the $\hat{a}_t$ are calculated and are shown in Table A 5.19, along with the confidence interval calculated as in (5.3.7).

All the sample autocorrelations of $\hat{a}_t$ lie inside the confidence limits and hence the autocorrelations of $\hat{a}_t$ can be taken as not significantly different from zero.

An overall check is performed by using the test statistic in (5.3.8) which is approximately distributed as χ^2 with k - p - q degrees of freedom. $\sum_{k=1}^{48} r_k^2(\hat{a}_t)$ is obtained as 0.4979. The observed value of Q is thus 60 x 0.4979 = 29.874 and χ^2 with 47 degrees of freedom are 67.5 and 71.42 at 5% and 2.5 significant levels, respectively.

Therefore, $Q < \chi^2_{47}$ at both levels of significance and the model is found to be appropriate. The model (5.5.3) is chosen to represent the series differenced with respect to the months and years, that is, $\nabla\nabla_{12}z_t$.

5.6 Stochastic Model for Inland Water Transport (Freight Ton) Series

The monthly data of Inland Water Transport (freight ton) series covers 7 years, from January, 1989 to December, 1995. The series consists of 84 observations. The nature of the series and seasonality, were discussed in chpater 1. In this section, a suitable stochastic model for this series will be found by using the model building procedure explained in section (5.2).

5.6.1 Identification

For this series, sample autocorrelations functions are found for

(a) the original series, z_t

(b) the series differenced with respect to months only, ∇z_t

(c) the series differenced with respect to years only, $\nabla_{12}z_t$

(d) the series differenced with respect to both months and years, $\nabla\nabla_{12}z_t$

and are shown in Table A 5.4 of Appendix A. They are displayed in Figures B 5.13 to B 5.16 of Appendix B along with the confidence limits calculated as in (5.3.1).

Sample autocorrelation of the original series show slow decay and are oscillating and indicate the need to difference the series. Sample autocorrelations of ∇z_t and $\nabla \nabla_{12} z_t$ have a cutoff at lag 1 and all the sample autocorrelations of $\nabla_{12} z_t$ lie inside the confidence limits.

We also find the sample partial autocorrelations $\hat{\phi}_{kk}$ (k=1, 2,...,24) for $\nabla \nabla_{12} z_t$ series. $\hat{\phi}_{kk}$ and respective confidence limits are shown in Table A 5.8 and displayed in Figure B 5.20 . The confidence limits for $\hat{\phi}_{kk}$ are caluclated as in (5.3.2).

Based on these values, two models are considered. MA (1) model is to be fitted to both ∇z_t and $\nabla \nabla_{12} z_t$. That is,

$$w_t = \delta + a_t - \theta_1 a_{t-1} \qquad (5.6.1)$$

where $w_t = \nabla z_t = z_t - z_{t-1}$

or

$$w_t = \delta + a_t - \theta_1 a_{t-1} \qquad (5.6.2)$$

where $w_t = \nabla \nabla_{12} z_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$ and δ is a contant term. That is ,

$$z_t = \delta + z_{t-1} + z_{t-12} - z_{t-13} + a_t - \theta_1 a_{t-1}$$

When the model (5.6.1) is fitted and checked for its adequacy, it is not accepted. Model (5.6.2) is fitted as follows and when checked, it is accepted.

The initial estimate of the parameter θ_1 is obtaind by substituting the sample estimate $r_1 = - 0.45$ in the place of ρ_1 in (4.2.13) . The initial estimate of θ_1 so obtained is $\hat{\theta}_1 = 0.63$. The preliminary estimate of the constant term is from (4.2.15), $\hat{\delta} = \bar{w} = -0.6479$. So, the tentatively chosen model using preliminary estimates is

$$w_t = -0.6479 + a_t - 0.63\, a_{t-1} \qquad (5.6.3)$$

5.6.2 Estimation

In estimating θ_1 of the model (5.6.2) , the unconditional sum of squares for $S(\theta_1)$ is used. The value of the sum of squares function $S(\theta_1)$ under the moedel (5.6.2) is calculated from the series for different admissible values of θ_1, with an interval of 0.05 where $-1 \le \theta_1 \le 1$ and are shown in Table A 5.12. Minimum value of $S(\theta_1)$ occurs at $\theta_1 = 0.45$.When $S(\theta_1)$ is calculated for $\theta_1 = 0.40$ to 0.45 with an interval of 0.01, the minimum value of $S(\theta_1)$ is obtained at $\theta_1 = 0.43$. The estimate of θ_1 is taken as $\hat{\theta}_1 = 0.43$ since it yields the lowest $S(\theta_1)$ value for $-1 \le \theta_1 \le 1$.

For large sample, the estimated variance of $\hat{\theta}_1$ is $\hat{V}(\hat{\theta}_1) \approx n^{-1}(1-\hat{\theta}_1^2)$. For $\hat{\theta}_1 = 0.43$and $n = 71$, we obtain $\hat{V}(\hat{\theta}_1) \approx 0.0115$ and the estimated standard error $\widehat{S.E}\,(\hat{\theta}_1) \approx 0.1079$. The estimated standard error of $\hat{\delta}$ is 1.1928. Therefore, the model (5.6.2) is then

$$\underset{(1.1928)}{w_t = -0.6479} + a_t - \underset{(0.1079)}{0.43}\, a_{t-1} \qquad (5.6.4)$$

From the model (5.6.4), the estimated random shock values are calculated as

$$\hat{a}_t = 0.6479 + w_t + 0.43\, a_{t-1} \qquad (5.6.5)$$

where $w_t = z_t - z_{t-1} - z_{t-12} + z_{t-13}$ and are shown in Table A 5.16.

5.6.3 Diagnostic Checking

The estimated autocorrelations of a_t are calculated and are shown in Table A 5.20, along with the confidence interval calculated as in(5.3.7).

All the sample autocorrelations of a_t lie inside the confidence limits and hence the autocorrelations of a_t can be taken as not significantly different from zero.

An overall check is performed by using the test statistic in (5.3.8) which is approximately distributed as χ^2 with k-p-q degrees of freedom. $\sum_{k=1}^{48} r_k^2(\hat{a}_t)$ is obtained as 0.5143. The observed value of Q is thus

71 x 0.5413 = 36.5153 and χ^2 with 47 degrees of freedom are 67.5 and 71.42 at 5% and 2.5 significant levels, respectively.

Therefore, $Q < \chi^2_{47}$ at both levels of significance and the model is found to be appropriate. The model (5.6.4) is chosen to represent the series differenced with respect to the months and years, that is, $\nabla\nabla_{12}z_t$.

5.7 Forecasting

In the previous sections of this chapter, models for the Airways (freight lb), Airways (number of passenger), Railways (freight ton) and Inland Water Transport (freight ton) series have been identified, estimated and checked for adequacy. The accepted models will be used to forecst the values for January to December of 1996.

Airways (Freight Lb) Series

The accepted model for this series covering January, 1989 to December, 1995 is

$$w_t = \underset{(23.78)}{-0.7917} + a_t + \underset{(0.1114)}{0.33}\, a_{t-1}$$

Using this model the forecasts for January to December, 1996 are as shown in Table (5.1).

Table 5.1

The Forecasts for January to December, 1996 of Airways (Freight Lb) Series

(thousand lb)

Jan	187	May	268	Sep	244
Feb	250	Jun	182	Oct	321
Mar	655	Jul	255	Nov	245
Apr	560	Aug	223	Dec	298

Airways (Number of Passenger) Series

The accepted model for this series covering January, 1989 to December, 1995 is

$$w_t = -0.1509 + a_t - 0.53\, w_{t-1}$$

(1.0366) (0.1005)

Using this model the forecasts for January to December, 1996 are as shown in Table (5.2).

Table 5.2

The Forecasts for January to December, 1996 of Airways (Number of Passenger) Series

(thousand ton)

Jan	65	May	60	Sep	51
Feb	60	Jun	50	Oct	57
Mar	65	Jul	52	Nov	60
Apr	62	Aug	51	Dec	63

Railways (Freight Ton) Series

The accepted model for this series covering January, 1989 to December, 1995 is

$$w_t = -1.4225 + a_t - 0.75\, a_{t-1}$$

(2.6514) (0.0787)

Using this model the forecasts for January to December, 1996 are as shown in Table (5.3).

Table 5.3

The Forecasts for January to December, 1996 of Railways (Freight Ton) Series

(in Million)

Jan	234	May	200	Sep	184
Feb	221	Jun	191	Oct	200
Mar	230	Jul	188	Nov	217
Apr	295	Aug	190	Dec	235

Inland Water Transport (Freight Ton) Series

The accepted model for this series covering January, 1989 to December, 1995 is

$$w_t = -\underset{(1.1928)}{0.6479} + a_t - \underset{(0.1079)}{0.43}\, a_{t-1}$$

Using this model the forecasts for January to December, 1996 are as shown in Table (5.4).

Table 5.4

The Forecasts for January to December, 1996 of Inland Water Transport (Freight Ton) Series

(in Million)

Jan	286	May	278	Sep	233
Feb	267	Jun	247	Oct	248
Mar	302	Jul	272	Nov	234
Apr	278	Aug	247	Dec	246

CONCLUSION

In the foregoing chapters, we have presented basic statistical characteristics of some monthly transport time series of Myanmar such as Airways (freight lb), Airways (number of passengers), Railways (freight ton) and Inland Water Transport (freight ton) series and the model building procedures for these series.

Moreover, the traditional methods of seasonal measurement were also discussed. Some methods are quite simple and easy to perform and others are quite lengthy and complicated. Some methods are valid under the additive model and some under multiplicative model. The investigator has to choose a suitable method to fufil his own purpose for seasonal measurement. Baside these methods, the seasonality can be measured by using the methods such as harmonic representations and spectral methods.

As an alternative to these methods, stochastic seasonal models such as seasonal AR models, seasonal MA models, seasoanl ARMA models and general multiplicative seasonal model can be used. They were also discussed together with their characteristics.

At the begining, we choose six monthly transport time series such as Airways (freight lb), Airways (number of passengers), Railways (freight ton), Railways (number of passengers), Inland Water Transport (freight ton) and Inland Water Transport (number of passengers) series. For these time series, model building procedures were developed in detail. Since ready made computer software was not available to us, it took a long time to build up computer programs and run them. In the estimation stage, maximum likelihood method was used and conditional sum of squares had to be minimized. This differed according to the model identified and a separate computer program had to be worked out. Since the function to be minimized was nonlinear, the values of the function for different values of the parameters in the admissible range had to be found. To save the time and work, this was done for the parameter values with two decimal places at the interval of 0.05. The program was ran again for parameter values with two decimal places at the interval of 0.01around the minimum point. In doing so, the minimum point was obtained only for five series out of the six series considered. Thus, stochastic models were fitted to five series namely, Airways (freight lb), Airways (number of passengers), Railways (freight ton), Railways (number of passengers) and Inland Water Transport (freight ton). But when diagnostic check was applied, the fitted models for four series were only accepted. Thus, model fitting for only four series were discussed in this thesis.

IMA model was found to be suitable for Airways (freight lb) series and Inland Water Transport (freight ton) series. Series obtained by differencing with respect to years for Airways (freight lb) series and differencing with respect to months as well as year for Inland Water Transport (freight ton) series were fitted with these models and were found to be adequate by using a diagnostic check. ARI model for Airways (number of passengers) series and SMA model for Railways (freight ton) series were found to be suitable to represent the series differenced with respect to months as well as years for both Airways (number of passengers) series and Railways (freight ton) series. The accepted models for the four series were then used to find forecasts for each series for the period, January to December, 1996. The forecasts clearly show the seasonal variation, experienced by each series.

The objective of model fitting to a time series is to use the model for forecasting. One has to choose a model from a set of suitable models and this is done by comparing their forecast performance. Due to lack of time, we are not able to assess the forecast performance of our fitted models compared to other models. Besides, there exists other models and methods which will be interesting to explore if time permits and if computational hazards can be avoided.

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APPENDIX A

Table A 1.1
Airways (Feright Ton) Series

(In Thousand)

Month	1989	1990	1991	1992	1993	1994	1995
Jan	199	212	264	175	426	328	187
Feb	201	171	276	703	744	294	250
Mar	282	539	200	796	984	506	655
Apr	662	496	205	185	734	791	560
May	155	280	162	218	327	528	268
Jun	118	398	568	409	230	196	182
Jul	166	201	202	239	254	523	255
Aug	135	197	167	243	212	260	223
Sep	230	428	207	157	263	257	244
Oct	192	248	205	289	244	220	321
Nov	858	899	225	266	236	247	245
Dec	547	885	496	295	337	248	298

Source : Central Statistical Organization (C. S O.)

Table A 1.2
Airways (Number of Passengers) Seires

(In Thousand)

Month	1989	1990	1991	1992	1993	1994	1995
Jan	27	36	40	35	39	49	62
Feb	24	34	40	34	42	46	56
Mar	28	39	48	43	47	57	61
Apr	29	38	22	43	50	59	58
May	28	34	44	39	38	54	56
Jun	22	23	28	34	30	38	46
Jul	21	25	29	31	30	42	48
Aug	21	27	30	32	37	38	47
Sep	24	28	31	29	43	38	47
Oct	26	34	35	30	42	50	53
Nov	28	38	41	31	51	56	56
Dec	31	37	23	39	46	57	59

Source: Central statistical Organization (C.S.O)

Table A 1.3
Railways (Freight Ton) Series

(In Thousand)

Month	1989	1990	1991	1992	1993	1994	1995
Jan	165	208	203	220	210	235	249
Feb	169	185	190	207	223	231	206
Mar	175	208	197	200	260	235	211
Apr	131	159	177	170	200	209	182
May	146	146	201	177	185	212	192
Jun	131	149	175	177	184	196	185
Jul	125	152	158	184	199	181	182
Aug	121	157	154	147	190	210	193
Sep	107	124	141	178	175	207	191
Oct	123	139	157	188	195	211	214
Nov	127	148	191	206	202	276	209
Dec	177	173	204	171	232	285	227

Source : Central Statistical Organization (C. S O.)

Table A 1.4
Inland Water Transport (Freight Ton) Series

(In Thousand)

Month	1989	1990	1991	1992	1993	1994	1995
Jan	194	237	221	230	272	276	289
Feb	170	224	206	228	251	265	270
Mar	198	261	228	259	308	297	306
Apr	195	220	213	235	277	284	281
May	190	219	228	234	276	287	281
Jun	180	211	212	233	283	268	250
Jul	200	219	201	242	262	242	275
Aug	233	216	210	217	232	240	250
Sep	200	190	202	225	226	232	236
Oct	196	181	216	241	253	260	251
Nov	211	187	208	239	260	264	237
Dec	227	191	218	267	269	252	249

Source: Central Statistical Organization (C.S.O)

Table A 1.5

ANOVA Table for Airways (Freight Lb) Series

Source	S.S	D.F	M. S	F . Ratio
Between months	1990882.3	11	180989.3	5.3
Between years	125813.4	6	20968.9	0.61
Error	2258731.2	66	34223.2	
Total	4375426.9	83		

Table A 1 .6

ANOVA Table for Airways (No of Passengers) Series

Source	S.S	D.F	M. S	F . Ratio
Between months	3052.5	11	277.5	13.3
Between years	3641.4	6	606.9	29.2
Error	1372.8	66	20.8	
Total	8066.7	83		

Table A 1.7

ANOVA Table For Railways (Freight Ton) Series

Source	S.S	D.F	M. S	F . Ratio
Between months	52311.6	11	4755.6	19.3
Between years	30693.6	6	5115.6	208
Error	16255	66	246.3	
Total	99261.0	83		

Table A 1 .8

ANOVA Table for Inland Water Transport (Freight Ton) Series

Source	S. S	D.F	M. S	F . Ratio
Between months	21807.5	11	1982.5	8.4
Between years	30991.8	6	5165.3	21.8
Error	15602.4	66	236.4	
Total	68401.7	83		

Table A 5.1

Estimated Autocorrelations of Airways (Freight Lb) Series

Lag k	Autocorrelations												Standard Error for Row
1-12	0.30	-0.02	-0.02	-0.13	-0.19	-0.14	-0.09	-0.01	-0.11	-0.21	0.04	0.38	0.1091
13-24	0.21	0.11	0.08	-0.16	-0.21	-0.15	-0.10	-0.11	-0.06	-0.13	-0.02	0.05	0.1603
25-36	0.14	0.22	0.23	0.10	-0.02	-0.15	-0.12	-0.13	-0.10	0.01	0.04	-0.01	0.1749
37-48	0.08	0.04	0.12	0.15	0.04	-0.09	-0.11	-0.12	-0.10	0.01	-0.02	0.01	0.1808
1-12	-0.27	-0.23	0.09	-0.06	-0.08	0.01	-0.02	0.13	-0.01	-0.22	-0.08	0.36	0.1098
13-24	-0.05	-0.04	0.15	-0.14	-0.08	0.00	0.04	-0.03	0.04	-0.12	0.04	-0.04	0.1495
27-36	0.00	0.07	0.09	-0.01	0.02	-0.13	0.04	-0.03	-0.06	0.06	0.07	-0.12	0.1540
37-48	0.11	-0.08	0.01	0.10	0.02	-0.08	-0.01	-0.02	-0.06	0.10	-0.04	0.02	0.1578
1-12	0.32	0.08	0.04	-0.08	-0.02	0.02	0.03	0.19	-0.15	-0.22	-0.20	-0.17	0.1179
13-24	0.06	0.00	-0.01	-0.14	-0.15	0.00	0.00	-0.18	0.08	-0.09	-0.04	-0.20	0.1551
25-36	-0.09	0.17	0.06	0.09	0.12	-0.07	0.11	-0.01	-0.04	0.11	0.09	-0.02	0.1640
37-48	0.03	-0.08	-0.04	-0.07	-0.03	0.04	0.06	0.07	0.04	0.06	0.00	0.00	0.1665
Z_t 1-12	-0.38	-0.10	0.06	-0.13	0.02	0.02	-0.10	0.33	-0.18	-0.07	0.01	-0.14	0.1187
13-24	0.20	-0.03	0.07	-0.08	-0.12	0.11	0.04	-0.15	0.22	-0.14	0.13	-0.18	0.1737
25-36	-0.10	0.25	-0.09	0.00	0.15	-0.25	0.20	-0.06	-0.11	0.11	0.06	-0.10	0.1928
37-48	0.10	-0.10	0.05	-0.06	-0.01	0.03	0.01	0.03	-0.04	0.06	-0.04	-0.03	0.1953

Table A 5. 2

Estimated Autocorrelationsof the Airways (Number of Passengers)Series

	Lag k	Autocorrelations												Standard Error for Row
Z_t	1-12	0.75	0.67	0.52	0.45	0.39	0.38	0.36	0.42	0.39	0.45	0.44	0.45	0.1091
	13-24	0.38	0.30	0.21	0.17	0.17	0.15	0.13	0.09	0.08	0.13	0.15	0.18	0.3009
	25-36	0.16	0.12	-0.04	-0.03	-0.06	-0.04	-0.07	-0.05	-0.06	-0.01	0.01	0.02	0.3032
	37-48	-0.04	-0.05	-0.11	-0.12	-0.13	-0.15	-0.16	-0.15	-0.17	-0.15	-0.13	-0.11	0.3106
∇Z_t	1-12	-0.39	0.20	-0.18	-0.03	-0.15	0.04	-0.23	0.21	-0.21	0.17	-0.04	0.20	0.1098
	13-24	0.04	0.02	-0.06	-0.10	-0.02	0.02	0.00	-0.08	-0.11	0.11	-0.04	0.12	0.1564
	27-36	0.03	0.12	-0.09	-0.01	-0.12	0.12	-0.16	0.03	-0.09	0.06	0.00	0.17	0.1654
	37-48	-0.06	0.14	-0.11	-0.01	-0.02	-0.03	-0.06	0.03	-0.07	0.02	0.01	0.15	0.1704
$\nabla_{12} Z_t$	1-12	0.15	0.16	-0.04	-0.15	-0.08	0.02	0.07	0.44	0.20	0.12	0.03	-0.42	0.1179
	13-24	-0.05	-0.17	0.01	-0.02	0.12	0.03	0.08	-0.29	-0.18	-0.18	-0.16	-0.13	0.1866
	25-36	-0.03	0.00	-0.03	-0.06	-0.16	-0.04	-0.17	-0.01	0.00	-0.01	0.01	0.01	0.1912
	37-48	-0.02	0.02	-0.04	0.03	0.09	0.00	0.07	0.05	0.04	0.03	0.05	0.06	0.1932
$\nabla\nabla_{12} Z_t$	1-12	-0.53	0.12	-0.04	-0.11	-0.01	0.02	-0.19	0.35	-0.10	0.00	0.22	-0.49	0.1187
	13-24	0.29	-0.18	0.14	-0.12	0.15	-0.08	0.25	-0.28	0.07	-0.01	0.01	-0.05	0.2111
	25-36	0.05	0.02	-0.02	0.06	-0.14	0.15	-0.17	0.10	0.01	-0.01	-0.12	0.03	0.2170
	37-48	-0.06	0.07	-0.09	0.02	0.09	-0.09	0.04	0.00	-0.01	-0.02	0.01	0.02	0.2193

Table A 5. 3
Estimated Autocorrelationsof Railways (Freight Ton) Series

	Lag k	Autocorrelations												Standard Error for Row
Z_t	1-12	0.79	0.63	0.43	0.30	0.21	0.14	0.17	0.23	0.34	0.46	0.50	0.54	0.1091
	13-24	0.45	0.33	0.20	0.11	0.06	-0.01	0.01	0.08	0.16	0.25	0.24	0.22	0.2797
	25-36	0.14	0.08	-0.04	-0.13	-0.16	-0.15	-0.10	-0.05	0.01	0.05	0.06	0.06	0.2841
	37-48	-0.01	-0.11	-0.18	-0.24	-0.26	-0.25	-0.20	-0.17	-0.13	-0.09	-0.10	-0.11	0.2986
∇Z_t	1-12	-0.01	0.10	-0.19	-0.10	-0.03	-0.28	-0.08	-0.13	-0.06	-0.24	0.03	0.32	0.1098
	13-24	0.07	0.07	-0.10	-0.12	0.08	-0.23	-0.14	-0.06	-0.03	0.25	0.04	0.17	0.1530
	27-36	-0.02	0.15	-0.06	-0.17	-0.12	-0.08	-0.03	-0.04	0.01	0.09	0.07	0.20	0.1636
	37-48	0.03	-0.02	-0.06	-0.08	-0.10	-0.09	0.03	0.00	-0.02	0.13	-0.02	0.20	0.1682
$\nabla_{12} Z_t$	1-12	0.42	0.09	-0.02	0.07	0.05	0.08	0.09	0.09	0.16	0.09	-0.11	-0.29	0.1179
	13-24	-0.03	0.04	0.11	0.15	0.17	-0.04	-0.09	-0.01	0.00	0.03	-0.06	-0.12	0.1756
	25-36	0.01	0.10	0.01	-0.17	-0.12	-0.03	0.00	-0.03	-0.02	0.02	0.13	0.19	0.1823
	37-48	0.06	-0.05	-0.01	0.05	0.00	0.10	0.06	-0.05	-0.09	-0.11	-0.12	-0.12	0.1849
[illegible] Z_t	1-12	-0.22	-0.16	-0.21	0.11	-0.04	-0.01	-0.01	-0.07	0.15	0.11	0.04	-0.37	0.1187
	13-24	0.20	-0.05	0.03	-0.02	0.18	-0.09	-0.09	0.01	-0.05	0.13	0.00	-0.13	0.1715
	25-36	-0.04	0.20	0.08	-0.13	-0.10	0.03	0.08	-0.04	0.00	-0.01	0.05	0.14	0.1810
	37-48	0.00	-0.13	0.00	0.05	-0.10	0.10	0.02	-0.02	-0.01	0.01	-0.06	-0.05	0.1845

Table A 5.4

Estimated Autocorrelations of Inland Water Transport (Freight Ton) Series

	Lag k	Autocorrelations												Standard Error for Row
Z_t	1-12	0.49	0.37	0.39	0.28	0.24	0.21	0.16	0.18	0.18	0.23	0.26	0.28	0.1091
	13-24	0.26	0.21	0.20	0.17	0.14	0.11	0.08	0.09	0.14	0.16	0.13	0.14	0.2076
	25-36	0.15	0.06	0.06	0.04	-0.03	-0.04	-0.06	-0.08	-0.04	-0.04	-0.06	-0.05	0.2106
	37-48	-0.05	-0.09	-0.10	-0.14	-0.17	-0.17	-0.18	-0.18	-0.19	-0.16	-0.15	-0.16	0.2255
∇Z_t	1-12	-0.39	-0.13	0.13	-0.08	-0.01	0.03	-0.08	0.00	-0.04	0.03	0.00	0.05	0.1098
	13-24	0.01	-0.03	0.01	0.00	0.00	-0.01	-0.03	-0.05	0.02	0.06	-0.04	0.01	0.1310
	27-36	0.09	-0.07	0.02	0.04	-0.05	0.00	0.01	-0.07	0.05	0.01	-0.02	0.01	0.1334
	37-48	0.05	-0.03	0.02	0.00	-0.03	0.01	-0.02	0.01	-0.03	0.01	0.04	-0.02	0.1341
$\nabla_{12} Z_t$	1-12	0.16	0.06	0.18	0.04	0.04	-0.02	-0.12	-0.08	-0.11	-0.10	-0.05	-0.09	0.1179
	13-24	-0.13	-0.06	-0.07	-0.08	-0.03	-0.07	-0.08	-0.03	-0.04	0.01	-0.02	-0.02	0.1358
	25-36	0.00	-0.05	-0.02	0.02	-0.02	-0.02	-0.01	-0.03	0.08	0.05	0.01	0.07	0.1376
	37-48	0.09	0.06	0.04	0.10	0.07	0.09	0.03	-0.03	-0.05	0.00	0.01	-0.05	0.1421
$\nabla\nabla_{12} Z_t$	1-12	-0.45	-0.12	0.15	-0.08	0.04	0.02	-0.07	0.07	-0.05	-0.03	0.06	-0.01	0.1187
	13-24	-0.06	0.05	-0.01	-0.04	0.05	-0.01	-0.04	0.03	-0.04	0.05	-0.01	-0.02	0.1489
	25-36	0.04	-0.05	-0.01	0.05	-0.03	0.00	0.03	-0.09	0.09	0.00	-0.06	0.03	0.1515
	37-48	0.03	-0.01	-0.04	0.05	-0.03	0.05	-0.01	-0.02	-0.04	0.02	0.04	0.00	0.1527

Table A 5.5

Estimated Partial Autocorrelation of $\nabla_{12} z_t$ Series of Airways (Freight Lb)

Lag k	Partial Autocorrelations $\hat{\phi}_{kk}$												Estimated Standard error
1-12	0.32	-0.11	-0.10	-0.14	0.02	0.04	0.04	0.10	-0.16	-0.15	-0.14	-0.02	0.1179
13-24	0.15	-0.10	0.02	-0.20	0.05	0.10	0.01	-0.12	-0.10	-0.12	-0.04	-0.21	

Table A 5.6

Estimated Partial Autocorrelation of $\nabla\nabla_{12} z_t$ Series of Airways (Number of Passengers)

Lag k	Partial Autocorrelations $\hat{\phi}_{kk}$												Estimated Standard error
1-12	-0.53	-0.15	-0.08	-0.23	-0.31	-0.21	-0.41	-0.06	0.12	-0.07	0.32	-0.19	0.1187
13-24	-0.02	-0.02	0.25	-0.09	-0.06	-0.14	0.03	0.18	0.02	0.04	0.11	-0.07	

Table A 5.7

Estimated Partial Autocorrelation of $\nabla\nabla_{12} z_t$ Series of Railways (Freight Ton)

Lag k	Partial Autocorrelations $\hat{\phi}_{kk}$												Estimated Standard error
1-12	-0.22	-0.22	-0.32	-0.08	-0.17	-0.15	-0.10	-0.22	0.03	0.08	0.23	-0.20	0.1187
13-24	0.17	-0.03	-0.08	0.06	0.12	0.00	-0.13	-0.08	-0.07	0.17	0.08	-0.28	

Table A 5.8

Estimated Partial Autocorrelation of $\nabla\nabla_{12} z_t$ Series of Inland Water Transoprt (FreightTon)

Lag k	Partial Autocorrelations $\hat{\phi}_{kk}$												Estimated Standard error
1-12	-0.45	-0.23	-0.11	-0.03	0.05	0.23	0.01	-0.03	-0.12	-0.23	0.19	-0.20	0.1187
13-24	-0.03	-0.06	0.02	-0.09	0.07	-0.08	0.01	0.06	-0.01	0.00	0.00	-0.23	

Table A 5.9

Sum of Squares Functions for Model Fitted to Airways (Freight Lb) Series

θ_1	$S(\theta_1)$	θ_1	$S(\theta_1)$
-0.99	108949448	-0.10	3007163
-0.95	8839497	-0.05	3086309
-0.90	7349793	0.00	3180857
-0.85	6238407	0.05	3291609
-0.80	5222093	0.10	3419676
-0.75	4440512	0.15	3566511
-0.70	3889766	0.20	3733983
-0.65	3513416	0.25	3924497
-0.60	3258435	0.30	4141181
-0.55	3086528	0.35	4388187
-0.50	2972266	0.40	4671124
-0.45	2899215	0.45	4997718
-0.40	2856876	0.50	5378769
-0.35	2838574	0.55	5829573
-0.34	2837395	0.60	6371998
-0.33	**2836980**	0.65	7037567
-0.32	2837307	0.70	7871960
-0.31	2839355	0.75	8941267
-0.30	2840106	0.80	10339155
-0.25	2858889	0.85	12189517
-0.20	2893417	0.90	14635944
-0.15	2942921	0.95	18179419
		0.99	27090469

Table A 5.10

Sum of Squares Functions for Model Fitted to Airways (Number of Passengers) Series

ϕ_1	$S(\phi_1)$	ϕ_1	$S(\phi_1)$
0.99	5092.16	-0.15	4684.55
-0.95	4899.63	-0.10	4902.52
-0.90	4727.42	-0.05	5145.13
-0.85	4494.07	-0.00	5417.00
-0.80	4331.88	0.05	5713.93
0.85	4494.07	0.10	6039.00
-0.80	4331.88	0.15	6390.95
-0.75	4193.69	0.20	6770.68
-0.70	4088.68	0.25	7176.19
-0.65	4004.95	0.30	7609.88
-0.60	3953.72	0.35	8069.65
-0.55	3926.83	0.40	8559.32
-0.54	3924.70	0.45	9071.33
-0.53	**3923.65**	0.50	9613.00
-0.52	3923.68	0.55	10181.23
-0.51	3924.80	0.60	10776.92
-0.50	3927.00	0.65	11399.35
-0.45	3952.13	0.70	12049.08
-0.40	4008.52	0.75	12725.69
-0.35	4088.05	0.80	13429.48
-0.30	4198.28	0.85	14160.25
-0.25	4332.19	0.90	14918.12
-0.20	4496.28	0.99	16350.44

Table A 5.11

Sum of Squares Functions for Model Fitted to Railways (Freight Ton) Series

Θ_1	$S(\Theta_1)$	Θ_1	$S(\Theta_1)$
0.99	387536.68	0.15	42835.48
-0.95	293663.65	0.20	41376.39
-0.90	218575.40	0.25	40018.69
-0.85	171881.76	0.30	38757.05
-0.80	141839.03	0.35	37587.36
-0.75	121708.54	0.40	36506.89
-0.70	107595.10	0.45	35514.53
-0.65	97222.36	0.50	34611.47
-0.60	89241.14	0.55	33802.60
-0.55	82838.57	0.60	33099.25
-0.50	77515.94	0.65	32524.23
-0.45	72961.21	0.70	32121.54
-0.40	68974.92	0.71	32068.06
-0.35	65426.50	0.72	32025.55
-0.30	62228.10	0.73	31995.09
-0.25	59318.70	0.74	31977.94
-0.20	56654.49	**0.75**	**31975.50**
-0.15	54202.80	0.76	31989.39
-0.10	51938.42	0.77	32021.48
-0.05	49841.28	0.80	32498.96
-0.00	47895.00	0.85	35713.82
0.05	46085.94	0.90	40985.92
0.10	44402.64	0.99	49161.54

Table A 5.12

Sum of Squares Functions for Model Fitted to Inland Water Transport(Freight Ton) Series

θ_1	$S(\theta_1)$	θ_1	$S(\theta_1)$
-0.99	2002770.51	0.15	19558.21
-0.95	315765.32	0.20	19181.24
-0.90	172013.68	0.25	18862.57
-0.85	114832.45	0.30	18603.66
-0.80	83495.00	0.35	18416.13
-0.75	64628.89	0.40	18306.06
-0.70	52742.14	0.41	18294.05
-0.65	45001.91	0.42	18285.92
-0.60	37586.84	**0.43**	**18281.92**
-0.55	35674.29	0.44	18287.04
-0.50	32766.57	0.45	18286.50
-0.45	30504.58	0.50	18395.44
-0.40	28692.46	0.55	18638.98
-0.35	27196.83	0.60	19068.18
-0.30	25933.43	0.65	19746.22
-0.25	24844.45	0.70	20729.97
-0.20	23892.52	0.75	22131.25
-0.15	23049.57	0.80	24094.27
-0.10	22297.62	0.85	26859.98
-0.05	21623.55	0.90	30829.94
-0.00	21018.00	0.95	36232.47
0.05	20474.37	0.99	51938.61
0.10	19988.58		

Table A5.13

Estimated Residual Values $\hat{a}_t = 0.7917 + w_t - 0.33\, a_{t-1}$ for Airways (Freight Lb) Series

33.85	-35.07	-112.04
-41.17	438.57	-413.03
270.59	451.27	-341.70
-255.29	-168.92	169.76
209.25	111.74	144.98
210.95	-195.88	-81.84
-34.61	101.64	296.01
73.42	42.46	-49.68
173.77	-64.01	10.40
-1.34	105.12	-27.43
41.44	6.31	20.05
324.32	-203.08	-95.62
-55.03	318.02	-109.45
123.16	-63.95	-7.88
-379.64	209.10	151.60
165.72	480.00	-281.03
-63.31	-49.40	-167.26
190.89	-162.70	41.20
-61.99	68.69	-281.59
-9.54	-53.67	55.93
217.85	123.71	-31.46
28.89	-85.82	111.38
-683.53	-1.68	-38.76
-163.43	42.55	62.79

Table A 5.14

Estimated Residual Values $\hat{a}_t = 0.1509 + w_t + 0.53\, w_{t-1}$ for Airways (Number of Passengers) Series

1.53	0.47	-4.41
-1.47	26.53	2.82
-4.06	-12.22	-2.18
-6.59	-2.78	6.47
0.35	1.83	-4.29
3.59	-2.12	-0.24
-0.94	-4.00	-8.88
2.94	-5.12	-11.83
4.12	-6.59	9.82
-2.94	23.35	3.89
-4.12	1.78	4.41
0.94	-2.36	5.18
4.06	-1.88	-1.94
-23.41	0.88	-7.59
12.75	-6.41	-8.18
8.75	-7.24	0.35
-3.65	1.41	7.59
-1.53	7.59	1.18
-0.53	12.18	1.94
-2.00	2.77	1.59
0.94	6.94	-6.00
-15.94	-8.76	-6.18
-0.01	-3.89	0.41
3.77		

Table A 5.15

Estimated Residual Values $\hat{a}_t = 1.4225 + w_t + 0.75a_{t-12}$ for Railways (Freight Ton) Series

6.00	48.31	-9.25
-10.25	-5.81	16.53
-3.25	5.56	-1.31
25.25	-71.06	-7.66
16.00	11.94	-16.09
-15.50	20.23	39.49
-13.25	31.67	4.68
-2.25	-23.30	-11.77
5.75	31.67	52.88
0.25	-23.30	-12.22
28.75	-25.75	-59.29
-30.75	9.78	-40.37
-14.75	18.55	-5.93
-7.69	1.98	9.40
-16.44	-9.77	6.02
8.94	5.64	3.25
-5.00	-6.83	-0.07
14.38	11.70	11.62
14.06	-27.05	4.51
-34.69	-1.82	10.17
-30.34	-0.17	

Table A 5.16

Estimated Residual Values $\hat{a}_t = 0.6479 + w_t + 0.43\, a_{t-1}$ for Inland Water Transport (Freight Ton) Series.

15.00	6.33	8.1
15.45	11.72	-21.50
-31.35	-3.96	8.75
-9.48	-17.70	7.76
-2.08	7.39	-19.66
-12.89	23.18	-16.45
-41.54	-24.03	20.92
-10.86	5.67	7.00
-9.67	4.44	4.01
-13.16	7.91	-1.28
-17.66	21.40	-21.55
12.41	2.20	20.73
3.33	-18.05	0.92
-13.57	18.24	4.39
20.17	0.84	-10.11
24.67	0.36	-7.35
2.61	5.16	-15.16
-17.88	-24.78	44.48
4.31	-15.66	-3.87
19.85	-20.73	-7.67
31.54	2.09	-16.30
-0.44	9.90	-25.01
5.81	-14.74	13.25
-15.50	-4.34	

Table A 5.17
Estimated Autocorrelations of $\hat{a}_t$ for Airways (Freight Lb) Series

Lag k	Autocorrelations $r_k(\hat{a}_t)$												Estimated standard error
1-12	0.02	0.03	-0.08	-0.16	-0.05	0.03	0.03	0.17	-0.06	-0.14	-0.17	-0.16	0.1179
13-24	0.17	-0.01	0.13	-0.12	-0.09	0.01	0.02	-0.05	-0.08	-0.10	0.01	-0.23	
25-36	0.11	0.21	0.02	0.16	0.07	-0.09	0.10	-0.07	-0.05	0.10	0.06	0.00	
37-48	0.05	-0.12	-0.05	-0.08	0.05	-0.01	0.06	0.06	-0.02	0.05	-0.02	-0.03	

Table A 5.18
Estimated Autocorrelations of $\hat{a}_t$ for Airways (Number of Passengers) Series

Lag k	Autocorrelations $r_k(\hat{a}_t)$												Estimated standard error
1-12	-0.03	-0.07	-0.03	-0.26	-0.07	0.07	0.13	0.00	-0.10	-0.08	-0.08	-0.18	0.1187
13-24	0.06	0.12	0.04	0.01	0.05	-0.04	-0.03	0.11	0.01	-0.06	-0.01	-0.06	
25-36	-0.02	-0.04	0.07	0.00	-0.11	0.07	-0.10	0.06	0.05	0.03	0.04	-0.03	
37-48	-0.06	0.05	-0.07	0.05	0.12	-0.08	0.00	0.02	-0.07	-0.05	0.01	0.05	

Table A 5.19

Estimated Autocorrelations of $\hat{a}_t$ for Airways (Freight Lb) Series

Lag k	Autocorrelations r_k ($\hat{a}_t$)												Estima standard
1-12	0.09	-0.18	-0.25	0.12	0.01	0.01	-0.10	-0.02	0.07	0.05	0.01	-0.04	
13-24	0.09	0.02	-0.07	-0.04	0.16	-0.04	-0.14	-0.06	0.11	0.13	-0.15	-0.14	
25-36	0.09	0.26	-0.04	-0.13	0.00	0.02	0.06	-0.16	-0.03	-0.04	0.09	0.11	0.129
37-48	0.07	-0.11	-0.01	-0.01	-0.05	0.04	0.08	-0.02	-0.08	-0.02	0.06	-0.02	

Table A 5.20

Estimated Autocorrelations of $\hat{a}_t$ for Inland Water Transport (Freight Ton) Series

Lag k	Autocorrelations r_k ($\hat{a}_t$)												Estimated standard err
1-12	0.07	-0.14	0.05	0.11	0.12	0.11	-0.15	-0.14	-0.07	-0.05	0.16	-0.23	
13-24	-0.15	0.02	0.03	-0.07	0.06	-0.13	0.04	0.12	-0.02	-0.01	0.11	-0.15	0.1187
25-36	0.09	0.04	-0.17	0.03	-0.18	0.01	-0.01	-0.05	0.04	0.05	-0.04	0.18	
37-48	0.08	-0.13	0.11	0.10	0.07	-0.09	-0.01	-0.02	0.10	0.06	0.03	-0.14	

APPENDIX B

Figure B 1.1
Graph for Airways (Freight Lb) Series

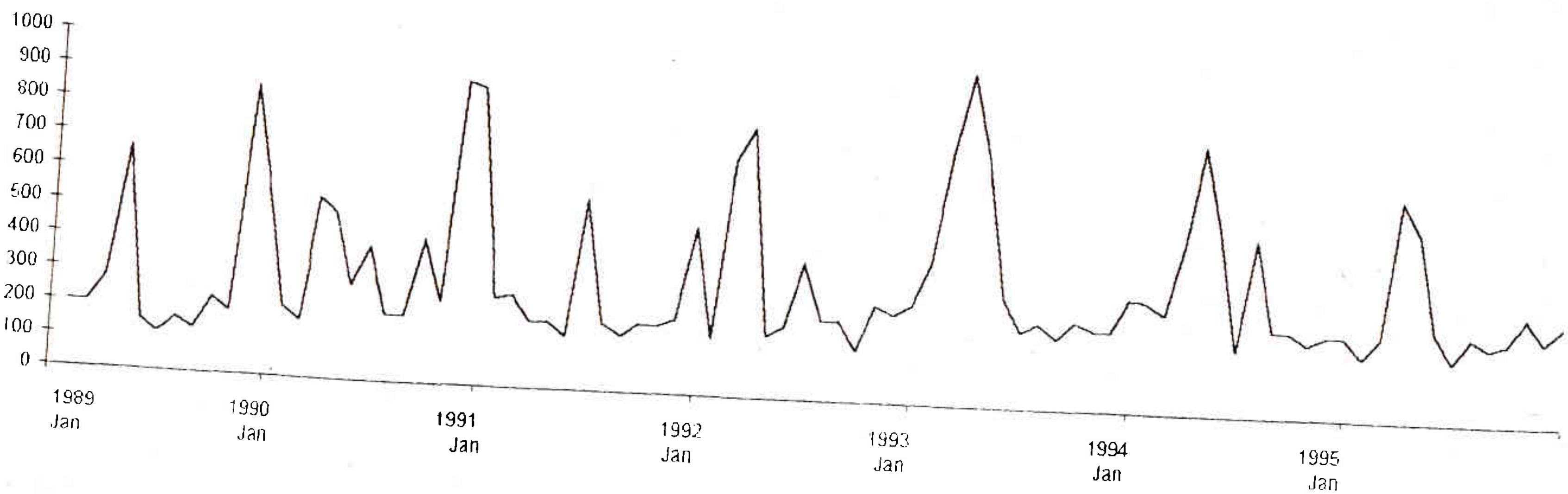


Figure B 1.2

Graph for Airways (Number of Passengers) Series

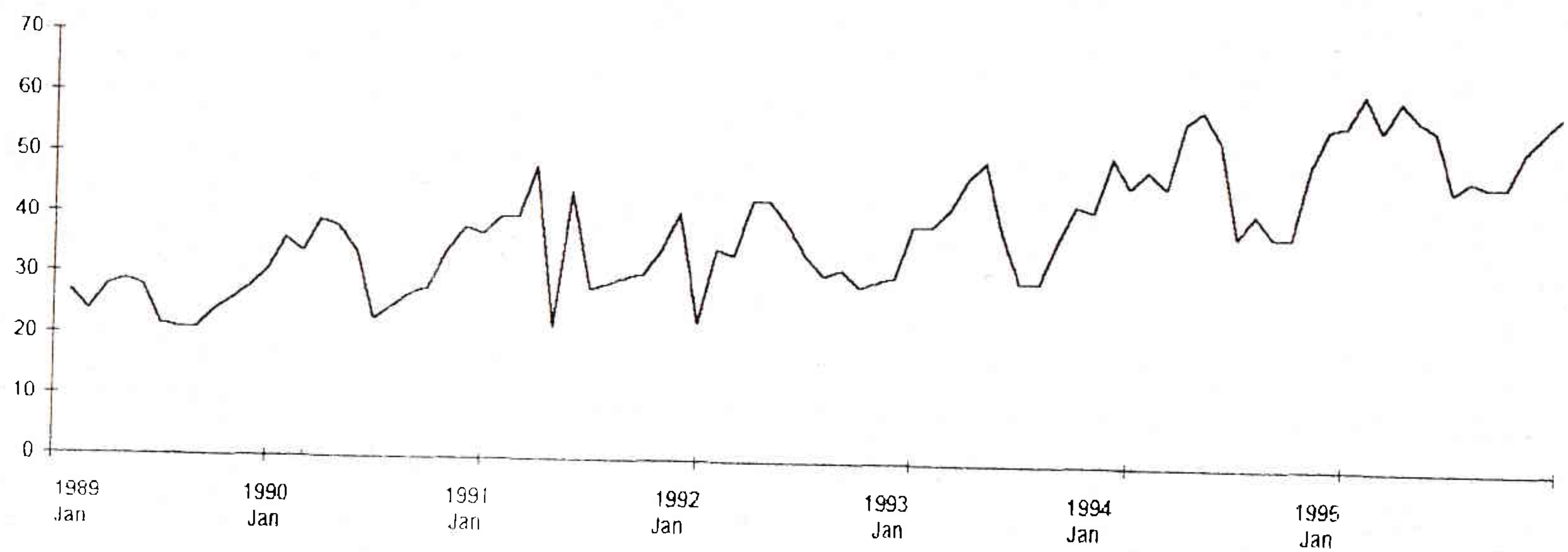


Figure B 1.3
Graph for Railways (Freight Ton) Series

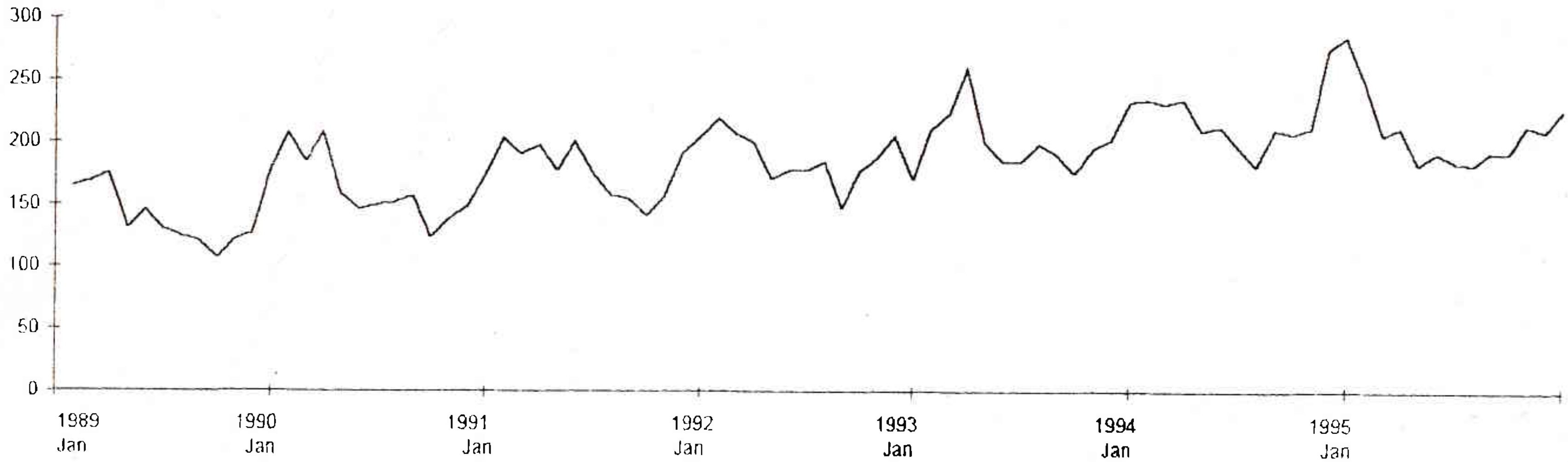


Figure B 1.4
Graph for Inland Water Transport (Freight Ton) Series

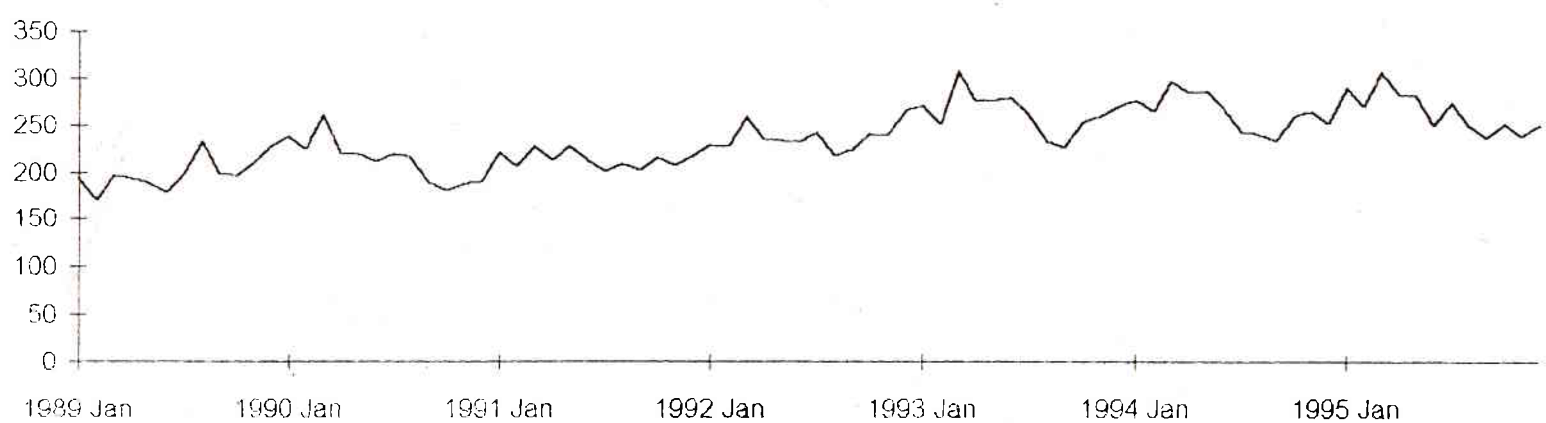


Figure B 1.5
Tier Chart for Airways (Freight Lb) Series

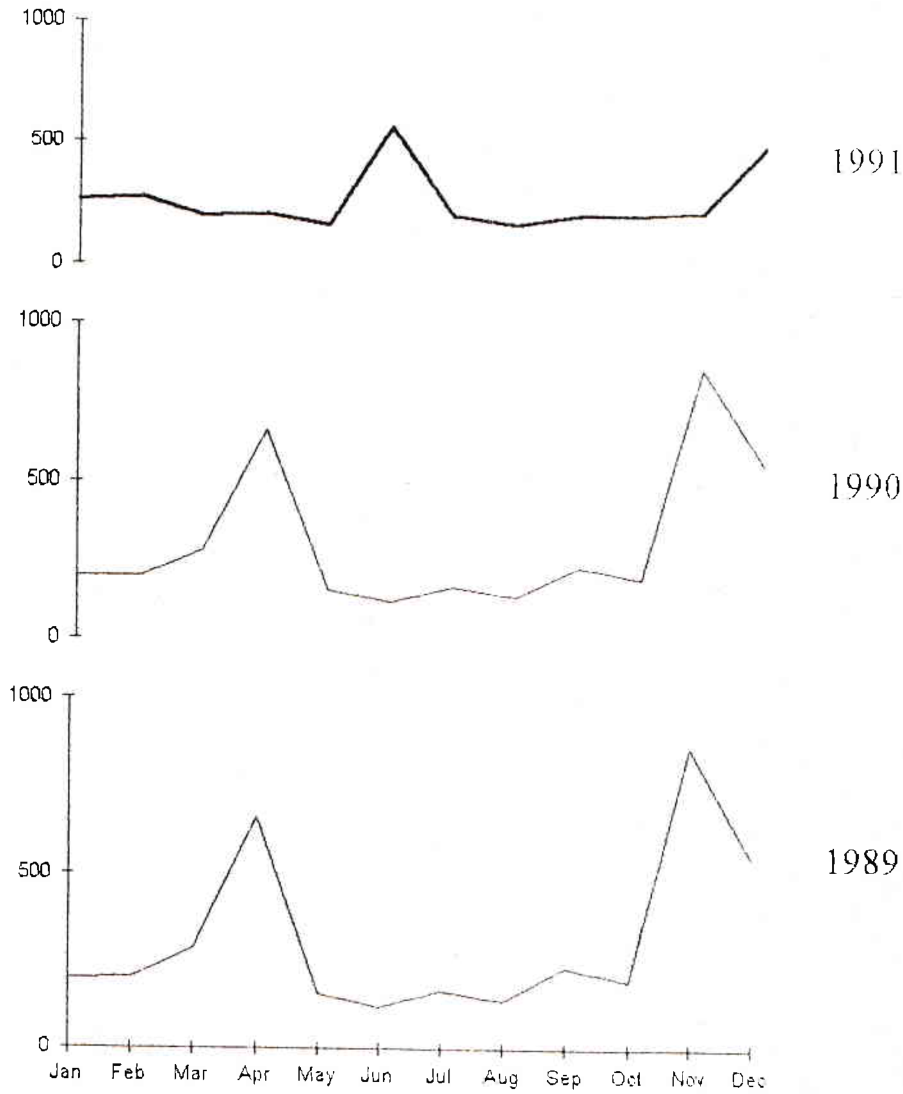


Figure B 1.5 (Con'd)

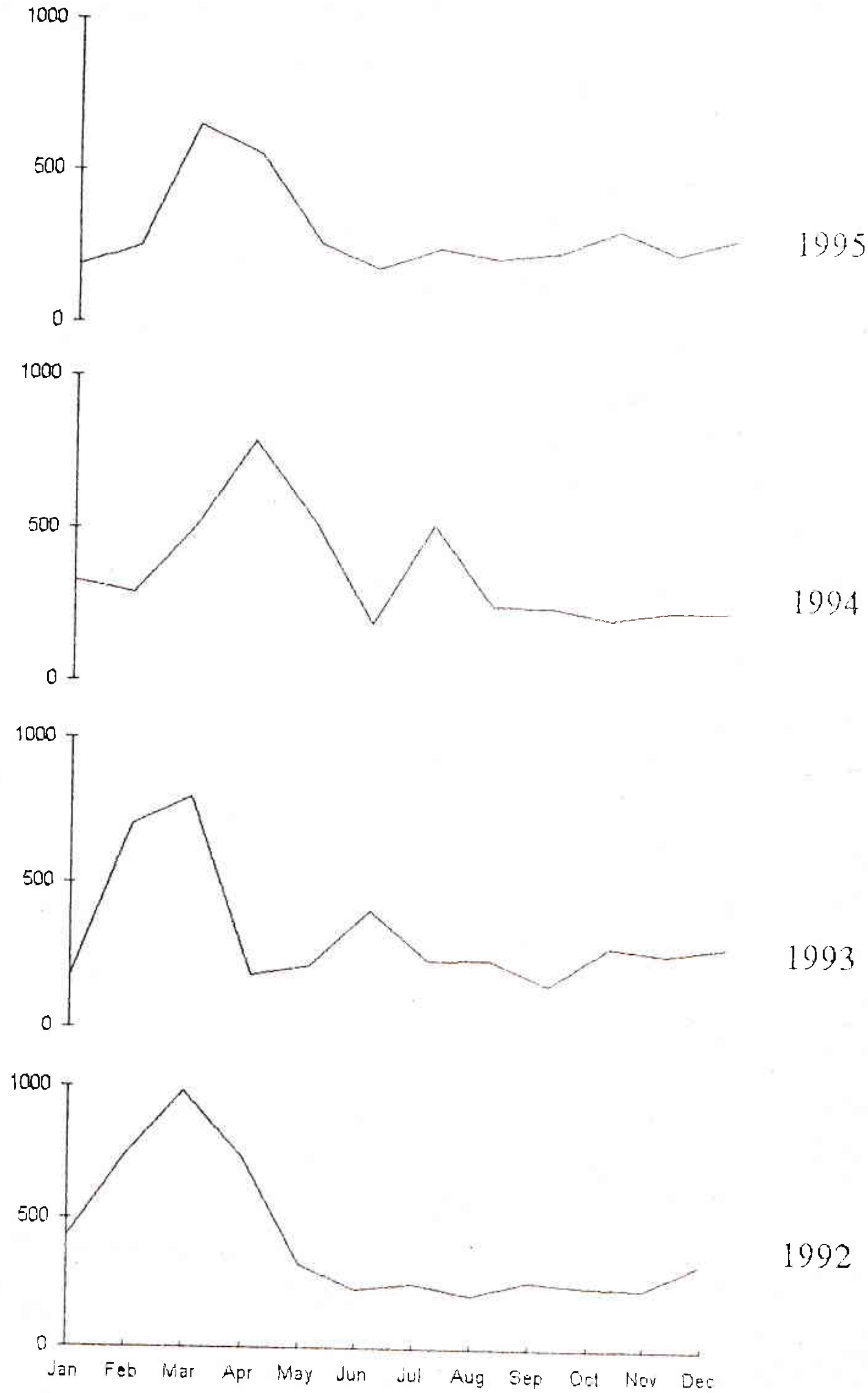


Figure B 1.6
Tier Chart for Airways (Number of Passengers) Series

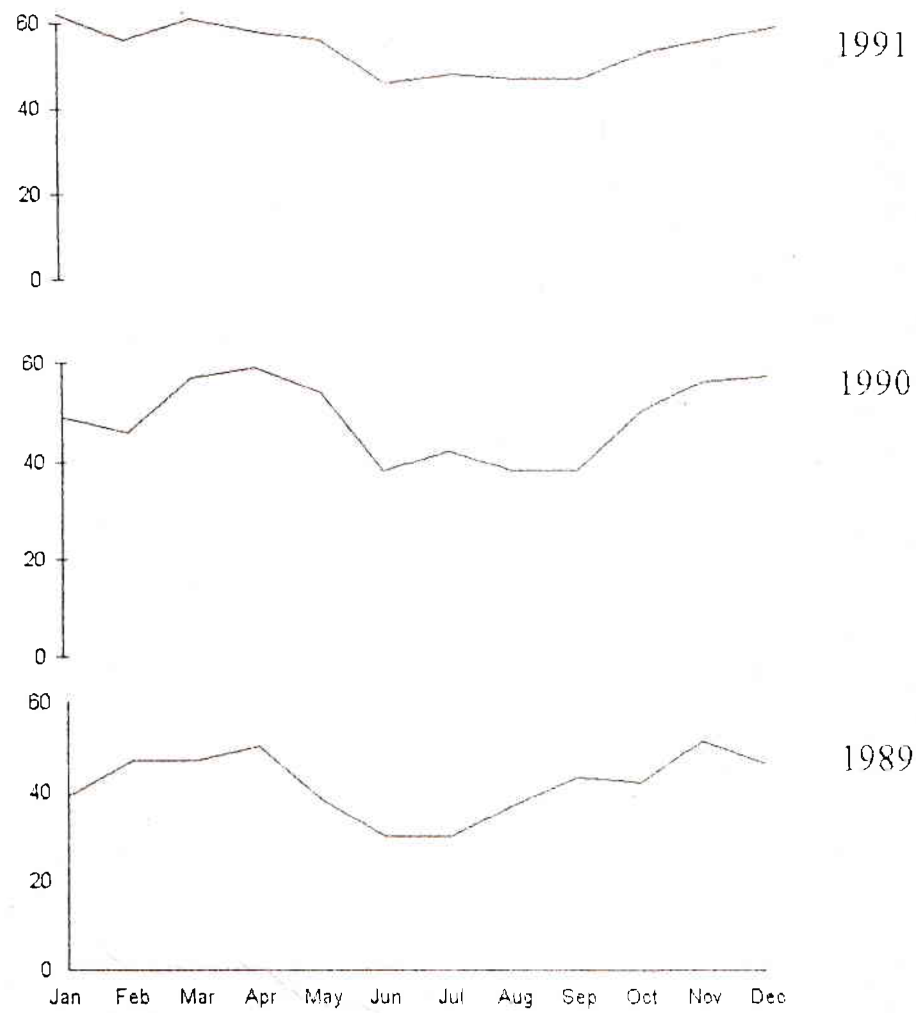


Figure B 1.6 (Con'd)

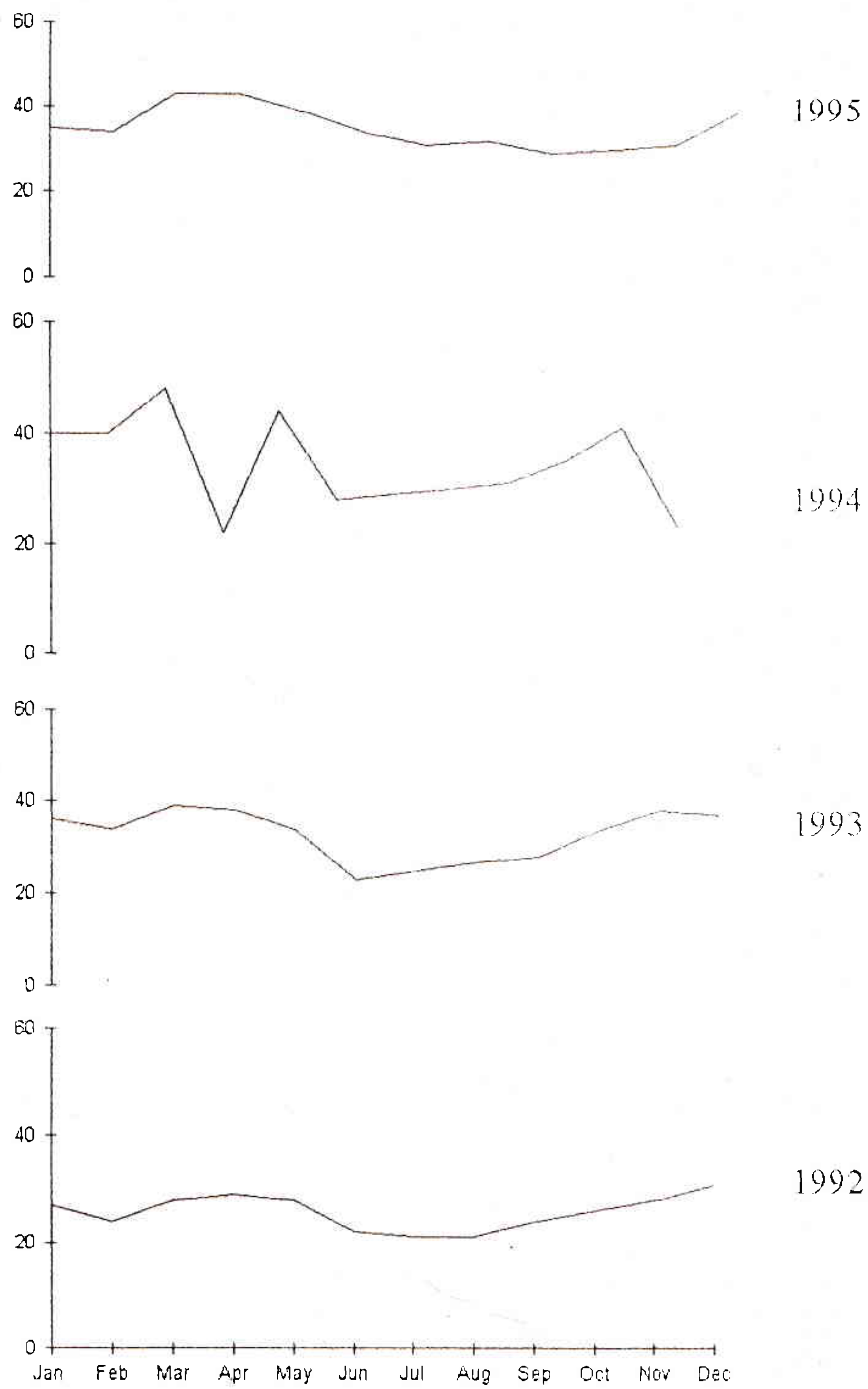


Figure B 1.7
Tier Chart for Railways (Freight Ton) Series

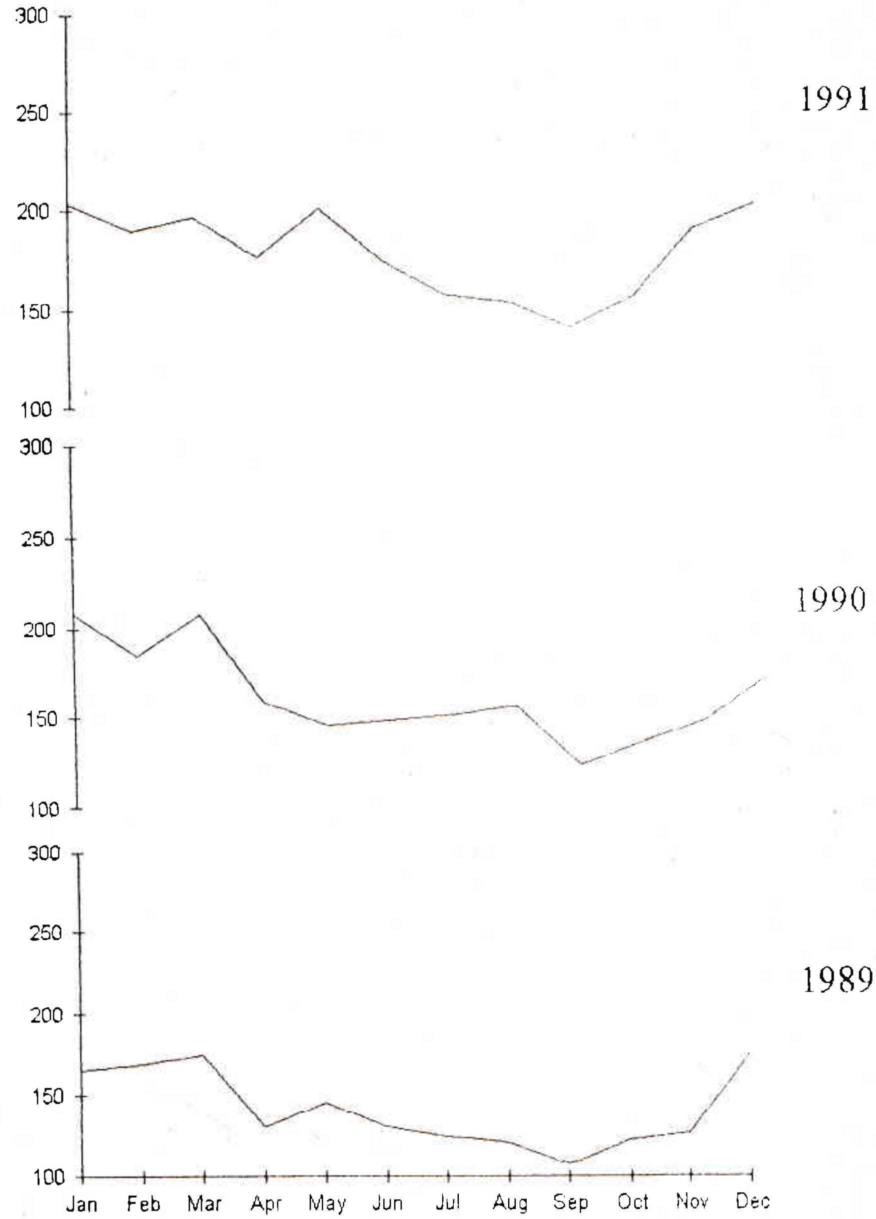


Figure B 1.7(Con'd)

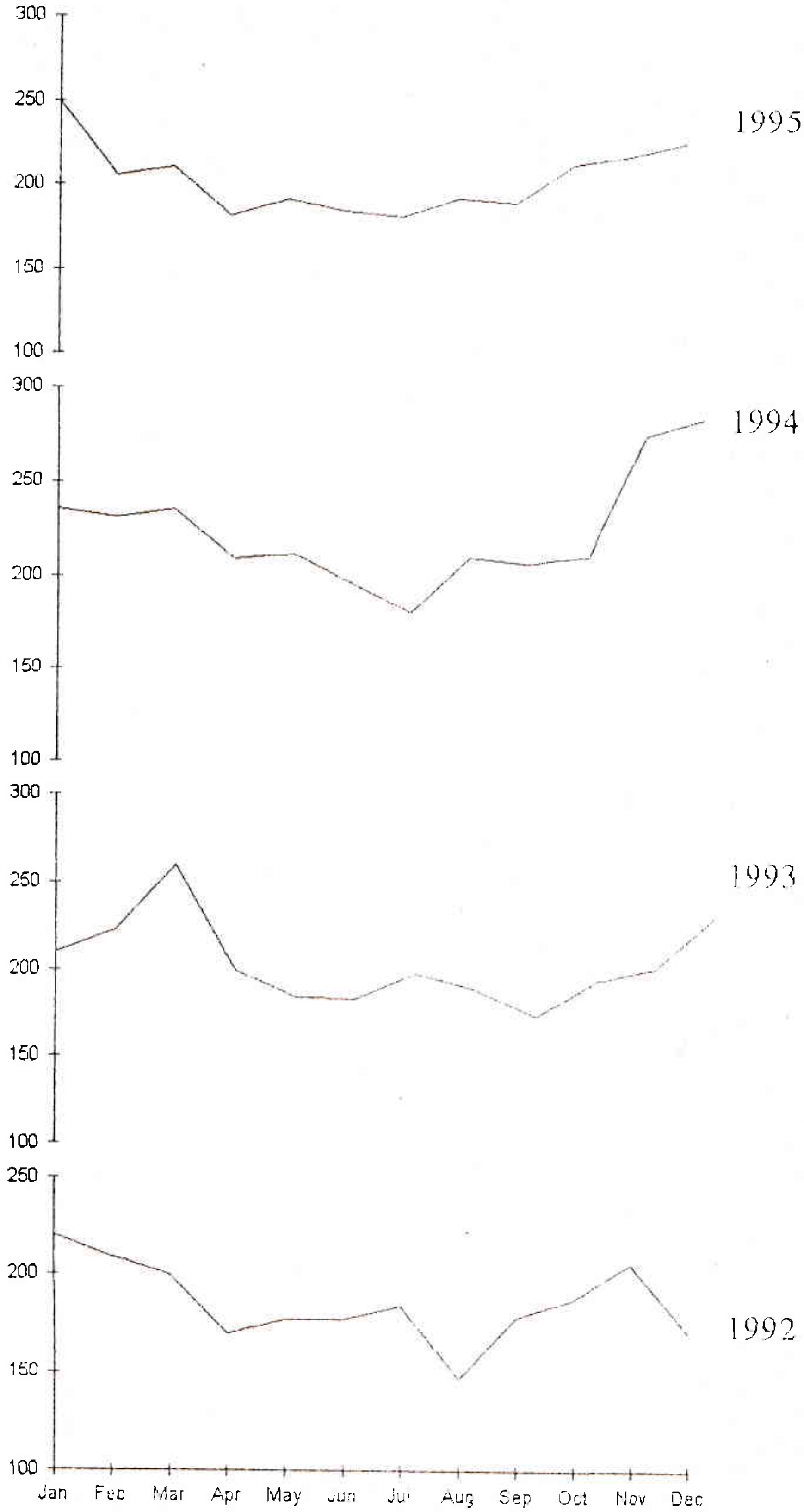


Figure B 1.8
Tier Chart for Inland Water Transport (Freight Ton) Series

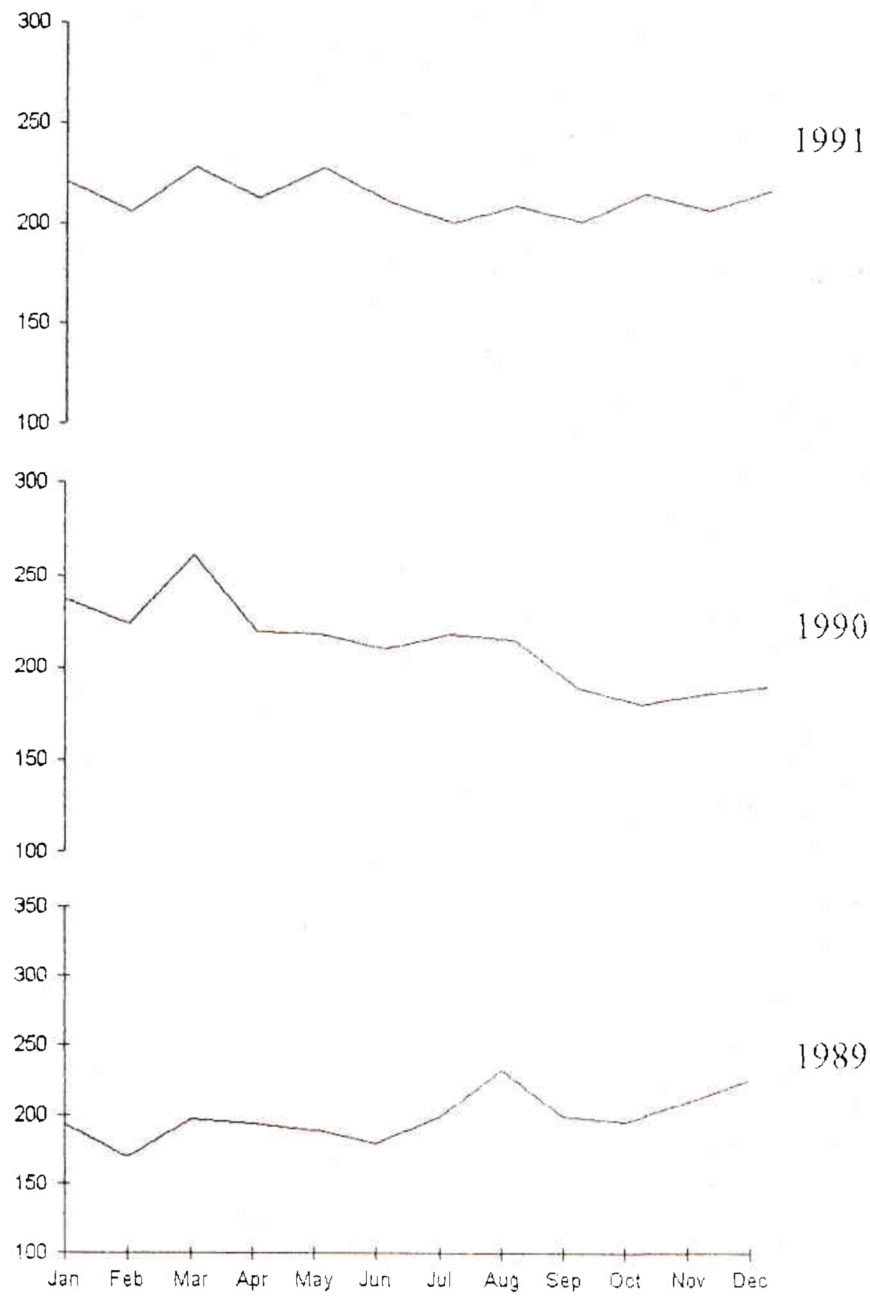


Figure B 1.8 (Con'd)

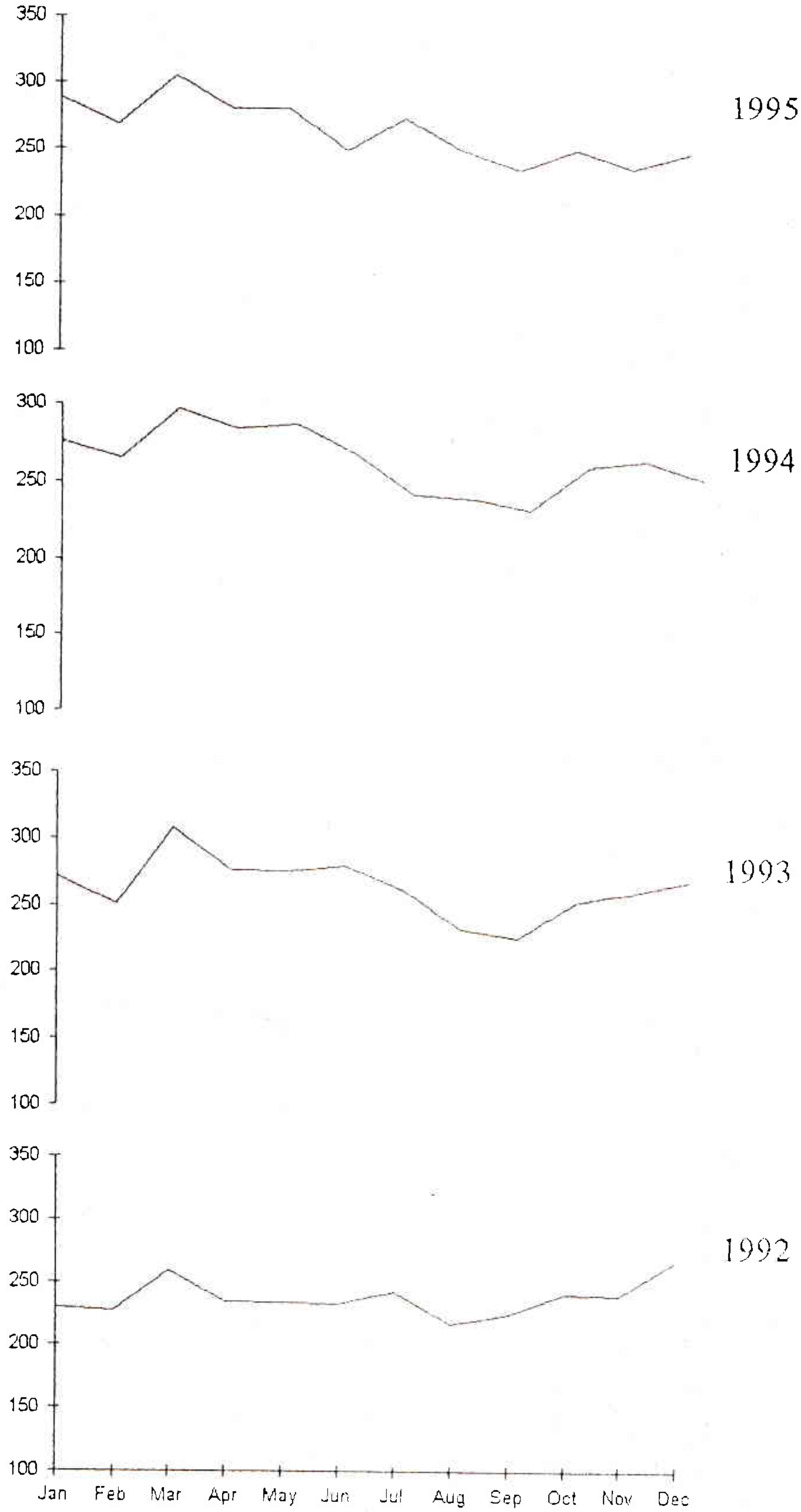


Figure B 5.1
Sample Correlogram for Orginal Series, z_t of Airlways (Freight Lb)

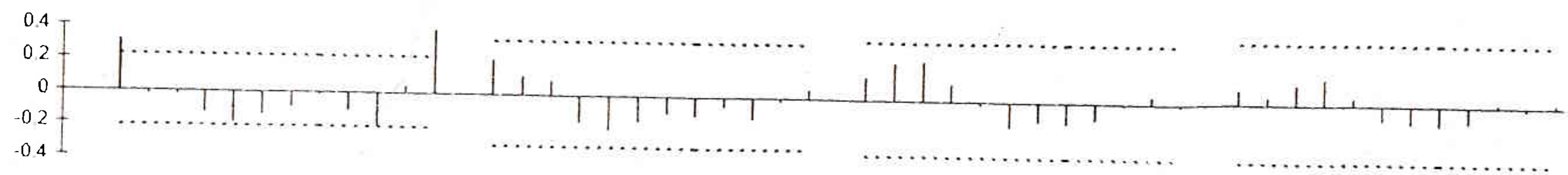


Figure B 5.2
Sample Correlogram for ∇z_t Series of Airways (Freight Lb)

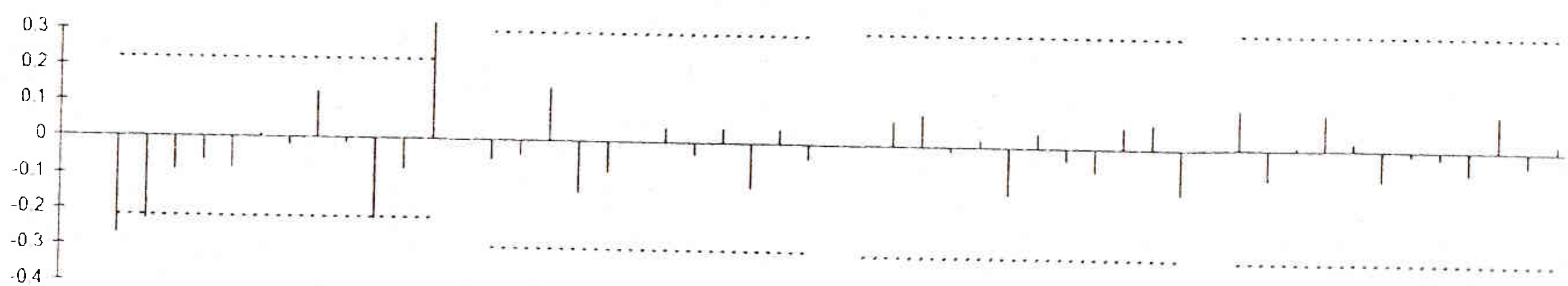


Figure B 5.3
Sample Correlogram for $\nabla_{12}z_t$ Series of Airways (Freight Lb)

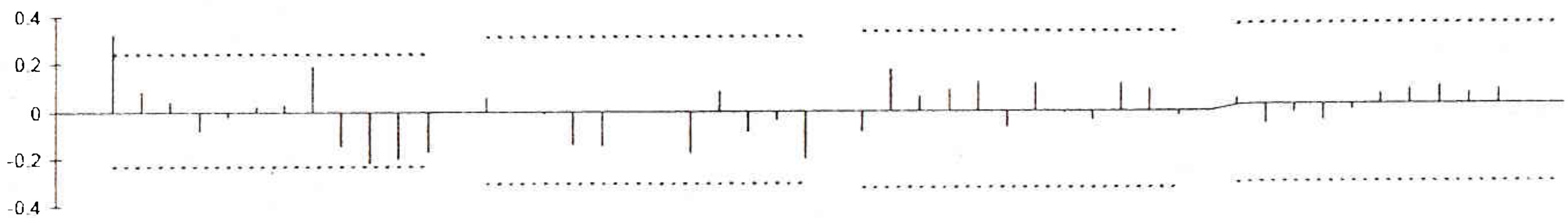


Figure B 5.4
Sample Correlogram for $\nabla \nabla_{12}z_t$ Series of Airways (Freight Lb)

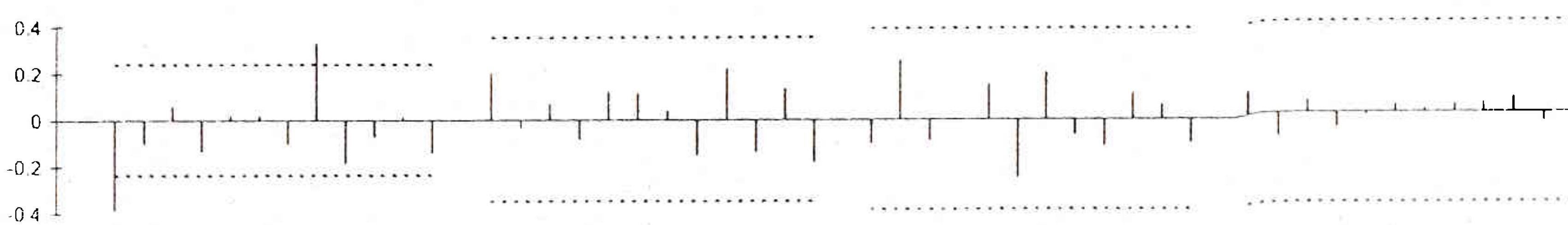


Figure B 5.5

Sample Correlogram for Original Series, z_t of Airways (Nuumber of Passengers)

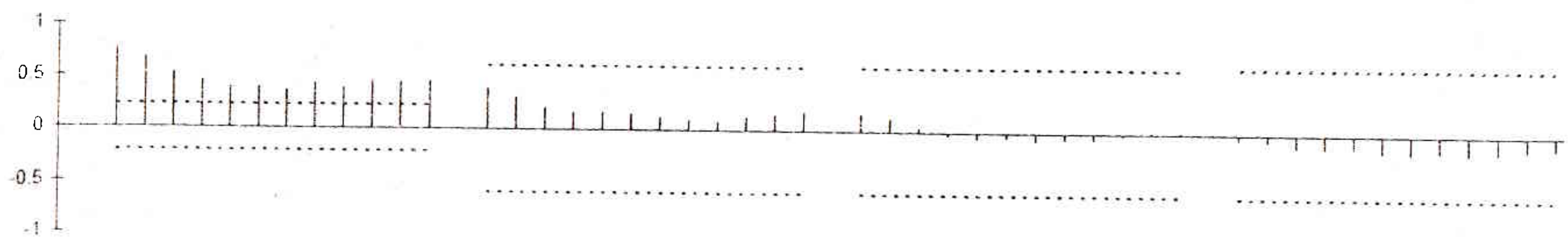


Figure B 5.6

Sample Correlogram for ∇z_t Series of Airways (Number of Passengers)

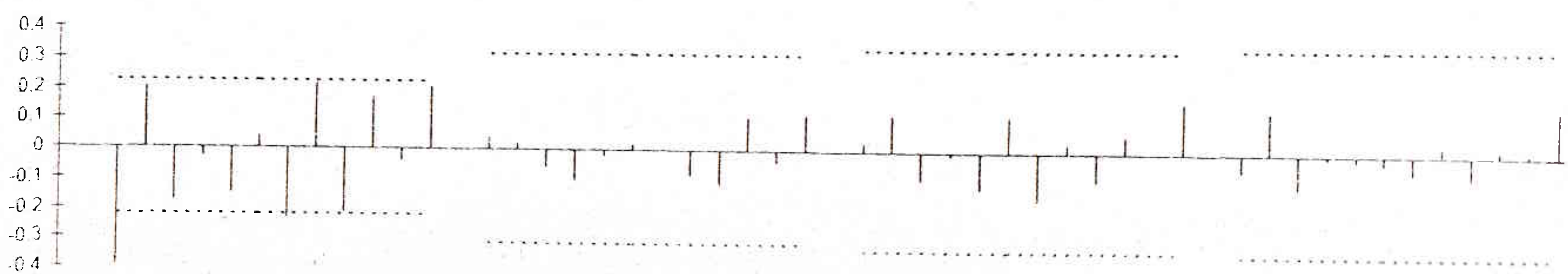


Figure B 5.7

Sample Correlogram for $\nabla_{12}z_t$ Series of Airways (Number of Passengers)

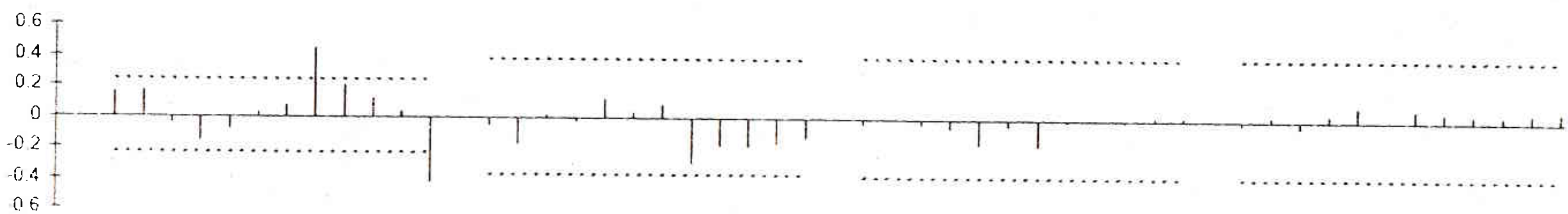


Figure B 5.8

Sample Correlogram for $\nabla \nabla_{12}z_t$ Series of Airways (Number of Passengers)

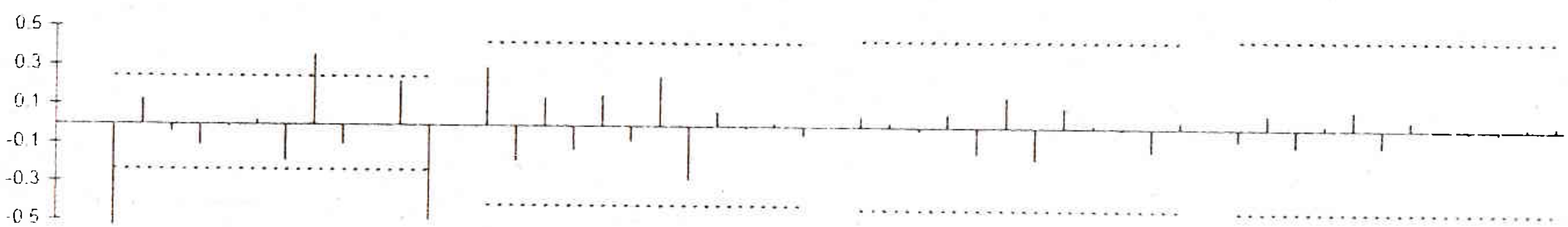


Figure B 5.9
Sample Correlogram for Original Series , z_t of Railways (Freight Ton)

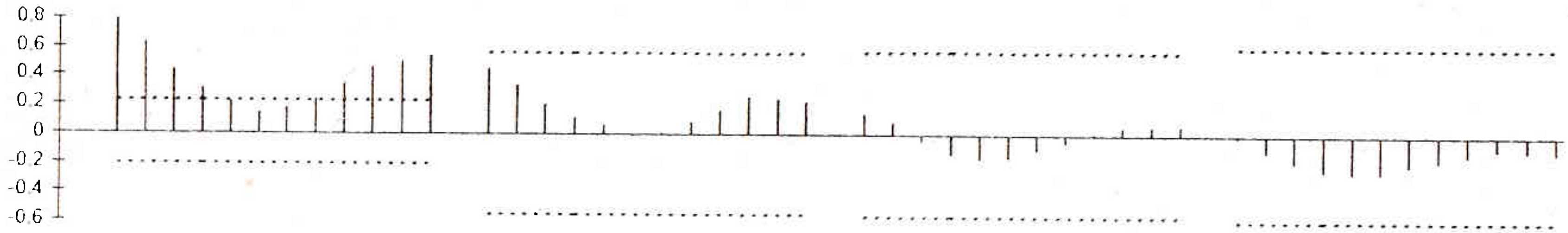


Figure B 5.10
Sample Correlogram for ∇z_t Series of Railways (Freight Ton)

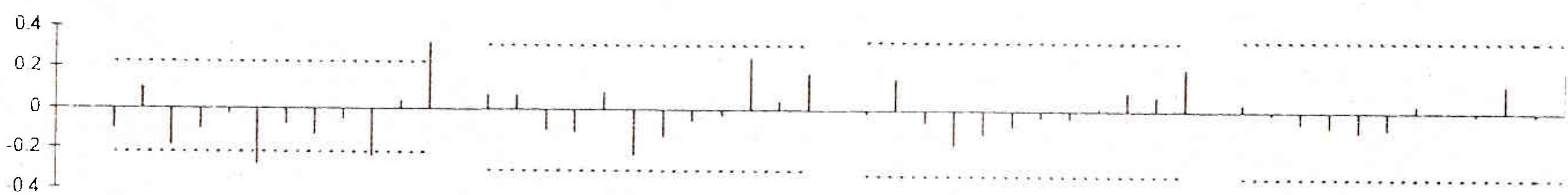


Figure B 5.11

Sample Correlogram for $\nabla_{12} z_t$ Series of Railways (Freight Ton)

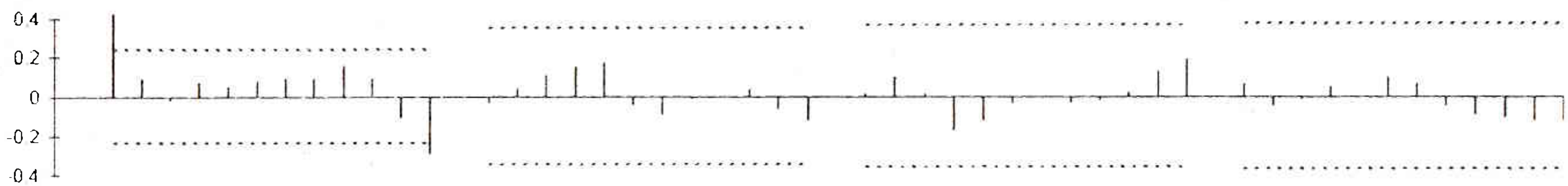


Figure B 5.12

Sample Correlogram for $\nabla \nabla_{12} z_t$ Series of Railways (Freight Ton)

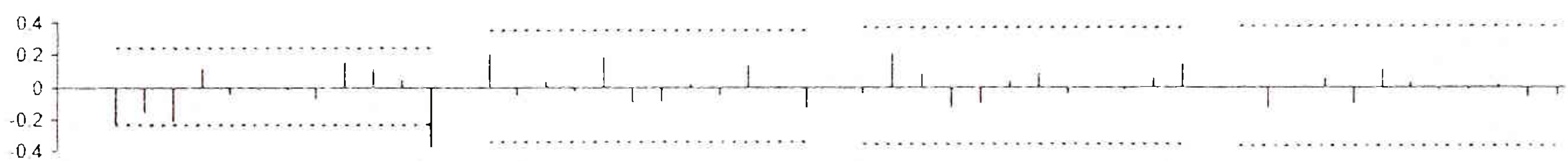


Figure B 5.13
Sample Correlogram for Original Series, z_t of Inland Water Transport (Freight Ton)

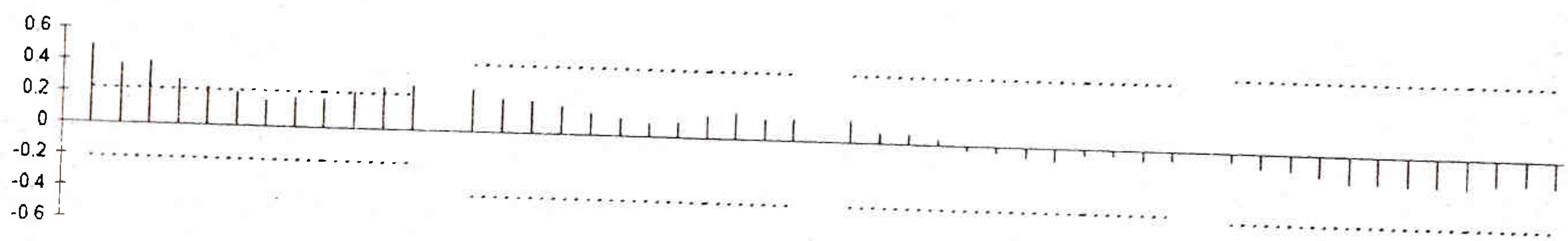


Figure B 5.14
Sample Correlogram for ∇z_t Series of Inland Water Transport (Freight Ton)

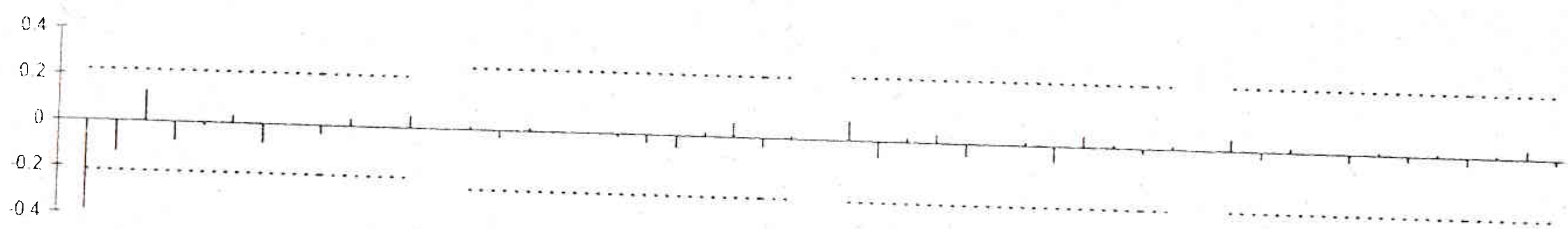


Figure B 1.15

Sample Correlogram for $\nabla_{12}z_t$ Series of Inland Water Transport (Freight Ton)

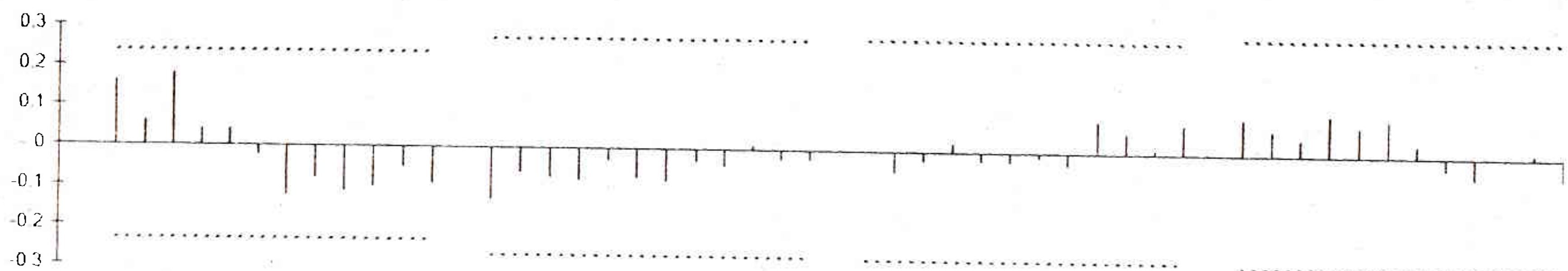


Figure B 1.16

Sample Correlogram for $\nabla\nabla_{12}z_t$ Series of Inland Water Transport (Freight Ton)

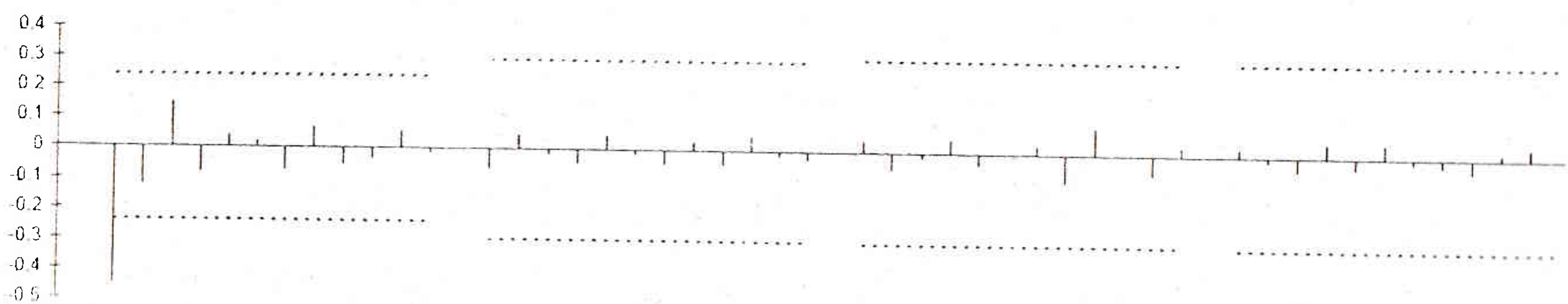


Figure B 5.17

Estimated Partial Autocorrelations for $\nabla_{12} z_t$ Series of Airways (Freight Lb)

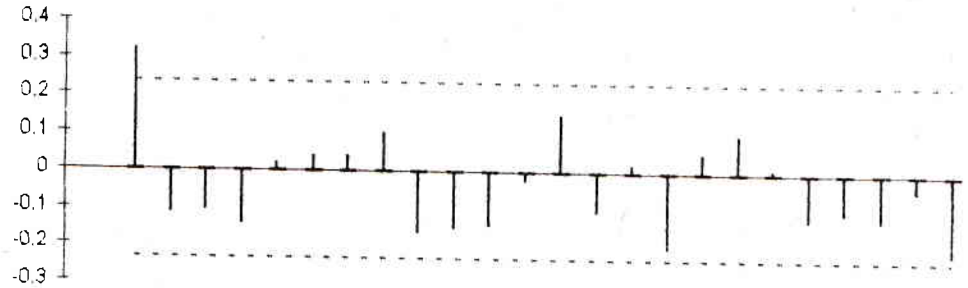


Figure B 5.18

Estimated Partial Autocorrelation for $\nabla\nabla_{12} z_t$ Series of Airways (Number of Passengers)

Figure B 5.19

Estimated Partial Autocorrelation for $\nabla\nabla_{12} z_t$ Series of Railways (Freight Ton)

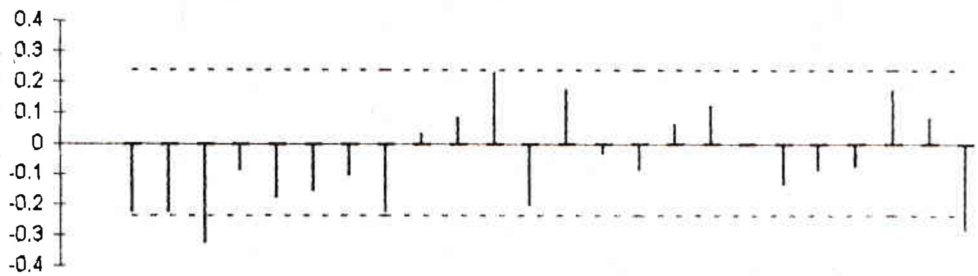


Figure B 5.20

Estimated Partial Autocorrelation for $\nabla\nabla_{12} z_t$ Series of Inland Water Transport (Freight Ton)

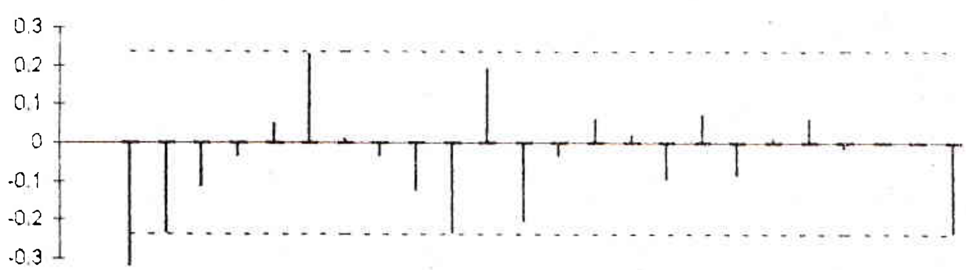


Figure B 5.21
Sample Correlogram of $\hat{a}_t$ for Airways (Freight Lb) Series

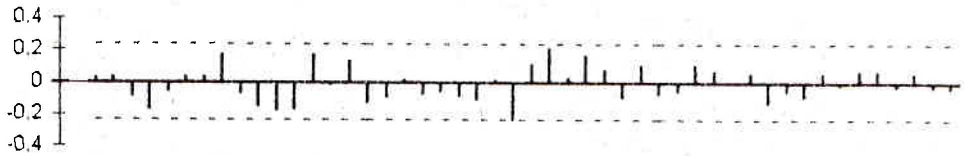


Figure B 5.22
Sample Correlogram of $\hat{a}_t$ for Airways (Number of Passengers) Series

Figure B 5.23
Sample Correlogram of $\hat{a}_t$ for Railways (Freight Ton) Series

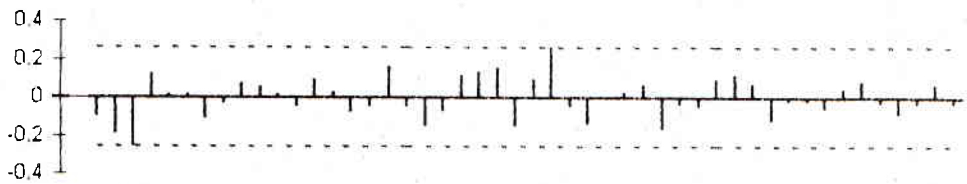
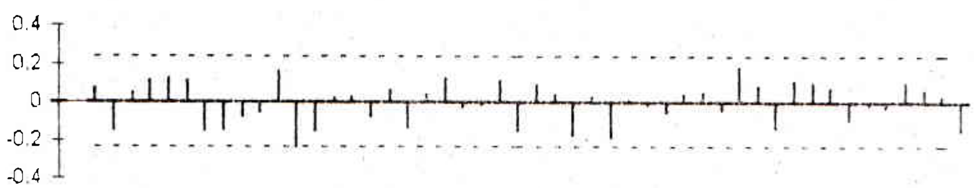


Figure B 5 . 24
Sample Correlogram of $\hat{a}_t$ for Inland Water Transport (Freight Ton) Series



Abbreviations

AR	=	Autoregressive
MA	=	Moving average
ARMA	=	Autoregressive moving average
ARIMA	=	Autoregressive integrated moving average
SAR	=	Seasonal autoregressive
SMA	=	Seasonal moving average
SARMA	=	Seasonal autoregressive moving average
Acf	=	Autocorrelation function
Pacf	=	Partial autocorrelation function
AIC	=	Akaike's Information Criterion
FPE	=	Final Prediction Error
HQ	=	Hannan and Quinn
BIC	=	Baysian Information Criterion