ADJUSTMENT DIAGNOSTIC FOR OUTLIERS IN TIME SERIES

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by

Soe Win

Department of Mathematics and Statistics
School of Mathematics and Computer/Information Sciences
University of Hyderabad
Hyderabad - 500 046

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Dedicated

to *My Mother*

and

Late Father.

Certificate

Department of Mathematics and Statistics
School of MCIS
University of Hyderabad
Hyderabad- 500 046.

Dated: 10 March 2004

This is to certify that I, Mr. Soe Win, have carried out the research embodied in the present thesis entitled **ADJUSTMENT DIAGNOSTIC FOR OUTLIERS IN TIME SERIES** for the full period prescribed under Ph. D. ordinance of the university.

I declare, to the best of my knowledge, that no part of this thesis was earlier submitted for the award of research degree of any university.

Candidate
(Soe Win)
Enrollment No. 98MMPP05

Head of the Department
(Prof. T. Amaranath)

HEAD
Department of Mathematics & Statistics
School of MCIS
University of Hyderabad
Hyderabad-500 046

Dean of the School
(Prof. R. Tandon)

Dean
School of Maths & Cis.
University of Hyd.
Hyderabad-500 134.

Supervisor
(Dr. S. M. Bendre)

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Chapter 1

Introduction

1.1 Introduction

A time series consists of a series of observations on a variable of interest collected sequentially in time. The analysis of time series is a necessary technique in many areas such as industrial research, economics, marketing, physical and chemical sciences, etc. One of the important aspects of such a series is the dependent structure of adjacent observations. For the satisfactory analysis of the series, it is necessary to construct an appropriate stochastic model which can further be used in various ways, depending on the field of applications.

The parametric approach to modeling the time series in terms of linear difference equations has led to an important class of models, namely autoregressive integrated moving average model with order p, d and q, popularly known as ARIMA(p,d,q) (Box and Jenkins, 1976). In particular, if $\{Z_t, t \in \kappa\}$, κ the set of integers, is a time series, then the ARIMA(p,d,q) model is

$$\phi(B)\,(1-B)^d Z_t = \theta(B)\, a_t, \qquad t \in \kappa \qquad (1.1)$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$ are polynomials of degree p and q in B, ϕ_i , $i = 1, 2, \ldots, p$ and θ_j, $j = 1,2, \ldots, q$ are the autoregressive and moving average parameters of the time series respectively and B is the backward shift operator, i.e., $B^j Z_t = Z_{t-j}$. In the model above, $\{a_t, t \in \kappa\}$ is the white noise, called the error series, and we assume throughout that a_t's are i.i.d normal with mean zero and variance σ_a^2, referred to as the error variance (or innovation variance).

We assume the series $(1-B)^d Z_t$ to be stationary, i.e., the roots of $\phi(B) = 0$ lie outside the unit circle, and invertible, i.e., the roots of $\theta(B) = 0$ lie outside the unit circle. For $d = 0$, (1.1) represents a stationary process ARMA(p,q), given by

$$\phi(B)\, Z_t = \theta(B)\, a_t. \qquad t \in \kappa \qquad (1.2)$$

For certain situations, the ARMA(p,q) process $\{Z_t, t\in\kappa\}$ can also be represented as a random shock model

$$Z_t = \psi(B)a_t, \qquad t \in \kappa \qquad (1.3)$$

where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ and the ψ weights are calculated by equating the coefficients of B in the equation $\phi(B)\psi(B) = \theta(B)$. For the series to be stationary, we assume that $\psi(B)$ converges for $|B| \le 1$, that is, on or within the unit circle. Alternatively, the ψ weights have the condition $\sum_{j=0}^{\infty}|\psi_j| < \infty$. Similarly, $\{Z_t, t\in\tau\}$ can also be represented as an inverted form of the model using the π weights as

$$\pi(B)Z_t = a_t, \qquad t\in\kappa \qquad (1.4)$$

where $\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \ldots$. The π weights are analogously obtained by equating coefficients of B in $\phi(B) = \theta(B)\pi(B)$. To satisfy the condition of invertibility, we assume that $\pi(B)$ converges on or within the unit circle.

Alternatively, the π weights are assumed to satisfy the condition $\sum_{j=0}^{\infty}|\pi_j| < \infty$ (Box, Jenkins and Reinsel, 1994, Section 3.1).

Following Box and Jenkins (1976), analysis based on these models has been extensively studied in the literature and for details we refer to Abraham and Ledolter (1983), Chatfield (1989), Kendall and Ord (1990), Box et al. (1994), Mills (1994) and Brockwell and Davis (1991, 1996). The inferential problems considered in the literature are identification of the order p, d and q in the model, estimation of time series parameters and error variance, diagnostic checking of the model, forecasting of future values, etc. Box and Jenkins suggested that the principle of *parsimony* is important in model building i.e., the number of parameters p, d, and q of the fitted model must be minimum (Box et al., 1994, p. 16).

In this thesis, we focus on the analysis of stationary and invertible time series ARMA(p,q) in the presence of outlier.

1.2 Presence of Outliers in Time Series

The time series data often encounters anomalous observations due to external disturbances or errors which disrupt the pattern of the time series. Such observations are called *outliers*. The presence of the outliers has significant influence on the analysis of the series. The outliers may influence adjacent observations due to the presence of correlation pattern in the series. Apart from affecting the estimates of model parameters, forecasting and so on, outliers can distort the model specification itself and the impact of outliers in time series modeling can be serious enough to affect the credibility of the model (Barnett and Lewis, 1994, Chapter 10).

Consequently, the typical ARMA model may not represent the set of these observations because of the presence of outliers. Thus the presence of outliers in the observed series manifests into the problem of efficiency and adequacy in fitting of the models. Thus the investigation into presence of outliers, identification of outliers, assessment of their effects on the analysis and the remedial measures to accommodate the outliers is a crucial aspect of time series analysis.

The dependent structure of time series observations makes the detection of outliers difficult as the presence of outliers may escape notice due to the dependent structure of the process. Also, unlike in case of general linear models, an outlier in time series need not necessarily be an extreme value (Barnett and Lewis, 1994, p. 395).

Sometimes, the business and economics time series are influenced by non-repetitive interventions such as strikes, outbreaks of war, monetary crises, implementation of a new regulation, major changes in political or economic procedure and so on. Such events usually bring in outliers into the time series data. Often in such cases, the position of outliers is known and an appropriate handling can be done using intervention analysis (Box and Tiao, 1975; Kendall and Ord, 1990, pp. 222-227; Wei, 1990, pp. 184-195; Box et al., 1994, pp. 461-469). However, we are not considering intervention analysis in this thesis.

In many situations, the presence of outliers is rarely known before hand. Hence the crucial step in analysis of time series in the presence of outliers is constructing an appropriate model representing outliers in the data.

For the construction of a useful model appropriately representing outliers, it is important to understand the nature of outliers and their impact on the time series. Following Fox (1972) and Abraham and Box (1979), the possible outliers in ARMA (p,q) model are divided into two types, Type I or *Additive Outliers* (AOs) and Type II or *Innovational Outliers* (IOs). The AOs are those which do not affect adjacent observations and hence can be visualized in terms of superimposing an isolated measurement or execution error on the standard process.

We denote by T, the time point at which a single outlier of either AO or IO type is present in the series and refer to it as 'outlier position'. Note that T may or may not be known. The model which accounts for an AO at time T in ARMA (p,q) (Fox, 1972; Abraham and Box, 1979) is

$$Y_t = Z_t + \omega\xi_t^{(T)}, \qquad t \in \tau \tag{1.5}$$

where $\{Y_t\ , t\in\tau\}$ is the observed series, $\{ Z_t, t\in\tau\}$ is an unobserved outlier free series as in (1.2), $\tau = \{1, \ldots, n\}$, ω is the *outlier parameter* $-\infty < \omega < \infty$, and $\xi_t^{(T)} =$ 1 if $t = T$ and $\xi_t^{(T)} = 0$ otherwise. Thus, AO does not have any "carry-over effect" on the succeeding observations. As a result, the presence of AOs is often dramatically manifested in a time sequence plot.

Alternatively, the IOs are those which indicate inherent form of contamination influencing successive observations through the correlation structure. As a result, the realization of an outlier often gets concealed by the observations succeeding it, which are affected by the "carry-over effect". The model which accounts for an IO at time T in ARMA (*p*,*q*) is specified by (Fox, 1972; Abraham and Box, 1979)

$$Y_t = Z_t + \omega\, \psi(B)\xi_t^{(T)}, \qquad t \in \tau \tag{1.6}$$

where, as before, $\{Y_t,\ t\in\tau\}$ is the observed series, $\{Z_t,\ t\in\tau\}$ is an unobserved outlier free series as in (1.2), ω is the outlier parameter $-\infty < \omega < \infty$, and $\xi_t^{(T)} = 1$ if $t = T$ and $\xi_t^{(T)} = 0$ otherwise. As per this model, the effect of outlier on time series begins at the observation Z_T at time T and the effect on successive observations decays with ψ weights.

In addition to these two main outlier types, two other types of outlier models are proposed in the literature to handle sudden level changes which may be of temporary or permanent type, called the *Temporary Change* (TC) model and *Level Shift* (LS) model (Tsay, 1988; Chen and Liu, 1993) respectively.

The TC model is expressed as

$$Y_t = Z_t + \frac{\omega}{(1-\delta B)}\xi_t^{(T)}, \qquad t\in\tau \tag{1.7}$$

where δ is the dampening factor with $0 < \delta < 1$. In this model, the temporary change is the initial effect on the observation at T and the effect on subsequent observations decays exponentially according to some dampening factor, δ. If $\delta=0$, the model is the same as the AO type model given by (1.5).

The LS model is same as (1.7) with $\delta = 1$, given by

$$Y_t = Z_t + \frac{\omega}{(1-B)}\xi_t^{(T)}, \qquad t \in \tau \tag{1.8}$$

and in this case, the effect on the series is the change in the series level by a constant magnitude ω starting from time point T till the end of the series.

It is not unusual to come across time series data with more than one outlier. The problem of handling multiple outliers in time series is more complicated, for the simple reason that the outliers could be of different types (Barnett and Lewis, 1994, p. 397). The outlier models presented earlier can be easily extended to the multiple outliers situation as follows.

$$Y_t = Z_t + \sum_{j=1}^{m} \omega_j D_j(B) \xi_t^{(Tj)}, \qquad t \in \tau \tag{1.9}$$

where 'm' is the total number of outliers present in the series, ω_j, $j = 1, 2, \ldots, m$ are the corresponding outlier parameters which may not be distinct. Further, based on outlier type present at time point T_j, $j = 1, 2, \ldots, m$, we define

$$D_j(B) = 1 \qquad \text{for an AO,}$$

$$= \psi(B) \qquad \text{for an IO,}$$

$$D_j(B) = \frac{1}{(1-\delta B)} \quad \text{for a TC,}$$

$$= \frac{1}{(1-B)} \quad \text{for a LS.}$$

The multiple outliers can occur at isolated time points $T_1, T_1, \ldots, T_m$ and are called isolated outliers. Alternatively, they can occur at consecutive time points $T_j = T+j-1$, $j = 1, 2, \ldots, m$ starting at time point T. Such outliers are called *patch outliers* of patch length m (Martin, 1979; Bruce and Martin, 1989). The occurrence of multiple outliers as patch outliers is a much more complicated phenomenon due to the masking effect (Chen and Liu, 1993; Justel, Peña and Tsay, 2001). To handle certain time series data, another type of outliers called *reallocation outliers* are also proposed in the literature by Wu, Hosking and Ravishanker (1993).

Among the various types of outliers which can occur in a time series, the AO and IO are considered most often in the literature and we focus on these two types in this thesis.

1.3 Review of Literature

As mentioned in Section 1.2, Fox (1972) introduced models to accommodate the presence of two types of outliers namely Additive and Innovational outlier (AO and IO respectively) in autoregressive time series data. A likelihood ratio test for outlier detection was also proposed.

The Bayesian approach was used in the time series models with AO and IO outlier types by Abraham and Box (1979). The models proposed by Fox (1972) were extended to their present form (see (1.5) and (1.6)) for ARMA(p,q) and Bayesian parametric inference for time series parameters as well as outlier parameter in the presence of both types of outliers in case of AR(p) was carried out. Abraham and Box also presented some numerical examples based on generated data from AR(1).

Using the mixed outlier model proposed by Fox, Muirhead (1986) introduced a likelihood ratio test for detection of single outlier and identification of the type of outlier. Muirhead also compared the proposed method with the corresponding Bayes rule proposed by Abraham and Box (1979) for identification of outlier.

Following Fox (1972), Chang and Tiao (1983) proposed an iterative detection procedure for outliers in ARMA models. This procedure is based on likelihood ratio test and involves the identification of the outlier types as well. Tiao (1985) illustrated this procedure using two real life data sets.

Chang, Tiao and Chen (1988) formally presented this iterative likelihood ratio test procedure for detection of IO and AO and estimation of time series parameters of ARMA in the presence of outliers. A detailed discussion on the performance of the iterative procedure based on simulation in the context of AR(1) and MA(1) in the presence of up to two outliers is also presented in this work.

The procedures discussed so far are based on the assumption that time series model is known, whereas Tsay (1986) proposed an iterative procedure for model specification of time series in the presence of outliers where the iterative outliers detection procedure proposed by Chang and Tiao (1983) was effectively used. Chen and Liu (1993) introduced a procedure for joint estimation of model parameters and outlier effect in ARMA where a similar iterative outlier detection and identification procedure based on likelihood ratio test is used. In this procedure, if an outlier is detected at any stage of iteration, the series is appropriately adjusted depending on the detection of outlier type. It was shown that in case an IO is detected at time point t, the adjusted observation at t is the conditional expectation of the original observation given the past; unlike the AO case, where the adjusted observation is an interpolation based on past and future observation. Chen and Liu also investigated the effects of different types of outliers on the observed series and pointed out that " ...the effect of an IO is more intricate than the effects of other types of outliers".

Schmid (1986, 1990) considered multiple outliers problem and derived a test of discordancy for AO type of outliers in an AR process. Further the asymptotic behaviour of the test is investigated. Earlier to that, Schmid (1989) proposed a UMPU test for AO identification in an AR model. Schmid (1996) introduced an alternative multiple outlier detection and identification test procedure for AR process which is based on observed and predicted values at each time instance. Asymptotic distribution of the test statistic under the hypothesis of outlier free model is derived in this paper. Schmid further presented a simulation based performance comparison of various procedures with the proposed procedure.

Tsay, Peña and Pankratz (2000) extended the outlier problem in univariate time series to a vector valued autoregressive integrated moving average (ARIMA) series. The effect of multivariate outlier and its impact on the joint and marginal models was discussed. It was pointed out that the effects depend not only on the outlier size and the model, but also on the interaction between the two. Two statistics for various types of outlier detection were proposed and investigated in this study.

The Bayesian approach to outliers in time series is addressed by a number of authors. As mentioned earlier, Abraham and Box (1979) were the first to present the Bayesian analysis which was followed by Muirhead (1986), who proposed a Bayes rule to distinguish between AO and IO type. Smith (1983) considered a general approach to robust Bayesian methods with specific consideration of outliers in time series. Bayesian forecasting methods in the presence of outliers from contaminated sources were presented by Ameen and Harrison (1985).

Alternatively, McCulloch and Tsay (1994), and Barnett, Kohn and Sheather (1996, 1997) used Markov Chain Monte Carlo (MCMC) methods to detect outliers and compute the posterior distribution of the parameters in case of ARIMA.

Several authors have considered robust estimation procedures for parameters of time series in the presence of outliers. Denby and Martin (1979) first proposed a class of generalized maximum likelihood estimates (GM-estimates) for AR(1) model in the presence of a single outlier of either type. It was shown that though GM-estimates perform moderately well in the presence of outliers AO and IO, the M-estimates perform much better in the presence of IO.

Martin (1979) extended the GM-estimates to AR(p) model. He also discussed some theory and methodology of robust estimation for time series with AO and IO as well as the problem of patch outliers. In addition, a formal significance test and a residual plotting diagnostic technique were proposed for determining the outlier type.

It was pointed out by Bustos and Yohai (1986) that the GM-estimator has a complicated asymptotic covariance matrix. They proposed two new robust estimators based on residual autocovariances (RA-estimators) and truncated residual autocovariances (TRA-estimators). The proposed estimators were compared with least squares (LS) estimator, M and GM estimators for AR(1) and MA(1) models with AO and IO outliers. Based on Monte Carlo results, it was shown that RA estimators are not qualitatively robust when the model has the moving average part but are much stable than LS and M estimators in the presence of AO.

A detailed review of robust estimators for ARMA model with outliers was presented by Martin and Yohai (1985). The influence curve of time series with

AO and IO from isolated to patch outliers was also briefly discussed in this paper. In a later paper, Maddala and Yin (1997) also review the outlier detection in time series model.

Based on re-weighted maximum likelihood estimator using Huber or redescending weights, Luceño (1998) proposed robust estimators in the presence of nonconsecutive multiple outliers in ARMA(p,q) series. By choosing appropriate weight for robust ARMA(p,q) fitting, a multiple outlier detection procedure was also introduced. Earlier to that, influence function based outlier detection procedure was proposed by Chernick, Downing and Pike (1982). The effect of outliers on stationary time series was investigated using the influence function of the autocorrelations. Another attempt in this direction was by Peña (1987) who discussed sample influence function for parameters in the presence of outliers in ARMA model.

Various diagnostic procedures for deletion of outliers and influential points in regression models have been discussed in the literature (Cook and Weisberg, 1982; Chatterjee and Hadi, 1988). Abraham (1987) and Peña (1987) were the first in adapting some of these procedures to time series models. Abraham discussed diagnostic tests based on deletion of suspected observations and in particular the impact of deletion on Q statistic (Draper and John, 1981) for AR(1) series.

Peña (1987) investigated the impact of outlier on parameter estimation through sample influence function by treating a single observation as missing in turn. The missing values were estimated using least squares predictors. The procedure is a natural generalization of leave-one-out technique in regression diagnostic.

Further, Abraham and Chuang (1989) investigated the effect of deletion of k observations on Q statistics in case of ARMA(p,q) and proposed an outlier deletion procedure based on Q statistic which is used for identification of outlier type as well. A model building strategy based on the investigation of the pattern of Q statistic was also presented.

An in-depth leave-k-out diagnostics approach to outliers in time series was proposed by Bruce and Martin (1989), where k consecutive observations were treated as missing and the parameters were estimated using Kalman filter in case of ARIMA(p,d,q). The effect of missing observations on sample influence function of time series parameters and error variance was investigated which led to various diagnostic tests. Since k consecutive observations were treated as missing, the proposed procedure was shown to handle outliers patch and avoid the masking effect. It was also shown that the diagnostics based on error variance is superior to that based on parameters of time series.

Another attempt at using deletion diagnostic for detecting outliers in time series was by Ledolter (1990). The impact of outlier on Cook D statistic (Cook and Weisberg, 1982, p. 185), which is equivalent to likelihood displacement criteria, was investigated by treating each observation as missing. Ledolter also investigated the behaviour of the proposed diagnostic procedure and presented simulation study for AR(1) process. Ledolter further investigated the additive outlier model and claimed that the sensitivity of variance estimate depends on the difference between observations and their interpolation values, which justifies the satisfactory behaviour of the deletion procedure in the presence of AO.

Ljung (1993) showed that analogous to the estimation of outlier parameter in regression model, the estimation of additive outlier in ARMA(p,q) is directly related to estimation of missing or deleted observation and established the relation between leave-k-out diagnostics procedure by Bruce and Martin and likelihood ratio criteria. Ljung, however, pointed out that deletion diagnostic measures are expected to perform well for AO but not IO.

In recent years, the procedure using Gibbs sampling was proposed by McCullogh and Tsay (1994) who showed that the Gibbs sampling provides satisfactory inference in case of AR process when the outliers do not occur in patches. Justel, Peña and Tsay (2001) showed that the procedure, however, is not efficient in the presence of patches of additives outliers in an autoregressive process. It was also pointed out that the leave-k-out procedure cannot efficiently determine the block size of the patches of outliers. Their procedure consists of modification of standard Gibbs sampling and the algorithm presented is shown to work effectively using real life and simulated data.

Another approach to diagnostic checking of outlying values, level shifts and switches using state space modeling of time series was done by De Jong and Penzer (1998).

1.4 Need and Outline of Present Study

The discussion in most of the available literature on outlier detection pertains to detection of presence of outliers, detection of position of outliers, estimation of parameters in the presence of outliers, accommodation of outliers and so on. It is well accepted in the literature that IO may have less influence on time series parameter estimation than AO (Abraham, 1987; Ljung, 1993) and the detection of such outliers is much more difficult.

As discussed in Section 1.3, various diagnostics procedures for handling of outliers in time series are being adapted. These procedures follow the approach of "deletion diagnostics", analogous to the deletion diagnostics in regression (Peña, 1987; Abraham and Chuang, 1989; Bruce and Martin, 1989; Ledolter, 1990; Ljung, 1993). In these proposed deletion diagnostic procedures, each observation is treated as missing in turn and is replaced by its least squares predictor, which is the weighted sum of adjacent observations.

The method of deletion diagnostics works well in the presence of AO but the IO poses a problem because of its dynamic nature (Chen and Liu, 1993; Ljung, 1993). The proposed deletion diagnostics methods, with the exception of Abraham and Chaung (1989), do not take into account the AO or IO type separately and hence do not identify the type of outlier, which is crucial in adapting an appropriate model for data analysis.

This thesis attempts to fill in this gap by proposing a different diagnostic procedure called *adjustment diagnostics*. In this procedure, two separate models for two types of outliers are considered. Further, each observation is treated as a possible outlier of each type in turn and the observed series is appropriately adjusted, taking into account the underlined model and estimation of parameters. The model adjustment is shown to be similar to deletion diagnostics in case of AO and is also shown to handle the influence of IO on successive observations. The adjustment diagnostics are presented both from theoretical and empirical points of view.

In Chapter 2, a detailed investigation of the effects of both types of outliers based on empirical study in case of AR(1) and MA(1) series is presented. The study suggests using estimate of error variance to analyze the outliers in time series. The maximum likelihood estimation of model parameters and outlier parameter used for further analysis is briefly discussed in Section 2.4. Section 2.5 heuristically shows that the deletion diagnostic procedure cannot satisfactorily handle the effect of IO. In Section 2.6, we investigate how the model can be appropriately adjusted to handle the effect of outliers, and suggest adjustment of the observed series as a possible way of analyzing the presence of outliers.

Chapter 3 analyzes the effect of series adjustment on the estimate of error variance. Two types of series adjustments corresponding to AO and IO are considered at each time point in turn. The expression for the estimates of error variance under the adjustment at correct and incorrect positions for correct type are derived in Section 3.3 and it is shown that the estimate of error variance is likely to be the smallest among all the estimates when the series is adjusted for the correct type of outlier at correct position. In Section 3.5 we investigate the impact on the estimate of error variance due to the incorrect type of adjustment at correct position. Computations supporting theoretical investigations are presented in Section 3.6 for specific AR(1) and MA(1) series.

Based on the findings of Chapters 2 and 3, an adjustment diagnostic procedure using Cook's likelihood displacement criterion (Cook, 1986, 1987) is derived for outliers in ARMA(p,q) series in Chapter 4. We call the proposed procedure *"Adjustment Diagnostic based on Variance" (ADV)* which, in addition to outlier detection, is can be used to identify the correct outlier type and the correct outlier position in the series. The problem of deriving the exact null reference distribution of the statistic in time series analysis is similar to that in case of regression diagnostics discussed in the literature (Abraham, 1987; Chatterjee and Handi, 1988; Bruce and Martin, 1989; Ljung, 1993).

Hence the critical values of the proposed procedure are computed based on extensive Monte Carlo simulations The simulation study presents upper 10% and 5% percentiles of the adjustment diagnostic procedure using 5000 replications in case of AR(1), AR(2), MA(1), MA(2) and ARMA(1,1) series of lengths n = 100(25)300 for a wide range of values of time series parameters.

Monte Carlo based performance evaluation of the proposed procedure in the presence of a single outlier and its comparative performance with the existing deletion diagnostic procedures is presented in Section 4.3 for AR(1) and MA(1) series with a single AO or IO outlier and outlier parameter ω = 0, 2(1)7. The

Monte Carlo simulations are carried out using IMSL (International Mathematical and Statistical Libraries) subroutines. The procedure here uses the maximum likelihood estimate of the time series parameters which is same as the least squares (LS) estimate since the errors are assumed to be Gaussian with mean 0 and variance σ^2. The tables and figures presented in Section 4.3 indicate that ADV provides a comprehensive diagnostic tool which performs satisfactorily better than the existing deletion diagnostics procedures irrespective of the type of outlier.

Since it is well known that the LS estimates are in general not robust in the presence of outliers or abnormal observations in the time series (Denby and Martin, 1979; Bustos and Yohai, 1986, Barnett and Lewis, 1994, p. 404), we further evaluate the performance of the proposed procedure using a robust estimate of time series parameter and compare it with that using LS estimate. The evaluation is again carried out on 1000 simulations of AR(1) and MA(1) for a wide range of time series parameter and is presented in Section 4.4. Based on the performance evaluation presented here it can be claimed that the detection of outlier and position identification is carried out with equal precision irrespective of whether the procedure uses robust or LS estimates of time series parameter. The performance of the procedure to identify the outlier type and its correct position marginally improves in the presence of AO type of outlier when a robust estimate is used. However, the performance using LS estimate is marginally better than that using robust estimate when IO is present in the series. The simulation study shows that the use of robust estimate of time series parameters for a contaminated series does not uniformly improve the performance of the procedure in the presence of any type of outlier.

Sections 4.5 to 4.7 address the problem of multiple outliers. An iterative procedure based on adjustment diagnostic is proposed for multiple outliers detection in Section 4.5. In Section 4.6 performance evaluation of multiple outliers detection procedure is presented based on Monte Carlo study of AR(1) and MA(1) in the presence of two outliers. We consider the presence of isolated and patch outliers of same type (2 AOs or 2 IOs) and different types (AO-IO or IO-AO) for evaluation. The simulation study shows that the procedure has satisfactory performance in the presence of two outliers as well, though it performs better in the presence of isolated outliers as against patch outliers. In Section 4.7, we critically address this issue and suggest possible ways of overcoming this drawback. We also present the effect of multiple outliers on the estimate of error variance when the outliers are isolated and when they occur in patches and claim that the identification of patch outliers will not be satisfactorily handled using estimates of error variance.

Chapter 5 presents analysis of some simulated data sets and real life data sets available in the literature using the proposed procedure. In Section 5.2, using simulated AR(1) and MA(1) series, the usefulness of proposed ADV plots in the presence of up to two outliers is demonstrated. Various simulated AR(1) and MA(1) series are considered in Section 5.3 to illustrate the performance of ADV. In Section 5.4, some real life data sets which are already discussed in the literature, namely, "Daily Average Number of Truck Manufacturing Defects Series" (Wei, 1990, p. 446), "Series A", "Series C", "Series D" and "Series J" (Box et al., 1994, pp. 541-545) are considered. The analyses of these data sets using proposed procedure are compared with the outlier analysis available in the literature.

The thesis ends with a short write up under the title Conclusions, which gives a summary of work presented and statements on performance comparison of the proposed procedure with procedures available in the literature. The possible future investigations are briefly outlined.

Appendix A presents upper 10% and 5% percentiles of the proposed procedure for AR(1), AR(2), MA(1), MA(2) and ARMA(1,1) series of lengths n = 100(25)300 for a wide range of values of time series parameters.

We have developed **STDS** (Statistical Time Series Diagnostic Software), a software to diagnose the outliers in time series. The main features of **STDS** are diagnostic plots and ADV procedures for ARIMA(p,d,q) model. The software provides the estimation of the magnitudes of outlier parameters, time series parameters and error variance after detecting the positions of outliers and identifying their types. The contents page of STDS manual is presented in Appendix B. The software STDS and its manual are provided in a Compact Disk (CD) at the end of this thesis. All the computer programs used for simulation and computations are also provided in the CD. The list of available programs is provided in Appendix C.

Chapter 2

The Effect of Outlier and Series Adjustment

2.1 Introduction

One of the important problems in time series analysis is the detection of the presence of an outlier and its position in the observed series. If the position of the outlier and also its type are known, the estimation of series parameters and the outlier parameter can be carried out simultaneously (Tiao, 1985; Tsay, 1986; Wei, 1990, Section 9.3; Box et al., 1994, Section 12.2). In order to have satisfactory outlier detection procedures, it is important to study the effect of outliers on statistical analysis of the series, in particular, the parameter estimation. Various authors have commented on the effect of outliers on the estimation of the time series parameters (Chang and Tiao, 1983; Bruce and Martin, 1989; Abraham and Chuang, 1989; Ljung, 1993; Wu et al., 1993; Luceño, 1998). Based on sample influence function, Bruce and Martin claimed that the effect of outliers on the estimation of error variance is more serious than that on the time series parameters. It is also mentioned that the effect of AO is more serious than that of IO on general parameter estimation (Chang and Tiao, 1983; Abraham 1987; Ljung, 1993). In Section 2.2, we discuss the model with two types of outliers and briefly describe the effect of both types of outliers on to original series and error series. In order to get a better understanding of these issues, we investigate them in more details based on empirical study in case of AR(1) and MA(1) series in Section 2.3. In Section 2.4, we briefly discuss estimation of time series parameters and outlier parameter (Chang et al., 1988; Box et al., 1994, Chapter 7) as we use these estimators for further analysis.

As mentioned in Section 1.3, most of the outlier detection procedures are based on deletion diagnostics, where each observation is deleted in turn and its predicted value based on remaining observations is substituted using missing value estimation (Peña, 1987; Ledolter, 1990; Ljung, 1993). The estimation of parameters of interest and the analysis of the time series is carried out using the substituted value. Ljung (1993) proved that this is equivalent to likelihood ratio procedure for outlier detection in case of AO. It is claimed in Section 2.5 that the deletion diagnostic procedure cannot satisfactorily handle the effect of an IO. In particular, it is argued based on hypothetical situation that the dynamic effect of an IO on the subsequent observations cannot be removed by treating only single observation as missing.

To overcome this drawback, in Section 2.6 we present an alternative method of handling the presence of outliers in time series. The method proposes the adjustment of the observed time series according to the underlying model, after estimating the outlier parameter. It is shown that the series adjustment takes care of the effect of outlier in case of both AO and IO types. Using ARMA(p,q), it is further shown that the method is equivalent to missing value estimation in AO case.

2.2 Effect of an Outlier on Original Series

In this section, we discuss the effect of an outlier on the original series (also called *outlier free series*) $\{Z_t,\ t\in\tau\}$ which is a stationary and invertible ARMA(p,q) process with error series $\{a_t,\ t\in\tau\}$ where $\tau = \{1, 2, \ldots, n\}$. The errors a_t's are assumed to be i.i.d normal with mean 0 and variance σ_a^2. Following (1.5) and (1.6), consider the observed series $\{Y_t,\ t\in\tau\}$, contaminated by a single outlier of AO or IO type, given by

$$Y_t = Z_t + \omega\xi_t^{(T)}, \qquad t\in\tau \quad \text{for AO}$$

$$Y_t = Z_t + \omega\psi(B)\xi_t^{(T)}, \qquad t\in\tau \quad \text{for IO.} \tag{2.1}$$

As mentioned in Section 1.2, the model in (2.1) allows only a single observation at time point t = T of the underlined series $\{Z_t,\ t\in\tau\}$ to be affected by AO, whereas the effect of IO at t = T is carried over to the subsequent observations of $\{Z_t,\ t\in\tau\}$ and decays with ψ's weights.

Suppose the observed series $\{Y_t,\ t\in\tau\}$ is treated as a typical stationary and invertible ARMA(p,q) series ignoring the presence of outlier. We then define the residual series of $\{Y_t,\ t\in\tau\}$ by $\{e_t,\ t\in\tau\}$ for $t\in\tau$,

$$e_t = \pi(B)Y_t, \tag{2.2}$$

where $\pi(B) = \psi^{-1}(B) = 1-\sum_{i=1}^{\infty}\pi_i B^i$.

The error series of $\{Z_t,\ t\in\tau\}$ is given by $\{a_t,\ t\in\tau\}$ where

$$a_t = \pi(B)Z_t \tag{2.3}$$

However, since $\{Y_t, t \in \tau\}$ is an outlier contaminated series, using (2.1) and (2.3), $\{e_t, t \in \tau\}$ can be expressed in terms of $\{a_t, t \in \tau\}$ as

For AO: $$e_t = \pi(B)Y_t = \pi(B)\{ Z_t + \omega\xi_t^{(T)} \}$$
$$= a_t + \omega\pi(B)\xi_t^{(T)}$$

For IO: $$e_t = \pi(B)Y_t = \pi(B)\{ Z_t + \omega\psi(B)\xi_t^{(T)} \}$$
$$= a_t + \omega\xi_t^{(T)} \qquad (2.4)$$

From (2.4), it can be seen that the presence of IO affects a single observation of $\{a_t, t \in \tau\}$ series at only one time point $t = T$, whereas the effect of AO on $\{a_t, t \in \tau\}$ is carried over to the subsequent observations and decays with π weights (Chang et al., 1988; Box et al., 1994, p. 471).

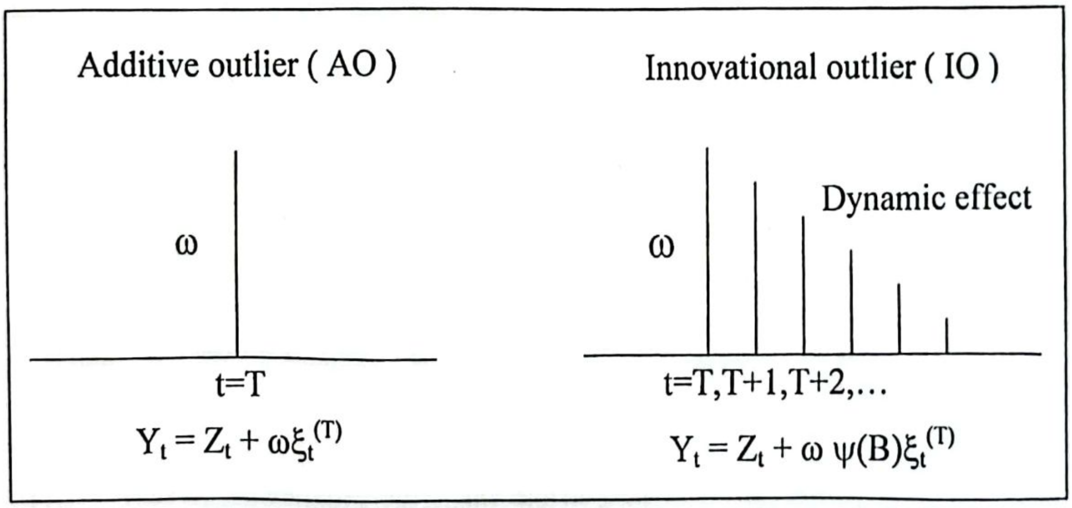


Figure 2.1: The Effect of an Outlier on $\{Z_t, t \in \tau\}$ Series

In Figures 2.1 and 2.2, we present a hypothetical situation representing the effect of outlier of either type on the original series $\{Z_t, t \in \tau\}$ and the error series

$\{a_t, t \in \tau\}$ respectively, with the outlier parameter of magnitude $\omega > 0$. Note that the dynamic effect of IO on observations and that of AO on error series can be either positive or negative depending on the signs of the weights ψ and π, and ω.

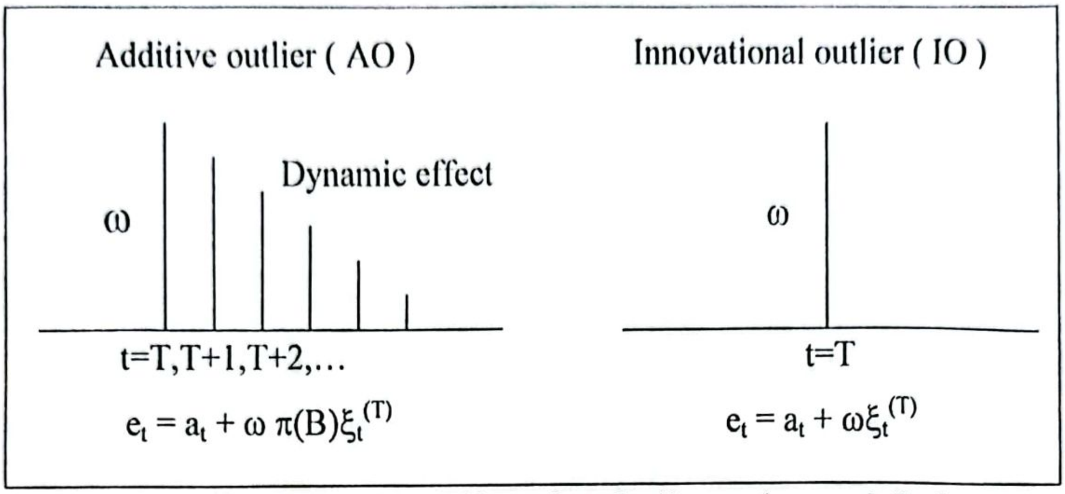


Figure 2.2: The Effect of an Outlier on $\{a_t, t \in \tau\}$ Series

To know the effect of an outlier on the analysis of the observed series, one must understand the effect on the estimators of time series parameters, error variance and some other statistics such as autocorrelation function. We discuss them in the next section.

2.3 The Effect of an Outlier on Some Estimates

In this section, we separately study the effect of the two types of outliers on the analysis of AR(1) and MA(1) series using simulated data. A wide range of values of outlier parameter is considered ($0 \leq \omega \leq 10$) to highlight the effect of AO and IO on the estimates of parameter, error variance and autocorrelation function, which are important statistics for the model building procedure in time series. The estimates considered are Maximum Likelihood (ML) estimates which are same as the Least Squares (LS) estimators under the assumption of Normal errors (Box et al., 1994, Chapter 7). The ML estimation procedure for all parameters is briefly discussed in Section 2.4. The simulation of the series and the estimation of the parameters is carried out using IMSL (International Mathematical and Statistical Libraries) subroutines under the Sun Operating System and the programs are given in the attached CD and listed in Appendix C. The IMSL subroutine NSLSE is an iterative procedure which gives non-linear LS estimators of the time series parameters . For all the computations presented in the thesis NSLSE is used and Yule Walker estimates (moment estimates) are selected as initial values. The Yule Walker estimators are not used in the thesis as final estimators of time series parameters since they are known to be 'unstable' (Wei, 1990, p. 137; Box et al., 1994, pp. 260-262).

AR(1) with One Outlier:

We now consider the observed series $\{Y_t, t \in \tau\}$ with AR(1) model and a single AO type outlier at T, given by

$$Y_t = Z_t + \omega \xi_t^{(T)}, \qquad t \in \tau \tag{2.5}$$

where $Z_t = \phi_1 Z_{t-1} + a_t$ is the AR(1) model without outlier, ω is the outlier parameter, ϕ is the autoregressive parameter, and a_t is the error term and has i.i.d normal with mean '0' and variance σ_a^2 and $\xi_t^{(T)} = 1$ if $t = T$ and $\xi_t^{(T)} = 0$ otherwise.

We can rewrite the model as

$$Y_t = \omega\xi_t^{(T)} + \phi Z_{t-1} + a_t, \qquad t \in \tau.$$

Hence, we can represent the AO model as

$$Y_t = \begin{cases} \omega + \phi Z_{t-1} + a_t, & t = T \\ \phi Z_{t-1} + a_t, & t \neq T. \end{cases} \qquad (2.6)$$

Similarly, to study the effect of IO type outlier on the observed series, we consider the IO outlier model

$$Y_t = Z_t + \omega\psi(B)\xi_t^{(T)}, \qquad t \in \tau$$

where Z_t is the AR(1) model in (2.5), ω is the outlier parameter and $\xi_t^{(T)}$ is as above. This model can be expressed as

$$Y_t = \omega\psi(B)\xi_t^{(T)} + \phi Z_{t-1} + a_t, \qquad t \in \tau$$

which can alternatively be written as

$$Y_t = \begin{cases} \phi Z_{t-1} + a_t, & t < T \\ \omega\psi_{t-T} + \phi Z_{t-1} + a_t, & t \geq T \end{cases} \qquad (2.7)$$

where $\psi_j = \phi^j$ for AR(1).

In order to study the effect of outliers in the series on various estimators, we generated 1000 outlier free AR(1) series for $n = 100$, $\phi = 0.6$ and $\sigma_a^2 = 1$ using the computer programs CP-1 and CP-2 as listed in Appendix C with the help of IMSL (International Mathematical and Statistical Libraries) subroutines under the

Sun Operating System. An AO and an IO with outlier parameter $\omega = 0(1)10$ was introduced at $t = 51$.

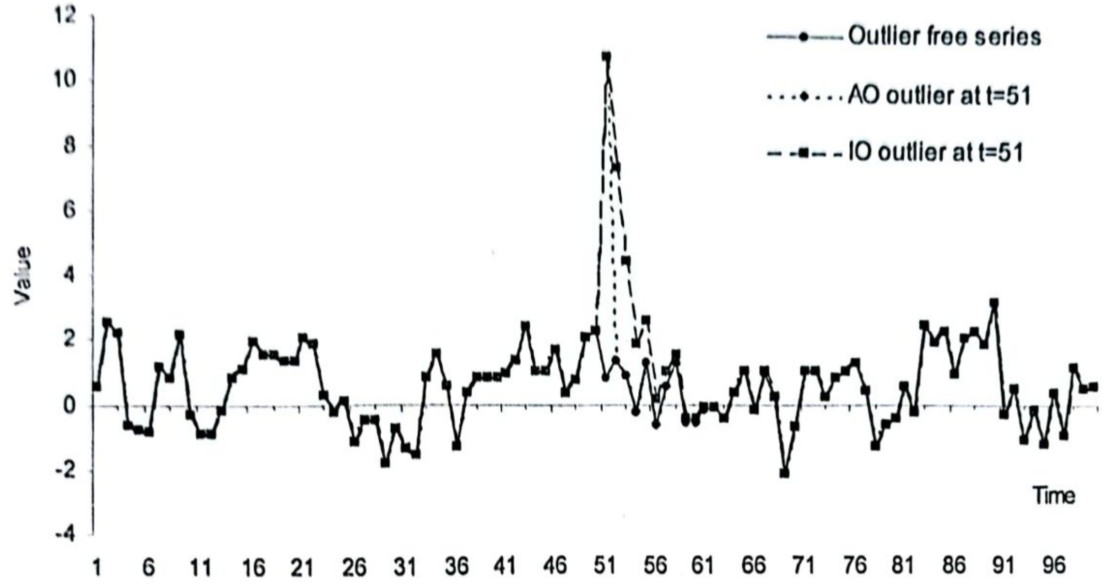


Figure 2.3: Generated AR(1) Series with an Outlier at $t = 51$
($n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$, $\omega = 10$)

For each of the generated series, the parameters of interest, namely the time series parameter ϕ, the error variance σ_a^2 and the lag-1 autocorrelation coefficient ρ_1 were estimated ignoring the presence of outlier.

Figure 2.3 presents the plots of one particular generated outlier free series along with the corresponding outlier contaminated series with AO and IO outlier at $t = 51$ for $\omega = 10$ and shows how the effect of AO is 'local' at $t = 51$ whereas IO affects the observation at $t = 51$ and the succeeding observations as well.

The estimates of parameters for these three series are as below:

	$\hat{\phi}$	$\hat{\sigma}_a^2$	$\hat{\rho}_1$
Outlier free series	0.57	1.04	0.48
Series with an AO	0.46	2.14	0.37
Series with an IO	0.67	1.91	0.61

It can be seen that the estimate of ϕ is different from that of ρ_1. Note that the estimate of ϕ is using the nonlinear LS estimator calculated iteratively where the initial value given is $\hat{\rho}_1$, which is the Yule Walker estimate of ϕ.

It is clear that an outlier of either type in the series has effect on the estimates of all the parameters, but the effect is noticeable in case of the estimated error variance $\hat{\sigma}_a^2$.

In Tables 2.1 and 2.2, we present the averages of the estimates of ϕ, σ_a^2, and ρ_1 for AO and IO respectively obtained from the 1000 simulated series for values of $\omega = 0(1)10$. In order to compare the changes in the estimates due to the presence of outlier of either type, a column of Relative Change (RC) is also presented for each estimate, where RC is defined as

$$\text{Relative Change} = \frac{A_\omega - A_0}{A_0} \times 100 \tag{2.8}$$

where A_ω is the average estimated value at $\omega>0$ (series with an outlier) and A_0 is the average estimated value of the same parameter at $\omega = 0$ (outlier free series).

Table 2.1

Average Values of $\hat{\phi}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: AR(1) with an AO at t = 51
(n = 100, ϕ = 0.6, σ_a^2 = 1; 1000 replications)

ω	$\hat{\phi}$		$\hat{\sigma}_a^2$		$\hat{\rho}_1$	
	Average	RC	Average	RC	Average	RC
0	0.5891	-	0.9897	-	0.5665	-
1	0.5911	0.33%	1.0066	1.71%	0.5674	0.15%
2	0.5775	−1.97%	1.0394	5.02%	0.5546	−2.11%
3	0.5547	−5.84%	1.1126	12.42%	0.5331	−5.90%
4	0.5343	−9.30%	1.2003	21.28%	0.5118	−9.65%
5	0.4995	−15.22%	1.3183	33.20%	0.4783	−15.57%
6	0.4764	−19.13%	1.4452	46.02%	0.4556	−19.57%
7	0.4442	−24.60%	1.6139	63.07%	0.4211	−25.66%
8	0.4174	−29.15%	1.7947	81.34%	0.3945	−30.36%
9	0.3789	−35.69%	1.9655	98.60%	0.3587	−36.68%
10	0.3525	−40.16%	2.2177	124.08%	0.3322	−41.35%

Note : RC=Relative Change

It can be seen from Tables 2.1 and 2.2 that with increasing magnitude of outlier parameter, the estimates of all the parameters change, though the effect on estimate of error variance is much more prominent and the error variance tends to get overestimated with higher values of outlier parameter ω, if the presence of outlier is ignored. This change is irrespective of the type of outlier, though the change is more prominent in case of AO. The relative change in the estimates of error variance $\hat{\sigma}_a^2$ is of higher absolute magnitude than that of time series parameter for both AO and IO. The estimated parameter $\hat{\phi}$ does not change significantly with change in ω in case of IO as compared to AO. The pattern in the decreasing values of estimated parameter $\hat{\phi}$ and estimated lag-1 autocorrelation $\hat{\rho}_1$ is similar since these are identical in case of AR(1) series.

Table 2.2

Average Values of $\hat{\phi}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: AR(1) with an IO at t = 51
(n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

ω	$\hat{\phi}$		$\hat{\sigma}_a^2$		$\hat{\rho}_1$	
	Average	RC	Average	RC	Average	RC
0	0.5901	-	0.9870	-	0.5685	-
1	0.5897	−0.06%	1.0055	1.87%	0.5663	−0.40%
2	0.5846	−0.93%	1.0338	4.73%	0.5627	−1.02%
3	0.5855	−0.77%	1.0795	9.37%	0.5633	−0.93%
4	0.5869	−0.54%	1.1413	15.62%	0.5653	−0.57%
5	0.5863	−0.65%	1.2488	26.52%	0.5643	−0.74%
6	0.5920	0.34%	1.3498	36.75%	0.5712	0.47%
7	0.5900	−0.01%	1.4729	49.22%	0.5683	−0.04%
8	0.5862	−0.66%	1.6293	65.07%	0.5649	−0.65%
9	0.5890	−0.18%	1.8098	83.35%	0.5700	0.26%
10	0.5896	−0.08%	1.9971	102.33%	0.5694	0.16%

Note : RC=Relative Change

We also present two plots of the absolute values of RC against ω for all the parameter estimates in case of an AO and an IO in Figures 2.4 and 2.5 respectively.

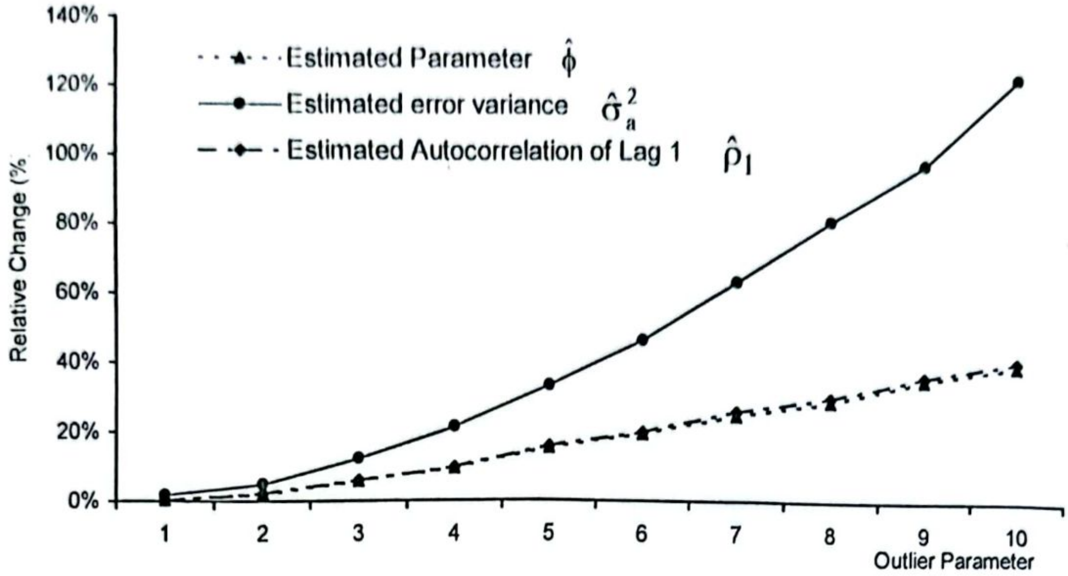


Figure 2.4: Absolute Relative Changes of $\hat{\phi}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: AR(1) with an AO at t = 51 (n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

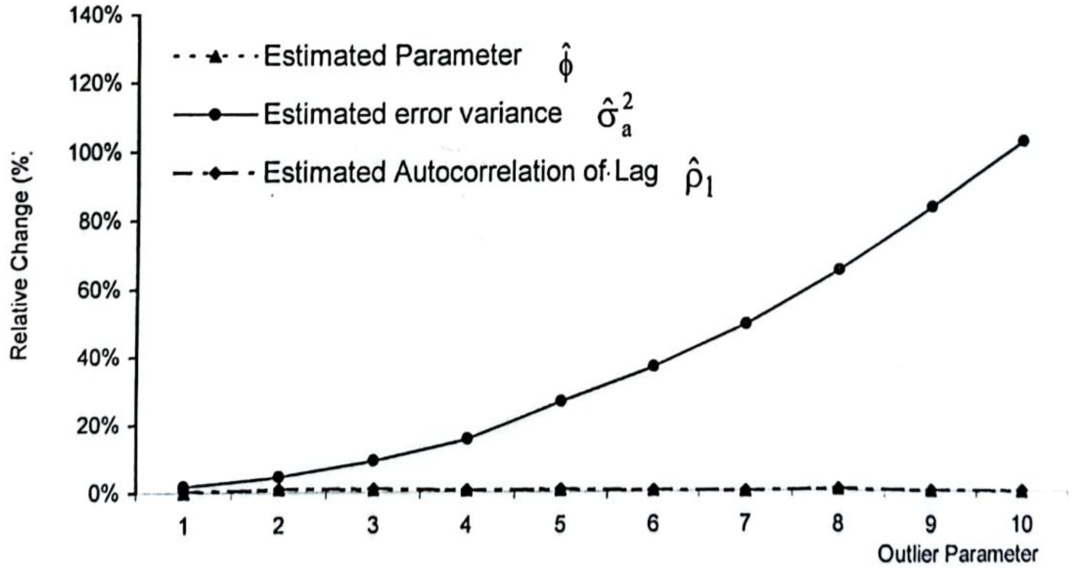


Figure 2.5: Absolute Relative Changes of $\hat{\phi}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: AR(1) with an IO at t = 51 (n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

It is clear that the presence of a single outlier of either type significantly overestimates the error variance. The finding provides empirical support to the claims in the literature that "the presence of outlier affects the estimate of error variance more than that of time series parameters" (for eg. Bruce and Martin, 1989; Ledolter, 1990, among others)

This also supports the justification for using estimate of error variance for outlier detection procedure in addition to its usefulness in verifying model adequacy. Also the empirical study shows that though the presence of AO affects the estimate of error variance more than that of IO, the effect of IO on the estimate is also high and needs to be taken into account, contrary to some of the claims in the literature (Chang and Tiao, 1983; Bruce and Martin, 1989; Ljung, 1993).

MA(1) with One Outlier:

In this section, we consider the MA(1) model and discuss the effect of the outlier on the estimates of parameters. Suppose that the observed series $\{Y_t, t \in \tau\}$ follows MA(1) with a single AO type outlier at T, which is modeled as

$$Y_t = Z_t + \omega\xi_t^{(T)}, \qquad t \in \tau \tag{2.9}$$

where $Z_t = a_t - \theta a_{t-1}$ is the MA(1) model for outlier free series, ω is the outlier parameter, θ is the moving average parameter, and a_t is the error term with mean '0' and variance σ_a^2 and $\xi_t^{(T)} = 1$ if $t = T$ and $\xi_t^{(T)} = 0$ otherwise. We can rewrite the model (2.9) as

$$Y_t = \omega\xi_t^{(T)} + a_t - \theta a_{t-1}, \qquad t \in \tau$$

It can be simplified to

$$Y_t = \begin{cases} \omega + a_t - \theta a_{t-1}, & t = T \\ a_t - \theta a_{t-1}, & t \neq T \end{cases} \qquad (2.10)$$

Similarly, we consider the MA(1) model with an IO type outlier as

$$Y_t = \omega\psi(B)\xi_t^{(T)} + a_t - \theta a_{t-1}, \qquad t \in \tau,$$

giving,

$$Y_t = \begin{cases} a_t - \theta a_{t-1}, & t < T \\ \omega + a_t - \theta a_{t-1}, & t = T \\ \omega\psi_1 + a_t - \theta a_{t-1}, & t = T+1 \\ a_t - \theta a_{t-1}, & t > T+1 \end{cases} \qquad (2.11)$$

where $\psi_1 = -\theta$.

Analogous to AR(1), we generated 1000 outlier free MA(1) series with n = 100, $\theta = -0.6$ and $\sigma_a^2 = 1$ using the computer programs CP-1 and CP-2 as listed in Appendix C, and introduced an AO and an IO at time point t = 51 for the values of outlier parameter ω = 0(1)10. For each of the generated series, we estimated the parameters of interest namely the time series parameter θ, the error variance σ_a^2 and the lag-1 autocorrelation coefficient ρ_1 ignoring the presence of outlier. One particular generated outlier free series and the corresponding outlier contaminated series for AO and IO for $\omega = 10$ are plotted in Figure 2.6.

Figure 2.6 shows that an AO outlier affects on the series $\{Z_t, t \in \tau\}$ at time point t = 51 only. An IO affects the series at t = 51 and its decaying effect can be

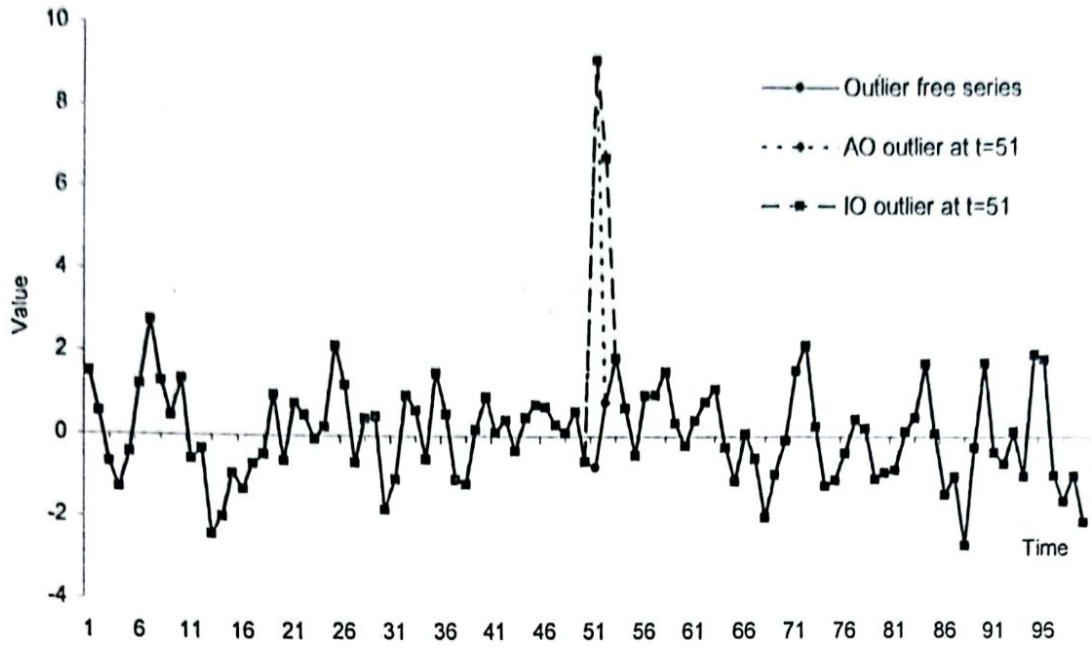


Figure 2.6: Generated MA(1) Series with an Outlier at t = 51
($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 10$)

seen at t = 52 as well. Since the weighs ψ_j of MA(1) are equal to zero when j>1, there is no IO effect after t = 52. The estimated parameters for the three series are as below:

	$\hat{\theta}$	$\hat{\sigma}_e^2$	$\hat{\rho}_1$
Outlier free series	−0.59	0.93	0.39
Series with an AO	−0.22	1.96	0.23
Series with an IO	−0.57	1.89	0.45

Analogous to the AR(1) case, the estimators $\hat{\theta}$ and $\hat{\rho}_1$ do not satisfy the parametric equation ($\rho_1 = \dfrac{-\theta}{1+\theta^2}$) since $\hat{\theta}$ is a nonlinear LS estimate calculated iteratively with initial value $\hat{\rho}_1$ where $\hat{\rho}_1$ is the moment estimate.

It is observed that the presence of AO affects the estimate of time series parameter more than that of IO, and both types give overestimate of the error variance.

Tables 2.3 and 2.4 present the averages of each estimated parameters along with their respective RCs based on 1000 simulated series for both types of outliers AO and IO for the parameter $\omega = 0(1)10$ respectively.

Table 2.3

Average Values of $\hat{\theta}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: MA(1) with an AO at t = 51
($n = 100, \theta = -0.6, \sigma_a^2 = 1$; 1000 replications)

ω	$\hat{\theta}$		$\hat{\sigma}_a^2$		$\hat{\rho}_1$	
	Average	RC	Average	RC	Average	RC
0	−0.6097	-	1.0112	-	0.4217	-
1	−0.5993	−1.71%	1.0337	2.23%	0.4140	−1.82%
2	−0.5722	−6.15%	1.0785	6.65%	0.4091	−2.98%
3	−0.5343	−12.37%	1.1597	14.69%	0.3902	−7.46%
4	−0.4950	−18.81%	1.2544	24.05%	0.3789	−10.15%
5	−0.4401	−27.82%	1.3564	34.14%	0.3494	−17.13%
6	−0.4028	−33.94%	1.4871	47.06%	0.3320	−21.27%
7	−0.3676	−39.71%	1.6441	62.59%	0.3088	−26.76%
8	−0.3251	−46.68%	1.8139	79.38%	0.2791	−33.82%
9	−0.2966	−51.36%	2.0047	98.24%	0.2572	−39.01%
10	−0.2632	−56.83%	2.1910	116.66%	0.2327	−44.83%

Note : RC=Relative Change

Table 2.4

Average Values of $\hat{\theta}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: MA(1) with an IO at t = 51
(n = 100, θ = –0.6, σ_a^2 = 1; 1000 replications)

ω	$\hat{\theta}$		$\hat{\sigma}_a^2$		$\hat{\rho}_1$	
	Average	RC	Average	RC	Average	RC
0	–0.6095	-	1.0194	-	0.4216	-
1	–0.6082	–0.21%	1.0261	0.65%	0.4221	0.13%
2	–0.6075	–0.32%	1.0648	4.45%	0.4262	1.09%
3	–0.6076	–0.31%	1.1003	7.93%	0.4203	–0.30%
4	–0.6035	–0.98%	1.1814	15.89%	0.4199	–0.40%
5	–0.6070	–0.40%	1.2748	25.06%	0.4175	–0.96%
6	–0.6084	–0.17%	1.3992	37.25%	0.4217	0.02%
7	–0.6011	–1.37%	1.5190	49.01%	0.4214	–0.04%
8	–0.6046	–0.80%	1.6600	62.84%	0.4243	0.64%
9	–0.6059	–0.58%	1.8314	79.65%	0.4238	0.53%
10	–0.6067	–0.45%	2.0428	100.39%	0.4243	0.65%

Note : RC=Relative Change

From Tables 2.3 and 2.4, it can be seen that the estimates of error variance $\hat{\sigma}_a^2$ increase with the value of ω and are much higher for higher values of ω, irrespective of the type of outlier. Also, analogous to AR(1), AO shows higher impact on $\hat{\sigma}_a^2$ than IO. The estimated parameter, $\hat{\theta}$ and the estimated lag-1 autocorrelation, $\hat{\rho}_1$ do not significantly change with varying values of outlier parameter ω in case of IO but show significant change in case of AO. It is clear from the simulated results that the effect of presence of outlier of either type on the estimate of error variance $\hat{\sigma}_a^2$ is much more than that on the other estimates $\hat{\theta}$ and $\hat{\rho}_1$.

To get a clearer picture of the tables presented for MA(1), we plot the absolute values of RC against ω for all the estimated parameters in case of AO and IO in Figures 2.7 and 2.8 respectively.

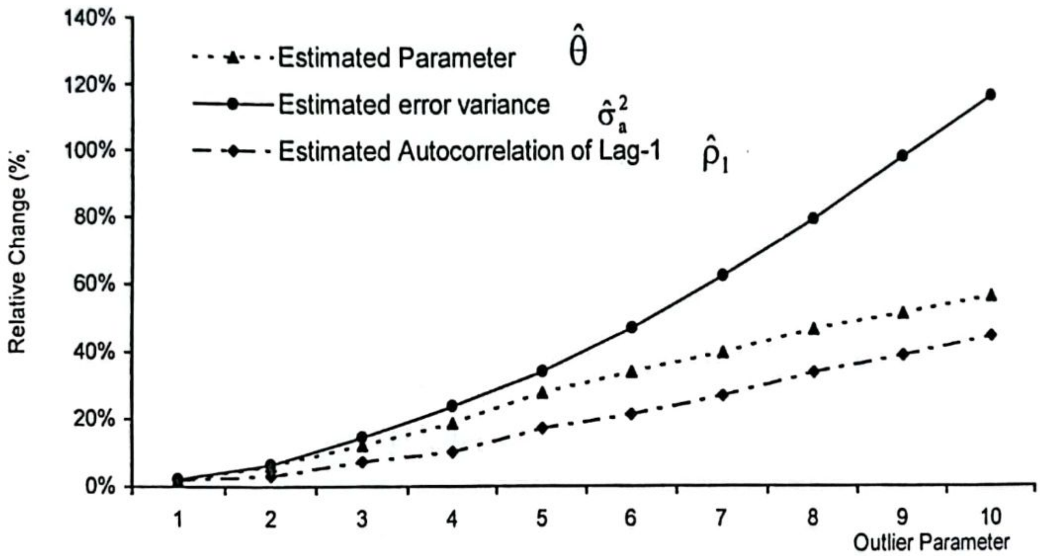


Figure 2.7: Absolute Relative Changes of $\hat{\theta}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: MA(1) with an AO at t=51 ($n = 100$, $\theta = -0.6$, $\sigma_a^2=1$; 1000 replications)

Based on the results presented, it is clear that for both AR(1) as well as MA(1), the estimate of error variance $\hat{\sigma}_a^2$ which gets affected if the presence of outlier in the series is ignored, irrespective of the type of outlier. The effect on the estimates of time series parameters $\hat{\theta}$ and lag-1 autocorrelation coefficient $\hat{\rho}_1$, however depends on both the type of series and the type of outlier.

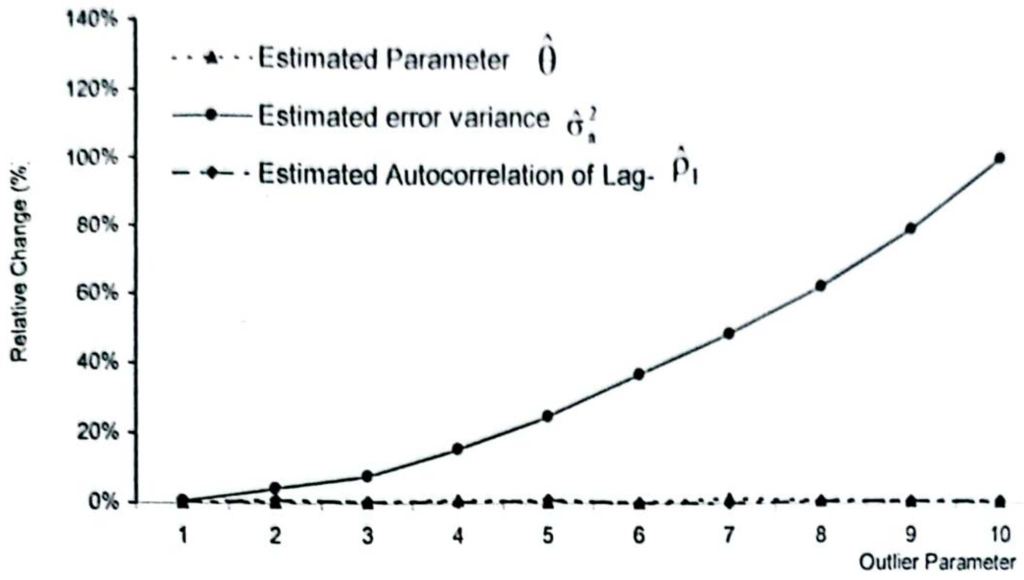


Figure 2.8: Absolute Relative Changes of $\hat{\theta}$, $\hat{\sigma}_a^2$ and $\hat{\rho}_1$: MA(1) with an IO at t = 51 (n = 100, $\theta = -0.6$, $\sigma_a^2=1$; 1000 replications)

The empirical study of a small number of generated AR(p) and MA(q) as well as ARMA(p,q) series for selected values of p and q also show analogous behavioural pattern of various estimates. A systematic empirical investigation of these series could not be carried out due to limited computational facilities and is not presented here. It is well known that a thorough theoretical justification of these facts is intractable due to a lack of close form expression of parameter estimators.

Abraham and Chuang (1989) claim that in time series analysis, some suspected outliers may have large residual but may not affect the parameter estimates, whereas others may not only have large residuals but also may affect model specification and parameter estimation. While investigating the usefulness of various estimators in outlier diagnostic procedures, Bruce and Martin (1989) show that the diagnostics based on error variance has better properties than the diagnostics based on time series parameters. The use of error variance to detect the presence of outlier in the time series has also been emphasized by Ledotler (1990). Based on the empirical study presented here, it can be concluded that the error variance is an appropriate statistic to investigate the presence of outlier of either type. Henceforth, the present work focuses on studying the estimation of error variance in the presence of outliers in time series.

We now discuss the estimation of various parameters present in the model.

2.4 Estimation of Parameters

In addition to the time series parameters $\beta = (\phi', \theta')'$ where $\phi = (\phi_1, \phi_2, \ldots, \phi_p)'$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_q)'$ and the error variance σ_a^2, we have an outlier parameter ω from models in (2.1). We first discuss the maximum likelihood estimation of outlier parameter and error variance when the time series parameters and outlier position are assumed to be known. The maximum likelihood estimation of time series parameters for ARMA(p,q) is briefly discussed subsequently.

2.4.1 Estimation of ω and σ_a^2

Consider the observed series $\{Y_t, t \in \tau\}$ in (2.1). The outlier position T and the parameters β of time series model are known. Chang et al. (1988) discussed least squares estimates of outlier parameters in ARIMA(p,d,q) model with AO and IO types of outliers (also see Box et al., 1994, p. 470-471). Ljung (1993) later established that the derived estimators are maximum likelihood estimators. We here briefly discuss the maximum likelihood estimation of ω and σ_a^2.

Consider the residual series defined in (2.2). Using (2.4), it can be expressed as

$$
\begin{aligned}
e_t &= a_t + \omega\,\pi(B)\xi_t^{(T)}, \qquad t \in \tau \quad \text{for AO} \\
e_t &= a_t + \omega\,\xi_t^{(T)}, \qquad t \in \tau \quad \text{for IO}
\end{aligned}
\tag{2.12}
$$

where a_t, $t = 1, 2, \ldots, n$ are i.i.d. normal with mean zero and variance σ_a^2 and ω is the outlier parameter. The joint probability density function of $a_1, a_2, \ldots, a_n$ is

$$
f(a_1, a_2, \ldots, a_n \mid \omega, \sigma_a^2, \beta) \propto (\sigma_a^2)^{-n/2} \exp[-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n} a_t^2]. \tag{2.13}
$$

For known β and T we express the likelihood function in terms of a_t, which is

$$
L(\omega, \sigma_a^2 \mid \beta, T, e) \propto (\sigma_a^2)^{-n/2} \exp[-SS(\omega)/2\sigma_a^2] \tag{2.14}
$$

where $SS(\omega)$ is the error sum of squares $\sum_{t=1}^{n} a_t^2$ and $e = (e_1, e_2, \ldots, e_n)'$. The error sum of squares will differ depending on the outlier type.

For AO model, from (2.12), the error sum of squares is

$$
SS_A(\omega) = \sum_{t=1}^{n} \{e_t - \omega\pi(B)\xi_t^{(T)}\}^2 \tag{2.15}
$$

which gives the log likelihood function

$$\ell(\omega,\sigma_a^2 \mid \beta, T, e) \propto -\frac{n}{2}\ln\sigma_a^2 - \frac{1}{2\sigma_a^2}\sum_{t=1}^{n}\left\{e_t - \omega\pi(B)\xi_t^{(T)}\right\}^2. \tag{2.16}$$

Thus, we get $\hat{\omega}_{A,T}$, the maximum likelihood estimator of the outlier parameter ω under AO model when the outlier position is T, given by

$$\hat{\omega}_{A,T} = \frac{\pi(F)e_T}{\sum_{j=0}^{n-T}\pi_j^2} \tag{2.17}$$

where $\pi(F) = \sum_{j=0}^{n-T} -\pi_j F^j$, $\pi_0 = -1$ and F is a forward-shift operator given by $Fe_t = e_{t+1}$. Further the maximum likelihood estimator of error variance $\hat{\sigma}_a^2$ is

$$\hat{\sigma}_{a,A}^2 = \frac{SS_A(\hat{\omega}_{A,T})}{n} = \frac{1}{n}\sum_{t=1}^{n}\left\{e_t - \hat{\omega}_{A,T}\pi(B)\xi_t^{(T)}\right\}^2. \tag{2.18}$$

We can analogously estimate the outlier parameter in the presence of an IO at time T. The error sum of squares for IO model from (2.12) is

$$SS_I(\omega) = \sum_{t=1}^{n} a_t^2 = \sum_{t=1}^{n} (e_t - \omega\xi_t^{(T)})^2 \tag{2.19}$$

and the corresponding log likelihood function is

$$\ell(\omega, \sigma_a^2 \mid \beta, T, e) \propto -\frac{n}{2}\ln\sigma_a^2 - \frac{1}{2\sigma_a^2}\sum_{t\neq T}^{n} e_t^2 - \frac{1}{2\sigma_a^2}(e_T - \omega)^2. \tag{2.20}$$

Hence, the maximum likelihood estimator of ω for IO model is

$$\begin{aligned}\hat{\omega}_{I,T} &= e_T \\ &= \pi(B)\, Y_T. \end{aligned} \tag{2.21}$$

We can also get the maximum likelihood estimate of error variance for IO model, which is

$$\hat{\sigma}^2_{a,I} = \frac{SS_I(\hat{\omega}_{I,T})}{n} = \frac{1}{n}\sum_{t=1}^{n}(e_t - \hat{\omega}_{I,T}\,\xi_t^{(T)})^2. \tag{2.22}$$

Thus, we can estimate the outlier parameter under both types of models when the position of outlier and time series parameters are assumed to be known. We use these estimates of ω to appropriately adjust the series in accordance with the assumed outlier model in Section 2.6.

2.4.2 Estimation of Parameters of ARMA(p,q)

As mention in Section 2.2 the series $\{Z_t, t\in\tau\}$ can be treated as a stationary and invertible ARMA(p,q) process. We consider the parameter estimation of a typical ARMA(p,q) process. Let $\{Z_t, t\in\tau\}$ be given by

$$\phi(B)Z_t = \theta(B)a_t, \qquad t\in\tau$$

where $\{a_t, t\in\tau\}$ are i.i.d. normal with mean 0 and variance σ_a^2. Let $\beta = (\phi', \theta')'$ be the parameter vector. The likelihood function for the parameters β and σ_a^2 is given by

$$L(\beta, \sigma_a^2 \mid \mathbf{Z}) \propto (\sigma_a^2)^{-n/2}\,|\Sigma|^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}\,\mathbf{Z}'\Sigma^{-1}\mathbf{Z}\} \tag{2.23}$$

where $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)'$ and $\text{cov}(\mathbf{Z}) = \sigma_a^2\Sigma$, Σ a function of β.

The maximization of likelihood function is achieved by two methods, leading to conditional and unconditional likelihood function. The methods are discussed in detail in literature (Abraham and Ledolter, 1983, Section 5.6; Wei, 1990, Section 7.2; Box et al., 1994, Chapter 7). We briefly sketch the two existing methods here.

In the subsequent chapters and the package Statistical Time Series Diagnostic Software (STDS) which is provided along, we use the unconditional likelihood function for parameters estimation leading to unconditional least squares estimates, which provides very close approximations to maximum likelihood estimates (Box et al., 1994, p. 229).

For the estimation procedure under conditional likelihood function, we assume the initial condition $\mathbf{Z}_*$ for $\mathbf{Z}$ and $\mathbf{a}_*$ for $\mathbf{a}$ which are also called the starting values. Conditional on the choice of $\mathbf{Z}_*$ and $\mathbf{a}_*$ for a given data set $\mathbf{Z}$, the log likelihood associated with the parameters β and σ_a^2 is

$$\ell_*(\beta, \sigma_a^2 \,|\, \mathbf{Z}_*, \mathbf{a}_*, \mathbf{Z}) \propto -\frac{n}{2}\ln(\sigma_a^2) - \frac{1}{2\sigma_a^2} SS_*(\beta) \tag{2.24}$$

where $SS_*(\beta) = \sum_{t=1}^{n} a_t^2 (\beta \,|\, \mathbf{Z}_*, \mathbf{a}_*, \mathbf{Z})$ is the conditional error sum of squares.

The estimates $\hat{\beta}$ can now be obtained from minimizing the sum of squares (2.24) by standard nonlinear least squares methods. These are known as conditional LS estimators in the literature (Box et al., 1994, p. 227).

In unconditional likelihood (exact likelihood) function approach, the unconditional log likelihood function denoted by $\ell(\beta, \sigma_a^2 | \mathbf{Z})$ is

$$\ell(\beta, \sigma_a^2 | \mathbf{Z}) \propto f(\beta) - \frac{n}{2} \ln(\sigma_a^2) - \frac{1}{2\sigma_a^2} SS(\beta) \qquad (2.25)$$

where $f(\beta)$ is a function of parameters β which is $\ln |\Sigma|^{-1/2}$, and the unconditional sum of squares function is given by

$$SS(\beta) = \sum_{t=1}^{n} [a_t | \beta, \mathbf{Z}]^2 + [u_*]^T \Sigma^{-1} [u_*]. \qquad (2.26)$$

where $[a_t | \beta, \mathbf{Z}] = E[a_t | \beta, \mathbf{Z}]$ is the conditional expectation of a_t given $\mathbf{Z}$ and β and $u_* = (Z_{1-p}, \ldots, Z_{-1}, Z_0, a_{1-q}, \ldots, a_{-1}, a_0)^T$. Alternatively the sum of squares in (2.26) can be expressed as

$$SS(\beta) = \sum_{t=-\infty}^{n} [a_t | \beta, \mathbf{Z}]^2,$$

and, in practice, the sum of squares can be calculated approximately by using the finite sum as

$$SS(\beta) = \sum_{t=1-Q}^{n} [a_t | \beta, \mathbf{Z}]^2 = \sum_{t=1-Q}^{n} [a_t]^2 \qquad (2.27)$$

where Q is a sufficiently large integer. The parameter estimates are calculated by minimizing the sum of squares (2.27). The estimates are called the unconditional or exact LS estimates (Box et al., 1994, p. 229). In (2.24) and (2.27), the two sum of squares $SS_*(\beta)$ and $SS(\beta)$ are not quadratic functions of the parameters β when the order of the moving average parameters $q > 0$. Hence, a nonlinear least squares estimation procedure must be used to get the estimates by minimizing the corresponding sum of squares. As a result, these estimators are also called nonlinear LS estimators or Box_Jenkins (BJ) estimators in the literature (Bustos and Yohai, 1986). We refer to Box et al. (1994, Chapter 7) for further details.

For an observed time series, the LS estimates can be obtained using IMSL subroutines. For computations presented in the thesis, the subroutine NSPE is first used to get the preliminary parameter estimates. These estimates are used as initial values for the estimation subroutine NSLSE. The subroutine NSLSE may be used to compute conditional or unconditional least-squares estimates of the parameters, depending on the choice of the backcasting length. The algorithms of these routines follow the approach of Box and Jenkins (1976, programs 2-4, pp. 498-509) and are discussed in details in IMSL(1997, Chapter 8).

2.5 Critical View of Deletion Diagnostics

As mentioned in Sections 1.3 and 2.1, in deletion diagnostic methods for detection of outliers, each observation is treated as missing in turn and is replaced by its predicted value. The various methods used in deletion diagnostics for computing the predicted value are least squares predictors by Brubacher and Wilson (1976) (Peña, 1987) and maximum likelihood predictors (Ljung, 1993). Bruce and Martin (1989) used Kalman filter estimates of parameters with missing values. These proposed deletion diagnostic methods do not take into account the type of outlier and for both AO as well as IO, same diagnostic technique is applied.

Various authors have raised doubts about deletion diagnostic methods in time series. In particular, in a discussion on Bruce and Martin (1989), Tong and Lawrence questioned "Is there a simpler but equally effective way of detective way of detecting outliers in time series without deletion?" and "Is deletion an appropriate way to handle diagnostics for time series?" respectively. Tsay in the same discussion was more forthright when he said "Deletion of all aberrant observations might not be the optimal way in time series ...". A hypothetical situation where an outlying observation (Figure 2.1) is treated as a missing value and replaced with its predicted value is shown in Figure 2.9.

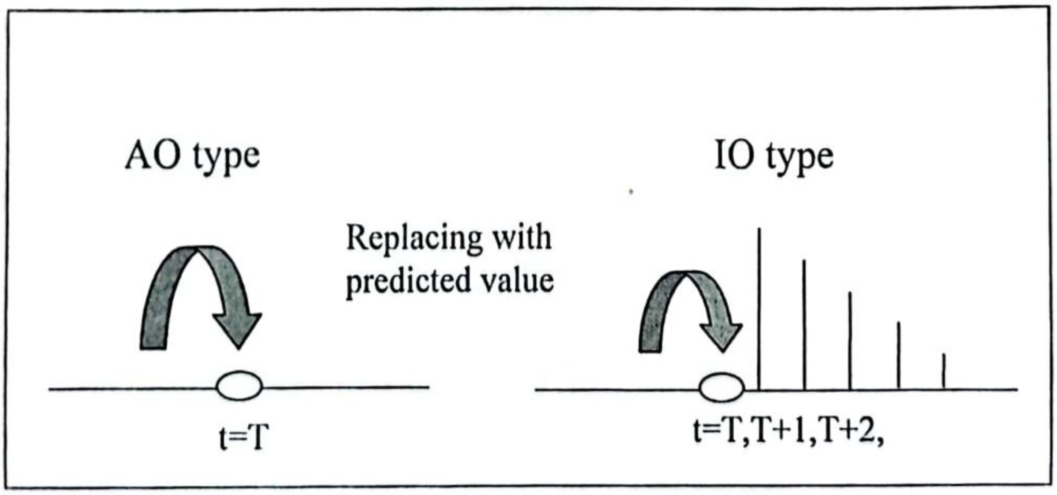


Figure 2.9: Treatment of Missing Value in $\{Y_t, t \in \tau\}$ Using Deletion Diagnostic Methods

We note two important points about the deletion diagnostic which substitutes an observation by its predictor.

Missing values estimation methods discussed above are so-called two-sided predictor because a predictor is computed from past and future values (Schmid, 1996). Thus the predicted value is often a weighted sum of adjacent observations as in case of least square predictors (Ledolter, 1990). Hence in the presence of an IO, the predicted value will be contaminated by the observations which are already affected by an IO. In the presence of AO, such a predictor will not be contaminated provided there is no outlier in the immediate neighborhood and the deletion diagnostic may work satisfactorily. Secondly, as can be seen from Figure 2.9, substituting the predicted value does not take care of the carry over effect of an IO on the subsequent observations. The outlier effect on the observations at the time points T+1, T+2, ..., etc still remains. Schmid (1996) also comments that "... testing for the existence of innovation outliers, tests using a *one-sided predictor* seem to be more appropriate."

While commenting on the analysis of time series in the presence of outliers, Chatfield (1989, p. 6) states "... the treatment of outliers is a complex subject in which common sense is as important as theory ... the outlier needs to be adjusted in some way before further analysis of the data". Noting these points of deletion diagnostics, it seems appropriate to adjust the effect of outlier in a proper way in order to detect the outlier and carry out further analysis. We now propose the adjustment of the outlier effect according to the underlying model which leads to an improved outlier detection procedure.

2.6 Series Adjustment to Handle the Outlier Effect

For the observed outlier series $\{Y_t, t\in\tau\}$, we consider the two single outlier models given by (2.1)

$$Y_t = Z_t + \omega\xi_t^{(T)}, \qquad t\in\tau \quad \text{for AO}$$

$$Y_t = Z_t + \omega\psi(B)\xi_t^{(T)}, \qquad t\in\tau \quad \text{for IO.} \tag{2.28}$$

We first consider the situation where the time series parameters of the underlying process $\{Z_t, t\in\tau\}$ are assumed known.

If, in addition to the type of outlier, ω and the outlier position T is also known, the unobserved outlier free series $\{Z_t, t\in\tau\}$ can easily be traced from the observed series using (2.28). If ω is unknown but the outlier position T is known, the maximum likelihood estimators of ω for both outlier types given by (2.17) and (2.21) can be considered and the effect of outlier can be 'removed' from the observed series by adjusting it using (2.28).

We extend this argument and consider the situation where both ω and T are unknown. Since T is unknown, we consider each time point $i\in\tau$ in turn and assume it to be the time point at which the outlier of one of the two types is present. Using (2.17) and (2.21), we can analogously obtain the maximum likelihood estimators of ω under both AO and IO outlier models for each $i\in\tau$, given by

$$\hat{\omega}_{A,i} = \frac{\pi(F)e_i}{\sum_{j=0}^{n-i}\pi_j^2}$$

and

$$\hat{\omega}_{I,i} = e_i\,, \tag{2. 29}$$

respectively, where the notation $\hat{\omega}_{S,i}$ stands for the maximum likelihood estimator of the outlier parameter ω when the outlier of type S is present at time point 'i', S = AO or IO, which we denote by A or I respectively and $i \in\tau$.

Using the estimated outlier parameter, we propose to 'remove' the effect of outlier by adjusting the observed series as follows.

Let $\{\ddot{Y}_{t(i),S},\ t\in\tau\}$ be the adjusted series corresponding to the observed series $\{Y_t,\ t\in\tau\}$ where $\ddot{Y}_{t(i),S}$ is the adjusted value of the original observation Y_t, adjusted by assuming that the outlier is present at i and is of the type S, where $i \in\tau$ and S = A or I. Henceforth i is called the *adjustment position*. For all $t\in\tau$, $Y_{t(i)}$ is defined by

$$\ddot{Y}_{t(i),A} = Y_t - \hat{\omega}_{A,i}\,\xi_t^{(i)} \qquad \text{for AO,}$$

$$\ddot{Y}_{t(i),I} = Y_t - \hat{\omega}_{I,i}\,\psi(B)\xi_t^{(i)} \qquad \text{for IO.} \tag{2.30}$$

Thus, corresponding to each time point i, we get the adjusted series $\{\ddot{Y}_{t(i),A},\ t\in\tau\}$ adjusted at position i for an AO type of outlier. We call this series AO adjusted series at adjustment position i. Analogously, $\{\ddot{Y}_{t(i),I},\ t\in\tau\}$ is called the IO adjusted series at adjustment position i.

Since all time points are considered in turn, for i = T we get the true position of outlier, which leads to correct adjustment position for the corresponding type of outlier. We define, for all $t\in\tau$,

$$\tilde{Z}_{t,A} = \ddot{Y}_{t(T),A} \qquad \text{for AO,}$$

$$\tilde{Z}_{t,I} = \ddot{Y}_{t(T),I} \qquad \text{for IO.} \qquad (2.31)$$

Note that these adjustments lead to predicted outlier free series which are denoted by $\{\tilde{Z}_{t,A},\ t\in\tau\}$ and $\{\tilde{Z}_{t,I},\ t\in\tau\}$ for AO and IO respectively when the time series parameters are known.

We now consider the two adjusted series $\{\ddot{Y}_{t(i),A},\ t\in\tau\}$ and $\{\ddot{Y}_{t(i),I},\ t\in\tau\}$ and illustrate how adjustment takes care of the effect of outlier in a stationary and invertible ARMA(p,q) model for both types of outliers at the correct adjustment position T. We consider the two models for two types of outliers separately assuming for the present that the outlier type is known.

2.6.1 Series Adjusted for AO

Suppose the observed series contains an AO at time point T. Then the series $\{\ddot{Y}_{t(i),A},\ t\in\tau\}$ for i = T is the AO adjusted series at adjustment position T, which from (2.30) is,

$$\ddot{Y}_{t(T),A} = \begin{cases} Y_t - \hat{\omega}_{A,T} & \text{for } t = T \\ Y_t & \text{for } t \neq T \end{cases} \tag{2.32}$$

where $\hat{\omega}_{A,T}$ can be obtained from (2.29) using $t = T$. Thus the AO adjusted series $\{\ddot{Y}_{t(T),A},\ t \in \tau\}$ is the same as the observed series $\{Y_t,\ t \in \tau\}$ at all time points $t \neq T$.

Consider the adjusted observation at time point T,

$$\ddot{Y}_{T(T),A} = Y_T - \hat{\omega}_{A,T}$$

$$= Y_T - \frac{\pi(F)e_T}{\sum_{j=0}^{n-T} \pi_j^2}.$$

Since $e_T = \pi(B)Y_T$,

$$\ddot{Y}_{T(T)A} = Y_T - \frac{\pi(F)\,\pi(B)Y_T}{\sum_{j=0}^{n-T} \pi_j^2}.$$

Without loss of the generality, we rewrite the expression as

$$\ddot{Y}_{T(T)A} = Y_T - \frac{\pi(F)\,\pi(B)Y_T}{\sum_{j=0}^{\infty} \pi_j^2} \tag{2.33}$$

Since $F = B^{-1}$ and the series is assumed to be stationary and invertible, we get,

$$\pi(F)\,\pi(B) = \pi(B^{-1})\,\pi(B)$$

$$= (1-\pi_1 B^{-1}-\pi_2 B^{-2}-\pi_3 B^{-3}- \dots)(1-\pi_1 B-\pi_2 B^2-\pi_3 B^3- \dots)$$

$$= (1+\pi_1^2+\pi_2^2+\pi_3^2+\dots)B^0$$

$$+ (-\pi_1+\pi_1\pi_2+\pi_2\pi_3+\dots)B^1 + (-\pi_1+\pi_1\pi_2+\pi_2\pi_3+\dots)B^{-1}$$

$$+ (-\pi_2+\pi_1\pi_3+\pi_2\pi_4+\dots)B^2 + (-\pi_2+\pi_1\pi_3+\pi_2\pi_4+\dots)B^{-2} + \dots$$

$$= (\sum_{i=0}^{\infty} \pi_i^2) B^0 + (\sum_{i=0}^{\infty} \pi_i \pi_{i+1}) B + (\sum_{i=0}^{\infty} \pi_i \pi_{i+1}) B^{-1} + (\sum_{i=0}^{\infty} \pi_i \pi_{i+2}) B^2$$

$$+ (\sum_{i=0}^{\infty} \pi_i \pi_{i+2}) B^{-2} + \ldots$$

$$= \sum_{k=-\infty}^{\infty} d_k B^k$$

where $d_k = d_{-k} = \sum_{i=0}^{\infty} \pi_i \pi_{i+k}$ and $\pi_0 = -1$. (2.34)

Hence, on substituting (2.34) in (2.33), the adjusted observation at time t = T after adjustment by AO is

$$\ddot{Y}_{T(T),A} = Y_T - \frac{\left(\sum_{k=-\infty}^{\infty} d_k B^k \right) y_T}{d_0}$$

$$= \frac{1}{d_0} \left(d_0 - \sum_{k=-\infty}^{\infty} d_k B^k \right) y_T$$

$$= - d_0^{-1} \sum_{k \neq 0} d_k \, Y_{T-k}$$

which can equivalently be written as

$$\ddot{Y}_{T(T),A} = - d_0^{-1} \sum_{k \neq 0} d_k \, Y_{T+k}. \tag{2.35}$$

The right hand side of (2.35) is the same as the least squares estimator of a single missing value (Brubacher and Wilson, 1976) as well as maximum likelihood estimator of a single missing value (Ljung, 1989). Thus in case of AO, the series adjustment procedure leads to the deletion and missing value estimation at position T using least squares estimate when the time series parameters are assumed known.

To illustrate, consider the observed series $\{Y_t, t \in \tau\}$ which is the contaminated version of an AR(1) series with an AO at T. Then on equating the coefficient of B in $\pi(B) = 1 - \phi B$ we get

$$\pi_1 = \phi$$

$$\pi_j = 0 \text{ for } j>1.$$

Hence,

$$\pi(F) = 1 - \phi F \text{ and } \sum_{j=0}^{n-T} - \pi_j^2 = 1 + \phi^2.$$

Using the first expression of (2.29) for i = T, the estimated outlier parameter is

$$\hat{\omega}_{A,T} = \frac{1}{1+\phi^2}(e_T - \phi\, e_{T+1})$$

where $e_t = \pi(B)Y_t = Y_t - \pi_1 Y_{t-1}$. Hence

$$\hat{\omega}_{A,T} = \frac{1}{1+\phi^2}(Y_T - \phi Y_{T-1} - \phi Y_{T+1} + \phi\, Y_T).$$

Therefore, the adjusted value at time point T is

$$\ddot{Y}_{T(T),A} = Y_T - \hat{\omega}_{A,T}$$

$$= \frac{\phi}{1+\phi^2}(Y_{T-1} + Y_{T+1}) \qquad (2.36)$$

which is the missing value estimator (Brubacher and Wilson, 1976; Ljung 1993; Box et al., 1994, p. 479).

Instead, if we consider any adjustment position i ≠ T, analogous to the argument above, the adjusted observation $Y_{t(i)}$ of the adjusted ARMA(*p,q*) series can be expressed as

$$\ddot{Y}_{t(i),A} = -d_0^{-1} \sum_{k \neq 0} d_k \, Y_{i+k}$$

giving the missing value estimate at point i.

In general, we get the adjusted series $\{ \ddot{Y}_{t(i),A}, t \in \tau \}$ for all $i \in \tau$ as

$$\ddot{Y}_{t(i),A} = \begin{cases} Y_t & \text{for } t < i \\ -d_0^{-1} \sum_{k \neq 0} d_k \, Y_{i+k} & \text{for } t = i \\ Y_t & \text{for } t > i. \end{cases} \tag{2.37}$$

Thus, we conclude that in the presence of AO, adjusting the series is equivalent to deleting the observation and replacing it by its least square estimate. As a result, the diagnostic method based on the adjusted series (*Adjustment Diagnostic*) will be same as the diagnostic method based on the missing value estimation (*Deletion Diagnostic*) when the outlier is AO.

The observed series $\{Y_t, t \in \tau\}$, however, can have either an AO or IO which is unknown, say. In such a situation, suppose the series is adjusted considering only the AO type of outlier, giving the adjusted series

$$\ddot{Y}_{t(i)A} = [\, Z_t + \omega\, \xi_t^{(T)} \,] - \hat{\omega}_{A,i}\, \xi_t^{(i)}, \qquad t,i \in \tau \quad \text{for AO} \tag{2.38}$$

$$\ddot{Y}_{t(i)A} = [\, Z_t + \omega\, \psi(B) \xi_t^{(T)} \,] - \hat{\omega}_{A,i}\, \xi_t^{(i)}, \qquad t,i \in \tau \quad \text{for IO.} \tag{2.39}$$

where T is the correct outlier position and i is the adjustment position which may or may not be correct. From (2.38), it can be seen that the AO adjustment leads to the reduction of the effect of ω in AO model when i = T and estimated value of $\hat{\omega}_{A,T}$ is close to $\omega_{A,T}$. When i $\neq$ T, the adjustment may give rise to a 'new' AO at time point i.

If instead, the outlier is IO, then as can be seen from (2.39), AO adjustment does not necessarily reduce the outlier effect, even in case of the correct adjustment position. For correct adjustment position and known outlier parameter, some 'carry-over' effect similar to a new IO type of outlier with outlier parameter $\omega^* = \omega\psi_1$ at T+1 still remains and the effects on the succeeding observations are $\omega^* \frac{\psi_2}{\psi_1}, \omega^* \frac{\psi_3}{\psi_1}, \ldots, \omega^* \frac{\psi_j}{\psi_1}, \ldots$. Thus the deletion diagnostics and 'missing value estimation' may not work well in case of IO.

2.6.2 Series Adjusted for IO

We now suppose that the observed series contains an IO at time point T. The IO adjusted series obtained on using the correct adjustment position i = T is, from (2.30),

$$\ddot{Y}_{t(T),I} = \begin{cases} Y_t & \text{for } t < T \\ Y_t - \hat{\omega}_{I,T} & \text{for } t = T \\ Y_t - \hat{\omega}_{I,T}\,\psi_{t-T} & \text{for } t > T \end{cases} \qquad (2.40)$$

where $\hat{\omega}_{I,T} = e_T = \pi(B)Y_T$. Hence,

$$\ddot{Y}_{T(T),I} = \pi_1 Y_{T-1} + \pi_2 Y_{T-2} + \pi_3 Y_{T-3} + \ \ldots \tag{2.41}$$

and the right hand side of (2.41) is a linear function of observations up to T−1, which are not affected by the presence of an IO at T. Further, it is same as the "lead 1 minimum mean squares error (MMSE) forecast" at origin T−1 (Box et al., 1994, pp. 148-149). Thus the adjusted observation at the correct time point T is the same as the lead 1 MMSE forecast value of the observation, treating the observation as missing. In addition, from (2.40) we get, for t>T

$$\ddot{Y}_{t(T),I} = Y_t - \hat{\omega}_{I,T}\,\psi_{t-T}.$$

Thus the subsequent observations also get adjusted. Further, for all t > T, the adjusted observation at time point t is a linear function of the original observation at t and the observations before T.

Even when the adjustment is not carried out at the correct position ($i \neq T$), analogous conclusions can be drawn. Hence, in general, the series adjustment in the presence of an IO is equivalent to replacing the value at the adjustment position by its lag 1 MMSE forecast at the adjustment position and subsequent values by appropriate linear functions obtained from (2.41) on replacing T by i. In some sense, proposed series adjustment in the presence of an IO has a 'smoothing' effect on the observed series.

In general, we get the adjusted series $\{Y_{t(i),I}, t\in\tau\}$ for all $i\in\tau$ as

$$\ddot{Y}_{t(i),I} = \begin{cases} Y_t & \text{for } t < i \\ \sum\limits_{j=1} \pi_j Y_{i-j} & \text{for } t = i \\ Y_t + \psi_{t-i} \sum\limits_{j=0} \pi_j Y_{i-j} & \text{for } t > i. \end{cases} \tag{2.42}$$

We now consider the IO adjustment of the series at i for both types of outlier models, which is for $t, i\in\tau$

$$\ddot{Y}_{t(i),I} = [\, Z_t + \omega\, \xi_t^{(T)}\,] - \hat{\omega}_{I,i}\, \psi(B)\xi_t^{(i)} \qquad \text{for AO,} \tag{2.43}$$

$$\ddot{Y}_{t(i),I} = [\, Z_t + \omega\, \psi(B)\xi_t^{(T)}\,] - \hat{\omega}_{I,i}\, \psi(B)\xi_t^{(T)} \qquad \text{for IO.} \tag{2.44}$$

From (2.43) it can be seen that in the presence of AO at time point T, the IO adjustment at i=T removes the effect of outlier at T, but also adjusts subsequent observations, which is not required. Thus the adjustment for IO in the presence of AO may not work satisfactorily. This issue is investigated in details in Chapter 3.

Also, (2.44) is the predicted series defined in (2.31) and for an estimate of ω close to the actual value of the parameter, the predicted series will be close to the unobserved outlier free series $\{Z_t, t\in\tau\}$.

Based on Subsections 2.6.1 and 2.6.2, it can be concluded that the series adjustment for a series with AO is equivalent to missing value estimation using least squares predictor and the series adjustment for a series with IO is equivalent to missing value estimation using lead 1 MMSE forecast, where the value at the adjustment point is treated as missing. The later observation is consistent with the observation made by Chen and Liu (1993). Thus the proposed series adjustment appropriately handles the two types of outliers separately.

However, in addition to lead 1 MMSE forecast, the subsequent observations also get adjusted in the presence of IO at time point i when the series is adjusted for IO (ref. (2.40)). As discussed in Section 2.5, in the presence of an IO at i, the subsequent observations also get affected and the proposed adjustment addresses this problem. The conclusions drawn here are under the assumption that the time series parameters are known and appropriate parameter estimation needs to be considered. Further investigations into the effect of series adjustment on the parameter estimation are postponed to Chapter 3.

Chapter 3

Adjustment Impact on Estimated Error Variance

3.1 Introduction

In the previous chapter, adjustment of the series is proposed to handle the presence of outliers. Since the position of outlier can rarely be assumed known, the series adjustment is proposed to be carried out at all adjustment positions in turn. In this chapter we investigate the impact of adjusted series on the estimate of error variance. Section 3.2 gives the estimates of error variance σ_a^2 based on the predicted outlier free series for the given outlier position of AO and IO types of outliers. Further, it establishes the relationship between these estimates and the estimate of σ_a^2 based on the observed outlier contaminated series when the presence of outlier is ignored. Sections 3.3 and 3.4 investigate and comment on the effect of adjustment on the estimate of error variance σ_a^2 due to correct and incorrect adjustment positions. The impact of incorrect type adjustment is discussed in Section 3.5. The simulation study is carried out to obtain the bias due to incorrect adjustment type. To illustrate these effects, an empirical study of AR(1) and MA(1) is presented in Section 3.6.

3.2 Estimate of Error Variance for Known Outlier Position

As discussed in Chapter 2, the presence of outlier significantly affects the estimate of error variance, irrespective of the type of outlier present in the series. Also, the impact of outlier on the time series parameters is not as serious and as has been proposed in the literature, the estimation of time series parameters $\beta = (\phi', \theta')'$ can be carried out based on the observed series itself, treating it as an outlier free series (Abraham and Ledolter, 1983, Section 8.2.2; Tsay, 1986; Chang et al., 1988; Wei, 1990, Section 9.3; Box et al., 1994, Section 12.2). Hence in this and the remaining sections we focus on the estimation of error variance σ_a^2.

Consider the outlier free series $\{Z_t, t \in \tau\}$, the corresponding error series $\{a_t, t \in \tau\}$ with error variance σ_a^2 and the observed series $\{Y_t, t \in \tau\}$ where we assume that an outlier occurs at a known position T. Thus we have from (2.3)

$$a_t = \pi(B) Z_t , \qquad t \in \tau$$

and from (2.2)

$$e_t = \pi(B) Y_t , \qquad t \in \tau.$$

(2.1) gives the following relationship between Z_t and Y_t

$$Z_t = Y_t - \omega\, \xi_t^{(T)} , \qquad t \in \tau \quad \text{for AO}$$

$$Z_t = Y_t - \omega\, \psi(B) \xi_t^{(T)}, \qquad t \in \tau \quad \text{for IO} \qquad (3.1)$$

where T is the correct outlier position in the observed series $\{Y_t, t \in \tau\}$. Also, (2.4) gives

$$a_t = e_t - \omega \pi(B) \xi_t^{(T)}, \qquad t \in \tau \quad \text{for AO}$$

$$a_t = e_t - \omega\xi_t^{(T)}, \qquad t\in\tau \qquad \text{for IO} . \tag{3.2}$$

We now consider the situation when the time series parameters, outlier parameter and error variance are unknown and T is known. Let $\hat{\phi}(B)$ and $\hat{\theta}(B)$ be the maximum likelihood estimators of $\phi(B)$ and $\theta(B)$ based on the observed series $\{Y_t, t\in\tau\}$ (Section 2.4.2). As mentioned above, we treat $\{Y_t, t\in\tau\}$ as an outlier free series for these estimates.

We consider the estimates of $\pi(B)$ and $\psi(B)$ given by (Chang et al., 1988)

$$\hat{\pi}(B) = \frac{\hat{\phi}(B)}{\hat{\theta}(B)} \quad \text{and}$$

$$\hat{\psi}(B) = \frac{\hat{\theta}(B)}{\hat{\phi}(B)} . \tag{3.3}$$

Corresponding to the residual series $\{e_t, t\in\tau\}$, we construct the predicted residual series $\{\hat{e}_t, t\in\tau\}$ where

$$\hat{e}_t = \hat{\pi}(B)Y_t, \qquad t\in\tau \tag{3.4}$$

(Box et al, 1994, p. 472). Hence, using (2.17) and (2.21) we get the estimated outlier parameters

$$\hat{\omega}_{A,T} = \frac{\hat{\pi}(F)\hat{e}_T}{\sum_{j=0}^{n-T}\hat{\pi}_j^2} \qquad \text{for AO and}$$

$$\hat{\omega}_{I,T} = \hat{e}_T \qquad \text{for IO} \tag{3.5}$$

where $\hat{\pi}(F) = 1 - \hat{\pi}_1F - \hat{\pi}_2F^2 - \ldots - \hat{\pi}_{n-T}F^{n-T}$ and F is the forward-shift operator given by $F\hat{e}_t = \hat{e}_{t+1}$.

Using (3.1) and (3.5), we now define

$$Z^{*}_{t,A} = Y_t - \hat{\omega}_{A,T}\,\xi_t^{(T)}, \qquad t\in\tau \quad \text{for AO and}$$

$$Z^{*}_{t,I} = Y_t - \hat{\omega}_{I,T}\,\hat{\psi}(B)\,\xi_t^{(T)}, \qquad t\in\tau \quad \text{for IO} \tag{3.6}$$

where the series $\{Z^{*}_{t,A}, t\in\tau\}$ stands for the predicted outlier free series obtained by adjusting the observed series $\{Y_t, t\in\tau\}$ with an AO outlier, at the correct adjustment position of AO using the estimated outlier parameter $\hat{\omega}_{A,T}$. The series $\{Z^{*}_{t,I}, t\in\tau\}$ stands for the predicted outlier free series obtained by adjusting the observed series $\{Y_t, t\in\tau\}$ with an IO outlier, at the correct adjustment position of IO using the estimated outlier parameter $\hat{\omega}_{I,T}$. The predicted error series $\{\hat{a}_{t,s}, t\in\tau\}$ where S is either A or I is defined as

$$\hat{a}_{t,A} = \hat{\pi}(B)\,Z^{*}_{t,A}, \qquad t\in\tau \quad \text{for AO,}$$

$$\hat{a}_{t,I} = \hat{\pi}(B)\,Z^{*}_{t,I}, \qquad t\in\tau \quad \text{for IO} \tag{3.7}$$

where $\hat{\pi}(B)$ is given in (3.3).

Hence, on substituting in (3.7) from (3.6) and (3.4), the predicted error series can be expressed in terms of the predicted residual series as

$$\hat{a}_{t,A} = \hat{e}_t - \hat{\omega}_{A,T}\,\hat{\pi}(B)\,\xi_t^{(T)}, \qquad t\in\tau \quad \text{for AO}$$

$$\hat{a}_{t,I} = \hat{e}_t - \hat{\omega}_{I,T}\,\xi_t^{(T)}, \qquad t\in\tau \quad \text{for IO.} \tag{3.8}$$

Using these errors series, we get

$$\hat{\sigma}^2_{a,A} = \frac{1}{n}\sum_{t=1}^{n}\hat{a}^2_{t,A} \quad \text{and}$$

$$\hat{\sigma}^2_{a,I} = \frac{1}{n}\sum_{t=1}^{n}\hat{a}^2_{t,I} \tag{3.9}$$

where $\hat{\sigma}^2_{a,A}$ and $\hat{\sigma}^2_{a,I}$ are the estimated error variances based on predicted outlier free series $\{Z^{*}_{t,A},\ t\in\tau\}$ and $\{Z^{*}_{t,I},\ t\in\tau\}$ respectively. These are maximum likelihood estimates of σ^2_a for given time series parameters (Ljung, 1993). For given β, the maximum likelihood estimate of $\sigma_a{}^2$ obtained on ignoring the outlier is denoted by $\hat{\sigma}^2_e$, and is given by

$$\hat{\sigma}^2_e = \frac{1}{n}\sum_{t=1}^{n}\hat{e}^2_t\ . \tag{3.10}$$

Henceforth, $\hat{\sigma}^2_e$ is referred to as the *residual variance.* This estimate of error variance is presented in Section 2.4 where the presence of outlier in the observed series is ignored.

(3.8), (3.9) and (3.10) lead to the relationships between the three estimates of error variance which are derived below in case of AO and IO separately.

For AO:

Consider the first expression in (3.8), which is

$$\hat{a}_{t,A} = \hat{e}_t - \hat{\omega}_{A,T}\,\hat{\pi}(B)\,\xi_t^{(T)}, \qquad t\in\tau$$

where $\hat{\pi}(B) = \sum_{j=0}^{\infty} -\hat{\pi}_j B^j$ and $\hat{\pi}_0 = -1$. The sum of squares of predicted errors based on the predicted AO outlier free series is

$$\sum_{t=1}^{n}\hat{a}_{t,A}^2 = \sum_{t=1}^{n}\left[\hat{e}_t^2 + \hat{\omega}_{A,T}^2\left\{\sum_{j=0}^{\infty} -\hat{\pi}_j\,\xi_{t-j}^{(T)}\right\}^2 - 2\hat{\omega}_{A,T}\left\{\sum_{j=0}^{\infty} -\hat{\pi}_j\,\xi_{t-j}^{(T)}\right\}\hat{e}_t\right]$$

$$= \sum_{t=1}^{n}\left[\hat{e}_t^2 + \hat{\omega}_{A,T}^2\left\{\sum_{j=0}^{\infty}\hat{\pi}_j^2\,\xi_{t-j}^{(T)} + \sum_{i\neq j}^{\infty}\sum\hat{\pi}_i\hat{\pi}_j\xi_{t-i}^{(T)}\,\xi_{t-j}^{(T)}\right\} - 2\hat{\omega}_{A,T}\left\{\sum_{j=0}^{\infty} -\hat{\pi}_j\,\xi_{t-j}^{(T)}\right\}\hat{e}_t\right]$$

$$= \sum_{t=1}^{n}\hat{e}_t^2 + \hat{\omega}_{A,T}^2\sum_{j=0}^{n-T}\hat{\pi}_j^2 - 2\,\hat{\omega}_{A,T}\,\hat{\pi}(F)\hat{e}_T$$

$$= \sum_{t=1}^{n}\hat{e}_t^2 + \hat{\omega}_{A,T}^2\;\hat{\eta}^2 - 2\,\hat{\omega}_{A,T}\,\hat{\pi}(F)\hat{e}_T$$

where $\hat{\pi}(F)$ is as specified earlier and $\hat{\eta}^2 = \sum_{j=0}^{n-T}\hat{\pi}_j^2$ which is greater than 1. Using (3.5), we get

$$\sum_{t=1}^{n}\hat{a}_{t,A}^2 = \sum_{t=1}^{n}\hat{e}_t^2 - \hat{\omega}_{A,T}^2\;\hat{\eta}^2.$$

It gives

$$\hat{\sigma}_{a,A}^2 = \hat{\sigma}_e^2 - \frac{1}{n}\hat{\omega}_{A,T}^2\hat{\eta}^2. \qquad (3.11)$$

The term $\frac{1}{n}\hat{\omega}_{A,T}^2\hat{\eta}^2$ is the bias in the estimate of error variance (Ljung, 1993) if the presence of outlier is ignored and it is clear that the error variance gets overestimated in the presence of an outlier of AO type, supporting the findings presented in Table 2.1 and 2.3 for AR(1) and MA(1) respectively.

For IO:

The second expression in (3.8) is

$$\hat{a}_{t,I} = \hat{e}_t - \hat{\omega}_{I,T}\, \xi_t^{(T)} \quad , \qquad t \in \tau$$

using which and (3.5), the sum of squares of predicted errors based on the outlier free series is

$$\sum_{t=1}^{n} \hat{a}_{t,I}^2 = \sum_{t=1}^{n} \left[\hat{e}_t^2 + \hat{\omega}_{I,T}^2 \xi_t^{(T)} - 2\hat{\omega}_{I,T} \xi_t^{(T)} \hat{e}_t \right]$$

$$= \sum_{t=1}^{n} \hat{e}_t^2 + \hat{\omega}_{I,T}^2 - 2\hat{\omega}_{I,T} \hat{e}_T$$

$$= \sum_{t=1}^{n} \hat{e}_t^2 - \hat{\omega}_{I,T}^2 .$$

Hence, we have

$$\hat{\sigma}_{a,I}^2 = \hat{\sigma}_e^2 - \frac{1}{n} \hat{\omega}_{I,T}^2 . \tag{3.12}$$

In (3.12), the term $\frac{1}{n}\hat{\omega}_{I,T}^2$ is the bias in estimate of error variance and it indicates that the error variance is overestimated due to the presence of an IO in observed series at T. It also supports the findings of Tables 2.2 and 2.4 for AR(1) and MA(1) respectively.

Based on (3.11) and (3.12), it can be concluded that irrespective of the type outlier, bias in the estimate of error variance does not depend on the sign of the estimate of outlier parameter ω, and increase in bias is proportional to the square of the estimate of ω. Also the bias is inversely proportional to the series length.

Hence an outlier in smaller data set will affect the estimate of error variance more than that in a large data set, which has also been pointed out by Bruce and Martin (1989) using the influence function.

3.3 Estimate of Error Variance for Unknown Outlier Position

It is clear from the above discussion that the adjustment of observed series at the correct outlier position T is crucial to the analysis. In this section, we investigate the adjustment effect of an outlier on the estimation of error variance when T is unknown.

Since T is unknown, we consider the estimation of outlier parameter for each time point in turn. Let for $i \in \tau$ $\hat{\omega}_{A,i}$ and $\hat{\omega}_{I,i}$ be the estimates of outlier parameter ω based on the AO and IO model respectively, as given by (3.5), when the position of outlier is fixed at T, which is unknown. Hence, from (3.5), we analogously get the general expression for estimated outlier parameters at position i, $i \in \tau$ as

$$\hat{\omega}_{A,i} = \frac{\hat{\pi}(F)\hat{e}_i}{\sum_{j=0}^{n-T} \hat{\pi}_j^2} \qquad \text{for AO and}$$

$$\hat{\omega}_{I,i} = \hat{e}_i \qquad \text{for IO.} \qquad (3.13)$$

Further, suppose the observed series is adjusted based on the adjustment position 'i', which may or may not be the correct position. In order to investigate the relation between estimate of error variance based on the predicted outlier free series and estimate of error variance based on adjusted series, we consider two cases:

when $i = T$, i.e. the correct adjustment position

when $i \neq T$, i.e. the incorrect adjustment position

for both AO and IO types.

Using (2.30), the two adjusted series $\{Y_{t(i),A},\ t\in\tau\}$ and $\{Y_{t(i),I},\ t\in\tau\}$ with adjustment position i for two types AO and IO respectively are defined as

$$Y_{t(i),A} = Y_t - \hat{\omega}_{A,i}\xi_t^{(i)}, \qquad t\in\tau \quad \text{for AO}$$

$$Y_{t(i),I} = Y_t - \hat{\omega}_{I,i}\hat{\psi}(B)\xi_t^{(i)}, \qquad t\in\tau \quad \text{for IO.} \tag{3.14}$$

Note that since maximum likelihood estimates of parameters are used, the observations $Y_{t(i),S}$ are maximum likelihood estimates of adjusted observations $\ddot{Y}_{t(i),S}$ defined in (2.30) for S = A, I. For the sake of brevity, we avoid using notation $\ddot{Y}_{t(i),S}$, S = A, I for $Y_{t(i),S}$ to indicate the estimates of observations.

We define the two corresponding predicted residual series after adjustment at i by

$$\hat{e}_{t(i),A} = \hat{\pi}(B)\,Y_{t(i),A}, \qquad t\in\tau \quad \text{for AO}$$

$$\hat{e}_{t(i),I} = \hat{\pi}(B)\,Y_{t(i),I}, \qquad t\in\tau \quad \text{for IO} \tag{3.15}$$

and the corresponding estimates of error variance based on the series adjusted at i by

$$\hat{\sigma}^2_{e(i),A} = \frac{1}{n}\sum_{t=1}^{n}\hat{e}^2_{t(i),A} \quad \text{and} \quad \hat{\sigma}^2_{e(i),I} = \frac{1}{n}\sum_{t=1}^{n}\hat{e}^2_{t(i),I} \tag{3.16}$$

corresponding to AO and IO respectively.

3.3.1 Correct Position $(i = T)$

For $i = T$, we get the estimate of outlier parameter $\hat{\omega}_{S,T}$ given by (3.13) for S taking value A and I respectively.

AO Model:

Since $i = T$, on considering the presence of an additive outlier AO in time series $\{Y_t, t \in \tau\}$ at T and the estimated outlier parameter $\hat{\omega}_{A,T}$ for AO, the series gets adjusted at the correct position T. Hence, (3.14) and (3.15) give us the adjusted series

$$Y_{t(T),A} = Y_t - \hat{\omega}_{A,T}\,\xi_t^{(T)}\,, \qquad t \in \tau$$

which is same as $Z^*_{t,A}$ given by (3.6) and the predicted residual series after the adjustment for AO as

$$\hat{e}_{t(T),A} = \hat{\pi}(B)\,Y_{t(T),A}\,, \qquad t \in \tau$$

which is same as $\hat{a}_{t,A}$ given by (3.7) for AO. Hence, from (3.9) we get the estimated error variance based on the AO adjusted series, at adjustment position T, given by

$$\hat{\sigma}^2_{e(T),A} = \hat{\sigma}^2_{a,A}\,. \tag{3.17}$$

$\hat{\sigma}^2_{e(T),A}$ is called the "AO adjusted at T" estimate of error variance.

IO Model:

On the analogous lines, for the observed series with an IO at T, and i = T, from (3.14) and (3.15), we have the adjusted series

$$Y_{t(T),I} = Y_t - \hat{\omega}_{I,T}\ \hat{\psi}\ (B)\xi_t^{(T)}, \qquad t \in \tau$$

which is same as $Z^*_{t,I}$ given by (3.6) and the predicted residual series based on adjusted series for IO

$$\hat{e}_{t(T),I} = \hat{\pi}(B)\ Y_{t(T),I}, \qquad t \in \tau$$

which is also same as $\hat{a}_{t,I}$ in (3.7) for IO. Hence, the estimated error variance based on the IO adjusted series, at adjustment position T, is given by

$$\hat{\sigma}^2_{e(T),I} = \hat{\sigma}^2_{a,I}. \tag{3.18}$$

$\hat{\sigma}^2_{e(T),I}$ is called the "IO adjusted at T" estimate of error variance.

From (3.17) and (3.18), we can see that the adjusted estimates of error variance at T are same as the estimated error variances based on predicted outlier free series for both AO and IO.

3.3.2 Incorrect Position ($i \neq T$)

Let $\hat{\omega}_{A,i}$ and $\hat{\omega}_{I,i}$ be the estimated outlier parameter of ω given by (3.13) based on the AO and IO model respectively, where $i \neq T$.

AO Model:

The AO adjusted series at adjustment position i, from (3.14)

$$Y_{t(i),A} = Y_t - \hat{\omega}_{A,i}\,\xi_t^{(i)}, \qquad t \in \tau \text{ and } i \neq T \tag{3.19}$$

and the predicted residual series based on the AO adjusted series given by (3.15)is

$$\hat{e}_{t(i),A} = \hat{\pi}(B)\, Y_{t(i),A}, \qquad t \in \tau \text{ and } i \neq T. \tag{3.20}$$

On substituting (3.19) in (3.20), we get

$$\hat{e}_{t(i),A} = \hat{e}_t - \hat{\omega}_{A,i}\,\hat{\pi}(B)\,\xi_t^{(T)}, \qquad t \in \tau \text{ and } i \neq T.$$

Using (3.13) and $\hat{\pi}(B) = \sum_{j=0}^{\infty} -\hat{\pi}_j B^j$ where $\hat{\pi}_0 = -1$ we get the sum of squares of $\hat{e}_{t(i),A}$ is

$$\sum_{t=1}^{n} \hat{e}^2_{t(i),A} = \sum_{t=1}^{n} \hat{e}^2_t - \hat{\omega}^2_{A,i}\,\hat{\eta}^2, \qquad i \neq T$$

where $\hat{\eta}^2 = \sum_{j=0}^{n-T} \hat{\pi}_j^2$ which is greater than 1. It gives

$$\hat{\sigma}_{e(i),A} = \hat{\sigma}^2_e - \frac{1}{n}\hat{\omega}^2_{A,i}\hat{\eta}^2, \qquad i \neq T. \tag{3.21}$$

Note that the expression holds for i = T as well, as can be seen from (3.17) and (3.11). On substituting $\hat{\sigma}^2_e$ from (3.11), we have

$$\hat{\sigma}_{e(i),A} = \hat{\sigma}^2_{a,A} + \frac{1}{n}\left(\hat{\omega}^2_{A,T} - \hat{\omega}^2_{A,i}\right)\hat{\eta}^2, \qquad i \neq T. \tag{3.22}$$

From (3.22), it is clear that the “AO adjusted at i” estimate of error variance depends on the adjustment position. For $\hat{\omega}^2_{A,T}$ greater than $\hat{\omega}^2_{A,i}$, the estimate of error variance based on adjusted series using incorrect adjustment position is expected to be larger than that based on adjusted series using correct adjustment position, i.e., $\hat{\sigma}^2_{e(i),A} > \hat{\sigma}^2_{e(T),A}$ for all $i \neq T$. Though $\hat{\omega}^2_{A,T}$ cannot necessarily be greater than $\hat{\omega}^2_{A,i}$, we expect it to be larger than $\hat{\omega}^2_{A,i}$ for large absolute values of the outlier parameter which will lead to an estimate of error variance being that could be relatively smaller when it is based on the series adjusted at the correct adjustment position. We can also expect that the effect of an AO outlier will not be satisfactorily removed with AO adjustment at incorrect position $i \neq T$.

IO Model:

The IO adjusted series with adjustment position i given by (3.14) is

$$Y_{t(i),I} = Y_t - \hat{\omega}_{I,i}\, \hat{\psi}(B)\, \xi_t^{(i)}\,, \qquad t \in \tau \text{ and } i \neq T \tag{3.23}$$

and the corresponding adjusted residual series given by (3.15) is

$$\hat{e}_{t(i),I} = \hat{\pi}(B)\, Y_{t(i),I}, \qquad t \in \tau \text{ and } i \neq T \tag{3.24}$$

On substituting (3.23) in (3.24), we get

$$\hat{e}_{t(i),I} = \hat{e}_t - \hat{\omega}_{I,i}\, \xi_t^{(T)}, \qquad t \in \tau \text{ and } i \neq T$$

the sum of squares of which is

$$\sum_{t=1}^{n} \hat{e}^2_{t(i),I} = \sum_{t=1}^{n} \hat{e}^2_t - \hat{\omega}^2_{I,i}. \qquad i \neq T$$

It gives

$$\hat{\sigma}_{e(i),I} = \hat{\sigma}_e^2 - \frac{1}{n}\hat{\omega}_{I,i}^2 \qquad i \neq T \qquad (3.25)$$

which from (3.18) and (3.12) is the expression for i = T as well.

On substituting the value of $\hat{\sigma}_e^2$ from (3.12), we have

$$\hat{\sigma}_{e(i),I} = \hat{\sigma}_{a,I}^2 + \frac{1}{n}\left(\hat{\omega}_{I,T}^2 - \hat{\omega}_{I,i}^2\right) \qquad i \neq T. \qquad (3.26)$$

Analogous to (3.22), we can conclude from (3.26) that for $\hat{\omega}_{I,T}^2$ greater that $\hat{\omega}_{I,i}^2$, the estimate of error variance based on an adjusted series which is adjusted at correct position will be larger than that based on an adjusted series which is adjusted at correct position. For a large absolute of outlier parameter ω, $\hat{\omega}_{I,T}^2$ is expected to be greater than $\hat{\omega}_{I,i}^2$.

In conclusion, the estimated error variance based on adjusted series will still be biased if the adjustment is at an incorrect position.

3.4 Comments on Estimates of Error Variance

We have derived various estimates of error variance σ_a^2 based on different series adjustment in the earlier sections and established the relationships between them. In this section, we list these relationships for comparison. For the sake of completion, we first list all the estimates and then compare them for each type of outlier separately.

$\hat{\sigma}_e^2$ = the estimated error variance based on observed series (residual variance)

$\hat{\sigma}_a^2$ = the estimated error variance based on outlier free series (unavailable)

$\hat{\sigma}_{a,A}^2$ = the estimated error variance based on predicted AO outlier free series

$\hat{\sigma}_{a,I}^2$ = the estimated error variance based on predicted IO outlier free series

$\hat{\sigma}_{e(i),A}^2$ = the "AO adjusted at i" estimate of error variance, $i \in \tau$

$\hat{\sigma}_{e(i),I}^2$ = the "IO adjusted at i" estimate of error variance, $i \in \tau$

AO Model:

From (3.11), (3.17), and (3.22), we can see the changes in the estimates of error variance σ_a^2 in case of AO adjustment of the series at the correct and incorrect adjustment positions. The various estimates are –

Estimate based on observed series (ignoring the outlier) $\quad \hat{\sigma}_e^2 = \hat{\sigma}_{a,A}^2 + \frac{1}{n}\hat{\omega}_{A,T}^2\hat{\eta}^2$

Estimate based on adjusted series

Adjustment at T $\quad \hat{\sigma}_{e(T),A}^2 = \hat{\sigma}_{a,A}^2$

Adjustment at $i \neq T$ $\quad \hat{\sigma}_{e(i),A}^2 = \hat{\sigma}_{a,A}^2 + \frac{1}{n}\left(\hat{\omega}_{A,T}^2 - \hat{\omega}_{A,i}^2\right)\hat{\eta}^2$

In these expressions, $\hat{\sigma}_{a,A}^2$, $\hat{\sigma}_{e(T),A}^2$ and $\hat{\sigma}_{e(i),A}^2$ are the estimated error variance based on predicted AO outlier free series and the "AO adjusted at i" estimates of error variance at correct adjustment position T and incorrect adjustment position $i \neq T$ respectively. It can be seen that the $\hat{\sigma}^2_{e(T),A}$ is same as the estimated error variance based on predicted AO outlier free series, $\hat{\sigma}^2_{a,A}$ since the adjustment is at the correct position in the observed series. The adjustment at correct position T is expected to give us the minimum variance among the AO adjusted estimates of error variance for all $i \in \tau$, provided $\hat{\omega}^2_{A,T} > \hat{\omega}^2_{A,i}$ which can be expected to hold true for large values of $|\omega|$. For small values of $|\omega|$, however $\hat{\sigma}^2_{e(i),A}$ may be smaller than $\hat{\sigma}^2_{e(T),A}$.

IO Model:

From (3.12), (3.18), and (3.26), the changes in the estimates of error variance σ^2_a in case of IO adjustment of the series at the correct and incorrect adjustment positions can be represented as follow.

Estimate based on observed series (ignoring the outlier) $\quad \hat{\sigma}^2_e = \hat{\sigma}^2_{a,I} + \frac{1}{n}\hat{\omega}^2_{I,T}$

Estimate based on adjusted series

Adjustment at T $\quad \hat{\sigma}^2_{e(T),I} = \hat{\sigma}^2_{a,I}$

Adjustment at $i \neq T$ $\quad \hat{\sigma}^2_{e(i),I} = \hat{\sigma}^2_{a,I} + \frac{1}{n}\left(\hat{\omega}^2_{I,T} - \hat{\omega}^2_{I,i}\right)$

Notice that the IO adjusted estimate of error variance at T, $\hat{\sigma}^2_{e(T),I}$, is same as the estimated error variance based on predicted IO outlier free series, $\hat{\sigma}^2_{a,I}$. We can expect the IO adjusted estimate of error variance to be minimum when the effect of outlier is adjusted at the correct position in the IO series. The empirical study investigating the adjustment impact on estimates of σ_e^2 due to correct and incorrect adjustment positions is presented in Section 3.6.

3.5 Adjustment Using Incorrect Type of Outlier

In this section, we investigate the impact on the estimate of error variance when the series is adjusted for an incorrect type of outlier. In particular, we consider two situations: when a series is contaminated by an AO but the series adjustment is carried out under the assumption of an IO and when the series is contaminated by an IO but the series adjustment is carried out under the assumption of an AO. We restrict the discussion to the correct adjustment position only and study the impact of incorrect type adjustment at correct position.

Suppose $\{Y_t, t \in \tau\}$ is the observed series contaminated with a single outlier at T. For any given series we can construct an AO adjusted series and an IO adjusted series, adjusted at i^{th} position and obtain the "AO adjusted at i" and "IO adjusted at i" estimates of error variance which from (3.21) and (3.25) with i = T are

$$\hat{\sigma}^2_{e(T),A} = \hat{\sigma}^2_e - \frac{1}{n}\hat{\omega}^2_{A,T}\hat{\eta}^2,$$

$$\hat{\sigma}^2_{e(T),I} = \hat{\sigma}^2_e - \frac{1}{n}\hat{\omega}^2_{I,T}.$$

Suppose the series is contaminated by an AO at T. In this case, from (3.17)

$$\hat{\sigma}^2_{e(T),A} = \hat{\sigma}^2_{a,A}$$

which is likely to be the smallest estimate of error variance. Hence, using (3.11), we express $\hat{\sigma}^2_{e(T),I}$ in terms of $\hat{\sigma}^2_{e(T),A}$ as

$$\hat{\sigma}^2_{e(T),I} = \hat{\sigma}^2_{e(T),A} + b_A(I) \tag{3.27}$$

where the term

$$b_A(I) = \frac{1}{n}(\hat{\omega}^2_{A,T}\hat{\eta}^2 - \hat{\omega}^2_{I,T}) \tag{3.28}$$

can be interpreted as the bias in estimate of error variance due to adjustment for IO type when the correct outlier type is AO.

On the same lines, if the outlier present in the series at time point T is an IO type, we get

$$\hat{\sigma}^2_{e(T),A} = \hat{\sigma}^2_{e(T),I} + b_I(A) \tag{3.29}$$

where

$$b_I(A) = \frac{1}{n}(\hat{\omega}^2_{I,T} - \hat{\omega}^2_{A,T}\hat{\eta}^2) \tag{3.30}$$

is the bias due to adjusting an AO when IO is the correct type.

(3.27) and (3.29) give the relation between estimates of error variance based on correct and incorrect type of series adjustment. The expressions do not lead to any conclusive evidence to say whether the incorrect type of series adjustment leads to an estimate of error variance larger than that based on correct type of series adjustment. The bias in both the expressions depends on the estimates of outlier parameter ω under the assumption of the presence of an AO type and an IO type of outlier. For further investigations, we carried out a simulation study of 1000 replications on an AR(1) and MA(1) series with parameters $\phi = 0.6$ and $\theta = -0.6$ respectively for $n = 100$ and $\omega = 1(1)5$. The average of empirical values of two biases $b_A(I)$ and $b_I(A)$ given in (3.28) and (3.30), due to incorrect type adjustments are computed using computer program CP-3 listed in Appendix C and are presented in Table 3.1.

Table 3.1

Average Bias $\times 10^2$ of Incorrect Type Adjustment for Outlier Series with an Outlier at $t = 51$
($n = 100$, $\sigma_a^2 = 1$; 1000 replications)

ω	AR(1) Series		MA(1) Series	
	$b_A(I)$	$b_I(A)$	$b_A(I)$	$b_I(A)$
1	0.2555	0.3470	0.3824	0.6165
2	0.9966	1.4580	1.2645	1.6744
3	2.3392	2.8228	2.8084	3.2606
4	4.0145	4.0855	5.3474	4.6704
5	6.3481	6.0859	9.1037	5.7481

From Table 3.1, it can be seen that the adjustment of incorrect type results in the larger estimated error variance than the correct type adjustment at position T since all biases are positive for both types in both series. We also observe that the

IO adjustment for AO series gives more bias than the AO adjustment for IO series particularly for MA(1).

Based on the theoretical and numerical results presented above, we can conclude that the adjustment of correct type and correct position is expected to give the smallest estimate of error variance among all adjusted estimates of error variance. Further exploration of this issue is postponed to Chapter 4 and Chapter 5 where simulation based study is presented.

3.6 Numerical Study of Adjustment Impact

In this section, we present the numerical study of adjustment impact on estimate of error variance discussed in the earlier section. The study presented uses generated AR(1) and MA(1) series of length $n = 100$ with a single outlier of AO or IO type at $t = 51$ for the value of outlier parameter $\omega = 4$. The outlier free series, which are used to generate the outlier series, are the same as in Figures 2.3 and 2.6 of Section 2.3 for AR(1) and MA(1) respectively. Thus for each generated series, we have two additional contaminated series corresponding to an AO and an IO outlier at $t = 51$. From each of these contaminated series on AO adjusted series and an IO adjusted series is obtained for every adjustment position i in turn, $i = 1, 2, \ldots, n$ and the estimates of error variance based on adjusted series are calculated.

We postpone the detailed simulation study to Chapter 4, where some additional issues will also be covered. The software STDS (Statistical Time Series

Diagnostic Software) which is developed during this study and is attached along is used for computations as well as plots which are presented in this section.

AR(1) with an AO:

We now consider the generated AR(1) series with an AO at t = 51 for n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$, and $\omega = 4$. The plots of AO and IO adjusted estimates of error variance for AO series are presented in Figure 3.1.

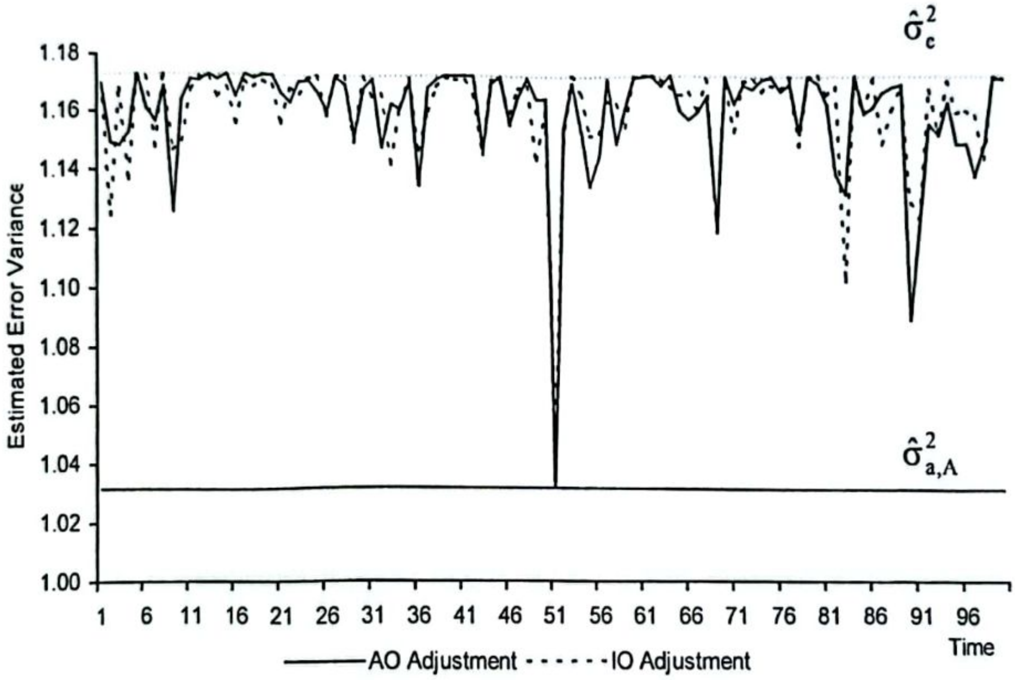


Figure 3.1: Adjusted Estimates of Error Variance: AR(1) with an AO at t = 51
(n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$, $\omega = 4$)

The estimated error variance based on predicted outlier free series $\hat{\sigma}_{a,A}^2$ and the residual variance $\hat{\sigma}_e^2$ are also shown as the straight lines in the figure.

In Figure 3.1, all estimated error variances based on each adjusted series are less than or equal to the residual variance $\hat{\sigma}_e^2 = 1.1730$ which is shown in the figure as the upper straight line. The estimated error variance based on the predicted AO outlier free series is $\hat{\sigma}^2_{a,A} = 1.0316$ as shown in the lower horizontal straight line in the figure. It is close to the estimated error variance $\hat{\sigma}^2_a = 1.0389$ obtained using the original outlier free series, as well as true error variance $\sigma^2_a = 1$.

From Figure 3.1, the two estimated error variances based on the adjusted series at adjustment position t = 51 are markedly different from others. Hence, t = 51 can be treated as the position of outlier in the series. The two estimates $\hat{\sigma}^2_{e(51),A}$ =1.0316 and $\hat{\sigma}^2_{e(51),I}$ = 1.0491 are the adjusted estimates of error variance based on each adjusted series at position t = 51. Since $\hat{\sigma}^2_{e(51),A}$ is less than $\hat{\sigma}^2_{e(51),I}$, it indicates that an AO outlier occurs at t = 51 in the generated AR(1) series.

AR(1) with an IO:

Next we consider a generated AR(1) series with an IO at t=51 for n=100, ϕ = 0.6, σ_a^2 =1, and ω=4. The adjusted estimates of error variance along with $\hat{\sigma}^2_{a,I}$ and $\hat{\sigma}^2_e$ are plotted in Figure 3.2.

From Figure 3.2, it can be seen that all the estimated error variances based on each adjusted series are less than or equal to the residual variance $\hat{\sigma}^2_e$= 1.1543. The lower straight line for $\hat{\sigma}^2_{a,I}$ = 1.0405 in the figure shows the estimated error variance based on predicted IO outlier free series. It is close to the estimated error variance $\hat{\sigma}_a^2$ = 1.0389 based on the original outlier free series as well as the true error variance $\sigma_a^2 = 1$.

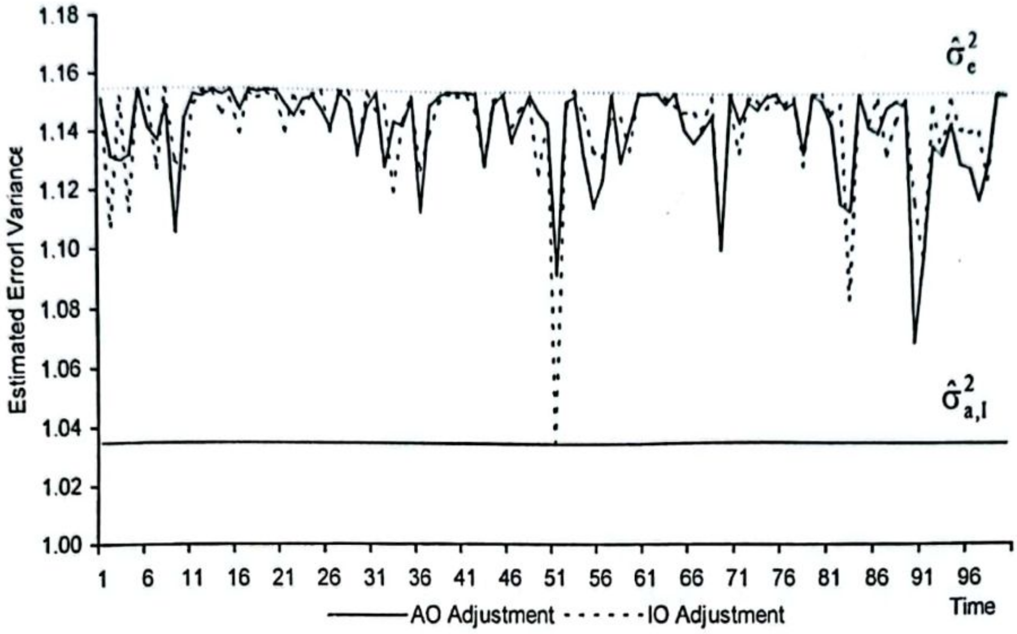


Figure 3.2: Adjusted Estimates of Error Variance: AR(1) with an IO at t = 51 ($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

From the graph, it can be seen that all of the adjusted estimates of error variance are greater than 1.06 except for the IO adjustment at t = 51. It indicates that an IO outlier occurs at t = 51 with the IO adjusted estimate of error variance at 51, $\hat{\sigma}^2_{e(51)I} = 1.0308$.

In Table 3.2, we summarize the values of adjusted estimates of error variance of the generated AR(1) series when the adjustment is made at correct or incorrect position and correct or incorrect type.

Table 3.2

Adjusted Estimates of Error Variance $\hat{\sigma}^2_{e(i)}$: AR(1) with an Outlier at t = 51
(n = 100, ϕ = 0.6, σ_a^2 = 1, ω = 4)

	Series with an AO		Series with an IO	
Adjustment type	Adjusted for AO	Adjusted for IO	Adjusted for AO	Adjusted for IO
Adjustment at i = T	1.0316 (51)	1.0491 (51)	1.0924 (51)	1.0348 (51)
Adjustment at i ≠ T				
Minimum value	1.0896 (90)	1.1028 (83)	1.0685 (90)	1.0832 (83)
Maximum value	1.1730 (41)	1.1730 (62)	1.1543 (19)	1.1543 (62)
Average value	1.1618	1.1622	1.1425	1.1436

Note: The adjustment positions are shown in parentheses.

It can be seen from the table that the estimated error variance based on the adjusted series is minimum with the correct adjustment position and correct type adjustment (the shaded cells in the table). It can be seen that when the adjustment is not at the correct position, i.e., i ≠ T, the average adjusted estimates of error variance are quite close to the maximum value of the adjusted estimate of error variance for i ≠ T, and are close to the residual variance $\hat{\sigma}^2_e$=1.1730 and 1.1543 for AO and IO respectively. It is clear that the adjustment at incorrect position cannot reduce the effect of an outlier in the observed series even when the adjustment is made for the correct outlier type.

Also, as can be seen from the table, when AO adjustment is carried out for a series with an IO type outlier, the smallest value among the adjusted estimates of error variance is $\hat{\sigma}^2_{e(90),A}$= 1.0685, which leads to the identification of incorrect outlier position. This indicates that AO adjustment may not work well for an IO contaminated series.

Table 3.3 shows that the adjustment impact on all the estimated parameters of AR(1) series. The correct type adjustment and correct adjustment position gives us the estimated error variances that are close to the estimated error variance of outlier free series.

Table 3.3

Adjustment Impact on Estimated Parameters: AR(1) with an Outlier at t = 51
($n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$, $\omega = 4$)

Estimated Parameter	Outlier Free Series	Series with an AO		Series with an IO	
		Without Adj.	AO Adj. at T	Without Adj.	IO Adj. at T
$\hat{\phi}$	0.5679	0.5790	0.5805	0.6433	0.5994
$\hat{\sigma}^2$	1.0389	1.1730	1.0316	1.1543	1.0405
$\hat{\omega}_T$	-	-	3.2529	-	3.4653

Hence, the adjustment at correct position with correct type can reduce the bias of estimated error variance in the presence of an outlier in time series. Although the estimates of ϕ show bias in comparison with the estimates based on original outlier free series, nothing can be said in general. The outlier parameters are underestimated compared to the true parameter $\omega = 4$.

MA(1) with an AO:

On analogous lines, we consider the MA(1) series of length n = 100 with $\theta = -0.6$, $\sigma_a^2 = 1$ and an AO with $\omega = 4$ at $t = 51$ and calculate the estimated error variances based on AO and IO adjusted series. The results are displayed in Figure 3.3.

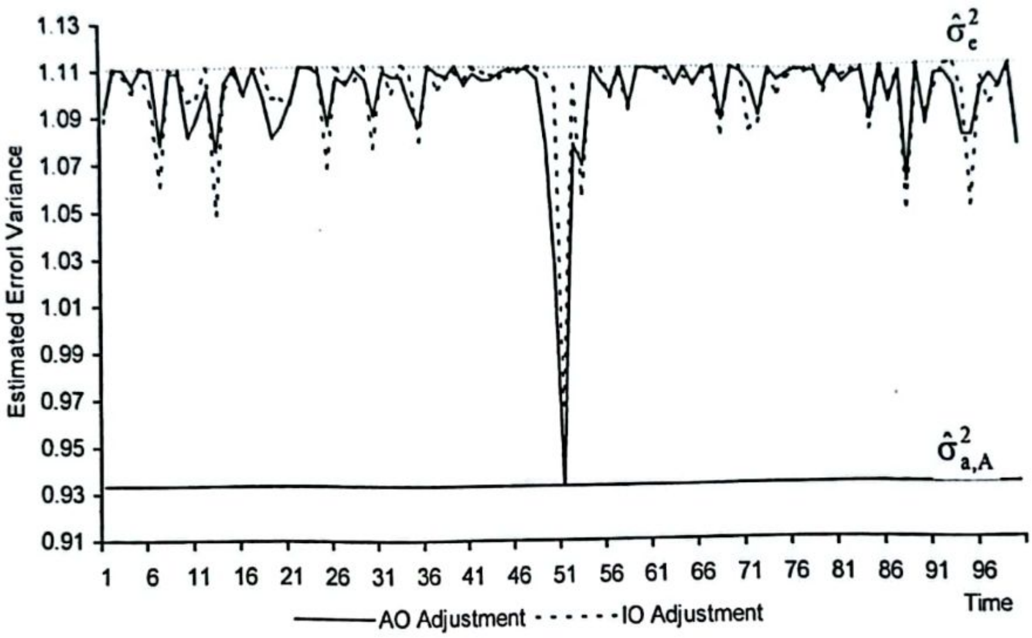


Figure 3.3: Adjusted Estimates of Error Variance: MA(1) with an AO at t = 51 ($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4$)

The estimated error variance based on predicted outlier free series and the residual variance are $\hat{\sigma}_{a,A}^2 = 0.9334$ and $\hat{\sigma}_e^2 = 1.1107$ respectively. The estimated error variance based on the outlier free series $\hat{\sigma}_a^2$ is 0.9342. From Figure 3.3, it can be seen that the minimum value of adjusted estimates of error variance over all i is $\hat{\sigma}_{e(51),A}^2 = 0.9334$ which is close to the estimated error variance based on the outlier free series. It indicates that an AO outlier occurs at $t = 51$ in the contaminated MA(1) series.

MA(1) with an IO:

Now we consider the MA(1) series with an IO at $t = 51$ for $n = 100$, $\theta = -0.6$, and $\omega = 4$. We get AO and IO adjusted the estimates of error variance for $i = 1, 2, \ldots, 100$ as shown in Figure 3.4.

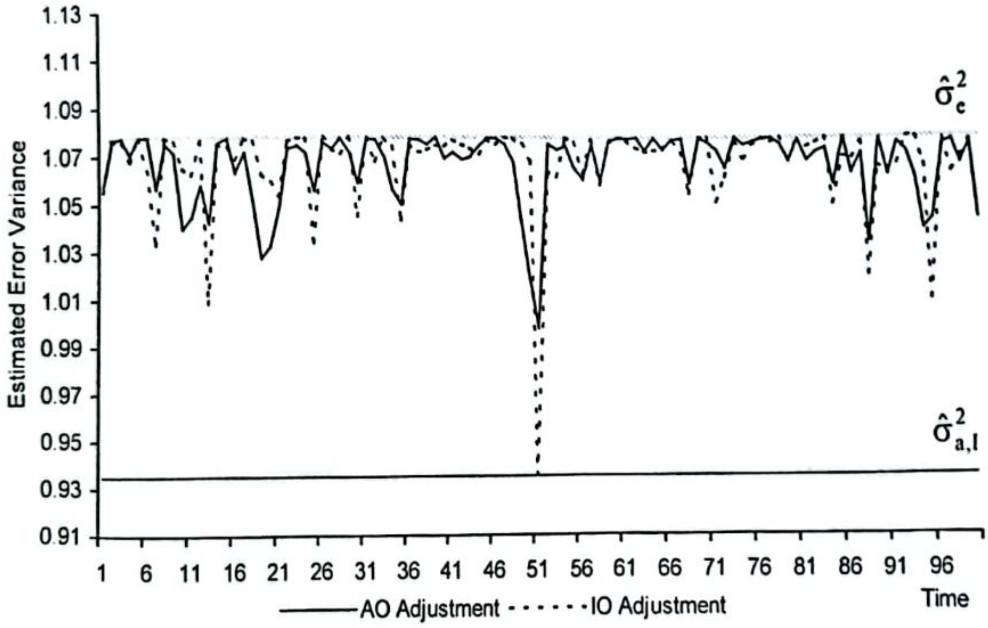


Figure 3.4: Adjusted Estimates of Error Variance: MA(1) with an IO at $t = 51$
($n = 100$, $\theta = -0.6$, $\sigma_a^2 = 1$, $\omega = 4$)

In Figure 3.4, all the adjusted estimates of error variance are greater than 0.99, except at $t = 51$. The adjusted estimates of error variance at $t = 51$ are $\hat{\sigma}^2_{e(51),A} = 0.9973$ and $\hat{\sigma}^2_{e(51),I} = 0.9348$ for AO and IO adjustment respectively. The IO adjusted estimate of error variance $\hat{\sigma}^2_{e(51),I}$ is less than $\hat{\sigma}^2_{e(51),A}$ and is close to the estimated error variance based on outlier free series $\hat{\sigma}_a^2 = 0.9342$, indicating the possibility of an IO outlier at $t = 51$ in the contaminated MA(1) series.

Table 3.4 shows the summary of estimates of error variance of MA(1) series based on adjustment at various positions and adjustment for different types of outliers.

Table 3.4

Adjusted Estimates of Error Variance $\hat{\sigma}_{e(i)}^2$: MA(1) with an Outlier at t = 51
($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4$)

	Series with an AO		Series with an IO	
Adjustment type	Adjusted for AO	Adjusted for IO	Adjusted for AO	Adjusted for IO
Adjustment at $i = T$	0.9334 (51)	0.9677 (51)	0.9973(51)	0.9348 (51)
Adjustment at $i \neq T$				
Minimum value	1.0283 (50)	1.0486 (13)	1.0177 (50)	1.0081 (13)
Maximum value	1.1107 (67)	1.1107 (78)	1.0777 (85)	1.0777 (31)
Average value	1.1006	1.1007	1.0672	1.0681

Note: The adjustment positions are shown in parentheses.

As in case of Table 3.2, Table 3.4 shows the estimates of error variance of the generated MA(1) series when the adjustment is made at correct or incorrect position and is of correct and incorrect type. The optimum values are obtained for AO and IO series with the adjustment of correct position and correct type (the shaded cells in the table). It can be seen that the average adjusted estimates of error variance for $i \neq T$ are close to the maximum values of corresponding adjusted estimates of error variance for $i \neq T$. These estimates are also close to the residual variances $\hat{\sigma}_e^2 = 1.1107$ and 1.0777 for AO and IO respectively.

Table 3.5
Adjustment Impact on Estimated Parameter: MA(1) with an Outlier at t = 51
($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4$)

Estimated Parameter	Outlier Free Series	Series with an AO		SEries with an IO	
		Without Adj.	AO Adj. at T	Without Adj.	IO Adj. at T
$\hat{\theta}$	-0.5915	-0.4388	-0.5795	-0.5730	-0.5993
$\hat{\sigma}^2$	0.9342	1.1107	0.9566	1.0777	0.9373
$\hat{\omega}_T$	-	-	3.6938	-	3.8048

Table 3.5 presents the estimates of all parameters involved using various series. It can be seen that after the adjustment of correct type and correct position is made, the estimated error variances are close to that of the original outlier free series. Hence, the adjustment method can be used to reduce the bias in estimated error variance and to detect the outlier position.

From Figures 3.1-3.4, we can notice that the bias in estimate of error variance can be substantially reduced by the adjustment not only at correct position but also at incorrect positions. It can mislead into selecting the incorrect outlier position when the outlier parameter is small. For instance, from Figure 3.2 and Table 3.2, the AO adjustment cannot properly indicate the position of the outlier when the observed series has an IO outlier. The IO adjustment seems to indicate the position of outlier even when there is an AO in observed series.

Though a detailed empirical study is postponed to later chapter, we conclude that the adjustment at correct position with correct type gives the minimum estimate of error variance for both types of outliers in time series. It agrees with the theoretical results in the previous sections and the estimates of error variance using adjustment method can be used for detection of outlier.

In the next chapter, we propose outlier detection method based on adjustment diagnostics.

Chapter 4

Adjustment Diagnostic for Outliers

4.1 Introduction

As mentioned in Sections 1.3 and 2.1, the existing diagnostic procedures in time series analysis are adapted from regression diagnostics and used in deletion diagnostic approach (Peña, 1987; Bruce and Martin, 1989; Abraham and Chuang, 1989; Ledolter, 1990). In previous chapters, we proposed series adjustment as a possible method to handle the diagnosis of outliers.

In this chapter we propose a diagnostic procedure for detection of outliers in ARMA(p, q) series which is based on the estimates of error variance using the observed series and the adjusted series. The procedure is derived in Section 4.2 using the likelihood displacement criterion proposed by Cook (1986, 1987). The proposed procedure is called *Adjustment Diagnostic based on Variance* (ADV).

Section 4.3 investigates the performance of ADV using ML estimator of β based on simulation study. As pointed out in Sections 1.4 and 2.5, most of the existing deletion diagnostic procedures do not distinguish between the types of outliers. It is shown that the proposed procedure performs better than the existing procedures in the presence of an IO and satisfactorily identifies the outlier type.

Section 4.4 presents the performance of the proposed procedure in the presence of a single outlier in a series using robust estimator of β and compares it with that using ML estimator. The simulation study shows marginal improvement in the performance of correct type identification using robust estimator only when an additive outlier is present in the series.

In Section 4.5, an iterative procedure for detection of multiple outliers is proposed. The performance of the proposed procedure in the presence of multiple outliers is presented in Section 4.6 where the contaminated series is assumed to have isolated or patch outliers. We critically evaluate the iterative procedure in Section 4.7 and suggest possible alternatives.

4.2 Adjustment Diagnostic Based on Likelihood Displacement

Cook (1986, 1987) introduced a general measure of model perturbation on parameter estimates using contours of log likelihood function. For any model M with parameter vector λ, let $\ell(\lambda)$ be the log likelihood function and $\hat{\lambda}$ is the maximum likelihood estimator of λ. Suppose we have a perturbation model $M(\omega)$ and let $\ell_\omega(\lambda)$ and $\hat{\lambda}_\omega$ be the log likelihood function of the perturbation model and the associated maximum likelihood estimator respectively. Thus we have two estimators $\hat{\lambda}$ and $\hat{\lambda}_\omega$ corresponding to the basic model and perturbation model respectively. The likelihood displacement or likelihood distance $LD_\omega(\lambda)$ proposed by Cook is

$$LD_\omega(\lambda) = 2[\,\ell(\hat{\lambda}) - \ell(\hat{\lambda}_\omega)\,] \tag{4.1}$$

which measures the changes in the log likelihood function due to the influence of perturbation on the parameter estimates. The likelihood displacement provides a theoretical foundation for a general measure to assess the influence of perturbation on the estimates of model parameters.

Cook also proposed modification of likelihood displacement in situation where a subset of parameters is of interest. In particular, let λ_1 be the parameters of interest when $\lambda = (\lambda_1', \lambda_2')'$ and let $\hat{\lambda}_\omega = (\hat{\lambda}_{1\omega}', \hat{\lambda}_{2\omega}')'$. Further, let $\hat{\lambda}_2(\lambda_1)$ be the maximum likelihood estimate of λ_2 obtained on maximizing $\ell(\lambda_1, \lambda_2)$ when λ_1 is fixed. Hence

$$\ell(\hat{\lambda}_{1\omega}, \hat{\lambda}_2(\hat{\lambda}_{1\omega})) = \max_{\lambda_2} \ell(\hat{\lambda}_{1\omega}, \lambda_2)$$

and the proposed likelihood displacement for λ_1 is

$$LD_\omega(\lambda_1) = 2\,[\ell(\hat{\lambda}) - \ell(\hat{\lambda}_{1\omega}, \hat{\lambda}_2(\hat{\lambda}_{1\omega}))]. \tag{4.2}$$

The likelihood displacement measure is often used in regression diagnostics (Cook and Weisberg, 1982, p. 181-188). In the regression setup, the model considered is $Y_{n\times 1} = X_{n\times k}\,\Theta_{k\times 1} + \varepsilon_{n\times 1}$ and the model perturbation is the deletion of i^{th} case from the observations. Cook and Weisberg also established that in case of regression model where the parameter Θ is of interest, the likelihood displacement is a monotone function of Cook's D statistic.

In case of time series observations, the likelihood displacement diagnostic measure is derived by Ledolter (1990) where a stationary and invertible ARMA (p, q) model

$$\phi(B)\, Y_t = \theta(B)\, a_t$$

is considered. The model perturbation considered by Ledolter is the deletion of i^{th} observation following deletion diagnostics proposed by Peña (1987) and Bruce and Martin (1989). The deleted observation is treated as an unknown parameter and its estimate is substituted in the observed series to estimate the parameters of interest. The estimate of deleted observation used by Ledolter is the weighted sum of adjacent observations which is the Brubacher and Wilson (1976) estimator discussed in Section 2.6. Based on the discussion presented in Section 2.6, the perturbation of model considered by Ledolter is similar to the proposed series adjustment under the presence of an AO type of outlier.

We investigate Cook's likelihood displacement in a more general setup where the model perturbation is the series adjustment and derive the diagnostic measure in the presence of outlier of AO and IO type. Alternatively, Akaike Information Criterion (AIC, Akaike, 1974) can also be considered.

Consider a stationary and invertible ARMA(p, q) process given by

$$\phi(B)\, Y_t = \theta(B)\, a_t$$

where $\{Y_t, t \in \tau\}$ is the observed series and a_t are i.i.d. normal with mean 0 and variance σ_a^2. We treat this as the original model M for the likelihood displacement. Let β denotes the time series parameter vector $(\phi', \theta')'$.

We consider two perturbation models $M_A(\omega)$ and $M_I(\omega)$ corresponding to AO and IO type respectively. In the presence of an AO, the adjusted series based on an AO adjustment at i = 1, 2, …, n is

$$\ddot{Y}_{t(i),A} = Y_t - \hat{\omega}_{A,i}\, \xi_t^{(i)}, \qquad i, t \in \tau.$$

From (2.37), we get the adjusted series $\{\ddot{Y}_{t(i),A}, t \in \tau\}$ where

$$\ddot{Y}_{t(i),A} = \begin{cases} Y_t & \text{for } t < i \\ -d_0^{-1} \sum\limits_{k \neq 0} d_k\, Y_{i+k} & \text{for } t = i \\ Y_t & \text{for } t > i. \end{cases} \tag{4.3}$$

Thus, under the perturbation model $M_{A,i}(\omega)$, observation at time point t = i is a weighted sum of adjacent observations, where the weights depend on the unknown time series parameters of the model. Hence the value of the observation at t = i needs to be estimated appropriately, using (3.14).

In case of the adjustment of the series with an IO type of outlier, for the adjustment position i in turn, i =1, 2, …, n, the proposed adjusted series is

$$\ddot{Y}_{t(i),I} = Y_t - \hat{\omega}_{I,i}\, \psi(B)\, \xi_t^{(i)}, \qquad i,t \in \tau$$

where, from (2.42),

$$\ddot{Y}_{t(i),I} = \begin{cases} Y_t & \text{for } t < i, \\ \sum_{j=1} \pi_j Y_{i-j} & \text{for } t = i, \\ Y_t + \psi_{t-i} \sum_{j=0} \pi_j Y_{i-j} & \text{for } t > i \end{cases} \tag{4.4}$$

and the π_j's and ψ_j's are the unknown model parameters. Thus, under the perturbation model $M_{I,i}(\omega)$, for all $t \geq i$, the observation Y_t at time point t is the weighted sum of the original observation at t and the observations prior to time point i. Since the weights depend on the unknown time series coefficients, we treat the series adjustment as estimation of adjusted observations $\ddot{Y}_{t(i),I}$, $t \geq i$ and obtain the estimates $Y_{t(i),I}$ appropriately, using (3.14). For both the adjustment models discussed above, we consider an iterative procedure to determine the estimates of the adjusted values, the time series parameters and the error variance.

We now obtain the likelihood displacement for the adjustment diagnostic in the presence of outlier in time series, using conditional log likelihood functions for simplicity. Suppose that the observed series $\{Y_t, t \in \tau\}$ follows model M. The conditional log likelihood function, denoted by $\ell(\beta, \sigma_a^2)$, is given by (See (2.24))

$$\ell(\beta, \sigma_a^2) \propto -\frac{n}{2}\ln(\sigma_a^2) - \frac{1}{2\sigma_e^2} SS(\beta) \tag{4.5}$$

where β is a set of parameters $(\phi', \theta')'$, σ_a^2 is the error variance and $SS(\beta) = \sum_{t=1}^{n} a_t^2$.

Then, under the model M, the maximum likelihood estimates $\hat{\beta}$ can be obtained by minimizing the conditional sum of squares function SS(β) of (4.5) as shown in Section 2.4.2.

We also obtain the maximum likelihood estimate of error variance, based on the observed series $\{Y_t, t \in \tau\}$. Using $\hat{\beta} = (\phi', \theta')'$ we obtain the estimated sum of squares function

$$SS(\hat{\beta}) = \sum_{t=1}^{n} (\hat{\pi}(B) Y_t)^2$$

where $\hat{\pi}(B)$ can be obtained by equating the coefficients of B in $\hat{\phi}(B) = \hat{\theta}(B)\hat{\pi}(B)$ and the maximum likelihood estimate of error variance is given by

$$\hat{\sigma}_e^2 = \frac{1}{n} SS(\hat{\beta}).$$

This is the residual variance introduced in (3.10). Thus, the log likelihood function required for likelihood displacement in (4.1) is obtained by substituting the values $\beta = \hat{\beta}$ and $\sigma_a^2 = \hat{\sigma}_e^2$ in (4.5), which gives,

$$\ell(\hat{\beta}, \hat{\sigma}_e^2) \propto -\frac{n}{2} \ln(\hat{\sigma}_e^2) - \frac{1}{2\hat{\sigma}_e^2} SS(\hat{\beta})$$

and it reduces to

$$\ell(\hat{\beta}, \hat{\sigma}_e^2) \propto -\frac{n}{2} \ln(\hat{\sigma}_e^2) - \frac{n}{2}. \tag{4.6}$$

To get the likelihood displacement under perturbation model, we need to obtain the maximum likelihood estimates of parameters β and the error variance

σ_a^2 under each of the perturbation models $M_{A,i}(\omega)$ and $M_{I,i}(\omega)$ based on the corresponding adjusted series given by (4.3) and (4.4) respectively for i = 1, 2, ..., n. As mentioned earlier, an iterative procedure must be used to obtain the maximum likelihood estimates of the adjusted observations $\ddot{Y}_{t(i),S}$, and those of the parameters of interest. We denote the estimates of parameters as $\hat{\beta}_{(i),S}$ and the estimated error variances as $\hat{\sigma}^2_{e(i),S}$ for S = A, I. The estimates of error variance, $\hat{\sigma}^2_{e(i),S}$, are obtained by replacing the original observations in the series by their estimates given in (3.14) and using $\hat{\beta}_{(i),S}$.

In particular, for $M_{A,i}(\omega)$, we iteratively obtain the maximum likelihood estimates of parameter vector β, $\hat{\beta}_{(i),A} = (\hat{\phi}'_{(i),A}, \hat{\theta}'_{(i),A})'$, based on estimates of adjusted series $Y_{t(i),A}$ and the estimates of time series parameters $\hat{\beta}$. The estimated error sum of squares is

$$SS_{(i)}(\hat{\beta}_{(i),A}) = \sum_{t=1}^{n} \left\{\hat{\pi}_{(i),A} Y_{t(i),A}\right\}^2 .$$

where $\hat{\pi}_{(i),A}$ can be obtained by equating the coefficients of B in $\hat{\phi}_{(i),A}(B)$ = $\hat{\theta}_{(i),A}(B)\ \hat{\pi}_{(i),A}(B)$. Hence, the maximum likelihood estimate of error variance, is given by

$$\hat{\sigma}^2_{e(i),A} = \frac{1}{n} SS_{(i)}(\hat{\beta}_{(i),A}).$$

Similarly, for $M_{I,i}(\omega)$, we iteratively get the maximum likelihood estimates of parameters and error variance based on the estimated adjusted series $Y_{t(i),I}$, given by

$$\hat{\beta}_{(i),I} = (\hat{\phi}'_{(i),I}, \hat{\theta}'_{(i),I})'$$

and

$$\hat{\sigma}^2_{e(i),I} = \frac{1}{n} SS_{(i)}(\hat{\beta}_{(i),I}).$$

For the sake of simplicity, we suppress the notation denoting two types of perturbations corresponding to AO and IO type of outliers and introduce a general notation to denote the maximum likelihood estimates under the perturbation model $M_i(\omega)$ by $\hat{\beta}_{(i)}$ for $\hat{\beta}_{(i),A}$ and $\hat{\beta}_{(i),I}$, and $\hat{\sigma}^2_{e(i)}$ for $\hat{\sigma}^2_{e(i),A}$ and $\hat{\sigma}^2_{e(i),I}$, given by

$$\hat{\beta}_{(i)} = (\hat{\phi}'_{(i)}, \hat{\theta}'_{(i)})',$$

and

$$\hat{\sigma}^2_{e(i)} = \frac{1}{n} SS_{(i)}(\hat{\beta}_{(i)})$$

where $SS_{(i)}(\hat{\beta}_{(i)})$ is the sum of squares function which uses $\hat{\beta}_{(i)}$ and is based on observations where the original observations are replaced by their estimates as per the suggested adjustments. We further introduce an additional estimate of error variance, denoted by $s^2_{e(i)}$, given by

$$s^2_{e(i)} = \frac{1}{n} SS(\hat{\beta}_{(i)})$$

where

$$SS(\hat{\beta}_{(i)}) = \sum_{t=1}^{n} \{\hat{\pi}_{(i)}(B)Y_t\}^2,$$

which is based on the original observed series without any adjustment of any observations, and using $\hat{\beta}_{(i)}$.

When the time series parameters β and error variance σ_a^2 are both of interest, the aim is to obtain the likelihood displacement introduced in (4.1), which reduces to

$$LD_i(\beta, \sigma_a^2) = 2\,\{\, \ell(\hat{\beta}, \hat{\sigma}_e^2) - \ell(\hat{\beta}_{(i)}, \hat{\sigma}_{e(i)}^2)\,\}, \qquad i \in \tau$$

where $\ell(\hat{\beta}, \hat{\sigma}_e^2)$ is given by (4.6) and $\ell(\hat{\beta}_{(i)}, \hat{\sigma}_{e(i)}^2)$ is obtained on substituting the estimates $\hat{\beta}_{(i)}$ and $\hat{\sigma}_{e(i)}^2$ in the log likelihood function (4.5) which gives

$$\ell(\hat{\beta}_{(i)}, \hat{\sigma}_{e(i)}^2) \propto -\frac{n}{2}\ln(\hat{\sigma}_{e(i)}^2) - \frac{ns_{e(i)}^2}{2\hat{\sigma}_{e(i)}^2}$$

Thus, the measure of interest $LD_i(\beta, \sigma_a^2)$ reduces to

$$LD_i(\beta, \sigma_a^2) = 2\left\{\left(-\frac{n}{2}\ln(\hat{\sigma}_e^2) - \frac{n}{2}\right) - \left(-\frac{n}{2}\ln(\hat{\sigma}_{e(i)}^2) - \frac{ns_{e(i)}^2}{2\hat{\sigma}_{e(i)}^2}\right)\right\}$$

$$= -n\ln\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2}\right) + n\left(\frac{s_{e(i)}^2}{\hat{\sigma}_{e(i)}^2} - 1\right), \qquad i \in \tau.$$

In order to obtain likelihood displacement for the error variance σ_a^2 of interest, the modified likelihood displacement introduced in (4.2) with $\lambda_1 = \sigma_a^2$ reduces to

$$LD_i(\sigma_a^2) = 2\ \{\ \ell(\hat{\beta}, \hat{\sigma}_e^2) - \ell(\hat{\beta}, \hat{\sigma}_{e(i)}^2)\ \}$$

where $\ell(\hat{\beta}, \hat{\sigma}_e^2)$ is as in (4.6) and $\ell(\hat{\beta}, \hat{\sigma}_{e(i)}^2)$ is obtained on substituting the estimates $\hat{\beta}$ and $\hat{\sigma}_{e(i)}^2$ in (4.5). The substitution gives

$$\ell(\hat{\beta}, \hat{\sigma}_{e(i)}^2) \propto -\frac{n}{2} \ln\,(\hat{\sigma}_{e(i)}^2) - \frac{n\hat{\sigma}_e^2}{2\hat{\sigma}_{e(i)}^2}.$$

Thus the measure of interest can be simplified to

$$LD_i(\sigma_a^2) = 2\ \{\ \ell(\hat{\beta}, \hat{\sigma}_e^2) - \ell(\hat{\beta}, \hat{\sigma}_{e(i)}^2)\ \}$$

$$LD_i(\sigma_a^2) = 2\left\{\left(-\frac{n}{2}\ln(\hat{\sigma}_e^2) - \frac{n}{2}\right) - \left(-\frac{n}{2}\ln(\hat{\sigma}_{e(i)}^2) - \frac{n\hat{\sigma}_e^2}{2\hat{\sigma}_{e(i)}^2}\right)\right\}$$

$$= -n\ \ln\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2}\right) + n\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2} - 1\right), \qquad i \in \tau. \tag{4.7}$$

As expected, the expression for the likelihood displacement is same as that derived by Ledolter (1990), except for the difference in the estimates, which in this case is based on adjusted series. Ledolter showed that this likelihood displacement is equivalent to the deletion diagnostic based on error variance (DV) developed by Bruce and Martin (1989). For $x = \frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2} - 1$, using the identity $\ln(1 + x) \approx x - \frac{x^2}{2}$, it was shown that (4.7) can be approximately expressed as

$$\mathrm{LD_i}(\sigma_a^2) \approx -n\left\{\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2}-1\right)-\frac{1}{2}\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2}-1\right)^2\right\}+n\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2}-1\right)$$

$$\approx \frac{n}{2}\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i)}^2}-1\right)^2 = \mathrm{DV_i} \quad , \qquad i\in\tau \qquad (4.8)$$

It is clear from the derivations that the diagnostics based on likelihood displacement will agree with the diagnostic proposed by Ledolter based on deletion. Further, since the series adjustment in the presence of AO is similar to deletion diagnostic, the procedures will coincide when the adjustment is carried out for AO type of outliers.

Before formally proposing the diagnostic procedure, we briefly discuss the situation when the parameter β is alone of interest. The likelihood displacement in this case reduces to

$$\mathrm{LD_i}(\beta) = 2\ \{\ \ell(\hat{\beta},\hat{\sigma}_e^2) - \ell(\hat{\beta}_{(i)}, s_{e(i)}^2)\ \}, \qquad i\in\tau$$

where $\ell(\hat{\beta},\hat{\sigma}_e^2)$ is as before and $\ell(\hat{\beta}_{(i)}, s_{e(i)}^2)$ is given by substituting the estimates $\hat{\beta}_{(i)}$ and $s_{e(i)}^2$ in (4.5), which reduces to

$$\ell(\hat{\beta}_{(i)}, s_{e(i)}^2) \propto -\frac{n}{2}\ln\left(s_{e(i)}^2\right) - \frac{n}{2}.$$

Hence, we get

$$\mathrm{LD_i}(\beta) = 2\left\{\left(-\frac{n}{2}\ln(\hat{\sigma}_e^2)-\frac{n}{2}\right)-\left(-\frac{n}{2}\ln(s_{e(i)}^2)-\frac{n}{2}\right)\right\}$$

$$LD_i(\beta) = -n\ln\left(\frac{\hat{\sigma}_e^2}{s_{e(i)}^2}\right), \qquad i\in\tau.$$

As claimed in Chapters 2 and 3, since the estimate of error variance is more sensitive than the estimates of the time series parameters in the presence of outliers (Bruce and Martin, 1989; Ledolter,1989, 1990), we return to the likelihood displacement diagnostic for the error variance σ_a^2, given by (4.7). Based on (4.7) and (4.8) and following Ledolter (1990) we consider the quantity $n\left(\frac{\hat{\sigma}^2}{\hat{\sigma}_{(i)}^2}-1\right)$ as the diagnostic measure based on error variance for outliers in time series. The displacement is equal to n times the influence on scale $D_v(i)$ proposed by Peña (1987).

Since we have two adjusted series for AO type and IO type of outlier, we introduce adjustment diagnostic measure based on error variance for both series.

Let $$ADV_{S,i} = n\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i),S}^2}-1\right), \qquad i\in\tau \tag{4.9}$$

where S is either A or I, $\hat{\sigma}_e^2$ is the estimated error variance based on the observed series, and $\hat{\sigma}_{e(i),S}^2$ is the estimated error variance based on adjusted series with adjustment position i, by outlier type S, S = A ,I.

The two measures introduced in (4.9) can be used for obtaining diagnostic plots. We propose two diagnostic plots of $ADV_{S,i}$ against i to get an initial idea about the contamination present in the available observations. The possible position of outliers will be denoted by the large values of $ADV_{S,i}$.

Hence, for diagnosing the presence of an AO or an IO type of outlier, respectively, in the observed series at an unknown position, we propose the criteria

$$\Omega_A = \max_i ADV_{A,i} = \max_i n\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}^2_{e(i),A}} - 1\right), \qquad i \in \tau \quad \text{for AO}$$

$$= ADV_{A,T} \text{ and}$$

$$\Omega_I = \max_i ADV_{I,i}, = \max_i n\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}^2_{e(i),I}} - 1\right), \qquad i \in \tau \quad \text{for IO}$$

$$= ADV_{I,T}. \tag{4.10}$$

The position of the outlier is given by the time point T at which the proposed statistic achieves the maximum provided the computed statistic Ω_A or Ω_I is significantly large.

Based on likelihood displacement criterion, we have two test statistics Ω_A and Ω_I for testing the possibility of an AO or an IO respectively. The type of outlier, however, is rarely known in practice and it is difficult to decide which detection test is more appropriate for a given situation. Also, as shown in Chapter 3, the series adjustment at correct adjustment position by correct type is crucial as it is more likely to yield the smallest adjusted estimate of error variance. Thus, even after detecting the presence of an outlier in the series, a criterion to distinguish between an AO from an IO is needed. One possible way is to compare the two statistics Ω_A and Ω_I at any suspected time point T. In particular, at any suspected time point T, the possible outlier is classified as an IO if $\Omega_A < \Omega_I$ and it is classified as an AO if $\Omega_A \geq \Omega_I$.

Alternatively, a comprehensive likelihood displacement based criterion involving both the statistics can be proposed on the lines of available literature (Abraham 1987; Wei, 1990, p. 199). The derivations and the discussions presented so far lead us to propose the criterion

$$\Omega^* = \max \{ \max_i ADV_{A,i} , \max_i ADV_{I,i} \}$$

$$= \max (\Omega_A, \Omega_I) \qquad (4.11)$$

We thus propose the following comprehensive outlier detection procedure:

If $\Omega^* > C$ and

$\Omega^* = \Omega_A = ADV_{A,T}$ then an AO type of outlier at time point T is identified;

$\Omega^* = \Omega_I = ADV_{I,T}$ then an IO type of outlier at time point T is identified,

where C is a predetermined positive constant.

We refer to the proposed procedure based on Ω^* by *Adjustment Diagnostic based on Variance* (ADV) procedure.

Remarks:

A few remarks on the proposed procedure follow.

a) The proposed test statistic Ω^* is a function of Ω_A and Ω_I where Ω_A and Ω_I (eqn (4.10)) are derived using likelihood displacement criteria under the perturbation models $M_{A,i}(\omega)$ and $M_{I,i}(\omega)$ using the adjusted time series defined in (4.3) and (4.4) respectively, for $i = 1, 2, \ldots, n$. The perturbation models considered in the derivation are the models which use adjusted time series (eqns (4.3) and (4.4)). These are not the deletion models considered in the literature so far (Ledolter, 1990).

The two statistics Ω_A and Ω_I are functions of $\hat{\sigma}_e^2$ and $\hat{\sigma}_{e(i),S}^2$ for S = A, I. Alternatively, it is possible to propose a procedure based on the maximum normed residual test statistic $\max\left|\frac{\hat{e}_{t(i),s}}{\hat{\sigma}_e^2}\right|$. This statistic, however, is different from that proposed and studied in the literature (Barnett and Lewis, 1994) since $\hat{e}_{t(i)A}$ and $\hat{e}_{t(i)I}$ involved in the expression are based on adjustment diagnostic and not deletion diagnostic. Note that $\hat{e}_{t(i)A}$ ($\hat{e}_{t(i)I}$) here is the residuals at t computed after adjusting the entire series under the assumption that AO (IO) type of outlier is present at position i (refer eqn (3.15)). As shown in Section 2.6, the series adjustment leads to adjusting the observations starting from i in the presence of IO. The proposed procedures available in the literature (Peña, 1987; Ledolter, 1990; Ljung, 1993) use the residual at t computed after deleting the i^{th} observation and substituting it by its least squares predictor. The motivation behind adjustment and the advantage in adjusting the series is presented in Sections 2.2 and 2.5 and Chapter 3.

b) The proposed procedure requires adjustment of the series at all time points t = 1, 2, ..., n since the position of the outlier is unknown. If knowledge of a few suspected positions is available, the computations can be significantly reduced by forming a lesser number of adjusted series corresponding to only the suspected positions. While such an approach will be cost effective, a difficulty in its implementation is the lack of such knowledge. As mentioned in Section 1.2, an outlier in a time series is not necessarily an extreme value (Barnett and Lewis. 1994, p. 395). As a result, for a given time series it is difficult to select the positions for adjustment while making sure that the actual outlier positions are not missed.

c) On the same lines, if the type of outlier present in the observed time series is known, the test procedure can be reduced to that based on Ω_A or Ω_I depending on whether it is an AO or IO respectively. In the presence of any additional knowledge of the outlier type, the procedure based on Ω_A or Ω_I will have better performance than that based on Ω^* since in this case there will not be any possibility of a wrong identification of outlier type. In case it is known that only an AO type of outlier is present in the series, the procedure will be similar to that proposed by Ledolter (1990) as anticipated in Section 2.6.1. In the next section a comparison of performance of the proposed procedure with that by Ledolter (1990) is presented. However, the type of outlier present in a contaminated data is rarely known in practice. Thus it is difficult to decide which detection test is more appropriate for a given situation.

The emphasis here is to provide a comprehensive test procedure which can be carried out even in the absence of any additional information and investigate its performance when the outlier type and its position is unknown.

Critical Values of Proposed Procedure

To obtain the cut-off point C for the proposed adjustment diagnostic procedure, the finite sample distribution of the statistic Ω^{*} in the null case is required. The theoretical derivation of the distribution is intractable due to the correlation between $ADV_{S,i}$'s since the test statistic depends on their maximum. The problems involved in the theoretical derivations here are similar to those discussed in the literature (Abraham, 1987; Ljung, 1993). Abraham and Chuang (1989), Chang et al. (1988), Ledolter (1990) used Monte Carlo simulation to determine suitable critical values C for their procedures. In the present study we follow the same approach and compute the critical values based on extensive Monte Carlo simulations. In addition to the critical values of Ω^{*}, the critical values of Ω_A and Ω_I are also provided for the situation when the type of outlier is known.

We carried out simulation study based on 5000 replications using IMSL subroutines to compute the critical values. The complete computer program CP-4 which can be used to compute the critical value is provided in the attached CD and is listed in Appendix C. For the series length n = 100(25)300 and percentiles 10% and 5%, AR(1), AR(2), MA(1), MA(2) and ARMA(1,1) series were considered for various values of parameters covering a wide range. The series considered are

a) AR(1) with $\phi = 0.2(0.1)0.9, -0.3$ and -0.6; $\sigma_a^2 = 1, 3, 5$

b) AR(2) with $(\phi_1, \phi_2) = (0.3, -0.3), (0.3, -0.6), (0.3, -0.9), (0.6, -0.3), (0.6, -0.6), (0.6, -0.9), (0.9, -0.3), (0.9, -0.6), (-0.3, 0.6)$ and $(-0.6, 0.3)$; $\sigma_a^2 = 1$

c) MA(1) with $\theta = 0.2(0.1)0.9, -0.3$ and -0.6; $\sigma_a^2 = 1, 3, 5$

d) MA(2) with $(\theta_1, \theta_2) = (0.3, -0.3), (0.3, -0.6), (0.3, -0.9), (0.6, -0.3), (0.6, -0.6), (0.6, -0.9), (0.9, -0.3), (0.9, -0.6), (-0.3, 0.6)$ and $(-0.6, 0.3)$; $\sigma_a^2 = 1$

e) ARMA(1,1) with $(\phi, \theta) = (0.3, -0.3), (0.3, -0.6), (0.3, -0.8), (0.6, -0.3), (0.6, -0.6), (0.6, -0.8), (0.9, -0.3), (0.9, -0.6), (-0.3, 0.6)$ and $(-0.6, 0.3)$; $\sigma_a^2 = 1$.

Tables A1 to A9 in Appendix A (pp. 229 - 237) provide the critical values for the series mentioned above. Below, we present the percentiles of Ω^*, Ω_A and Ω_I in Table 4.1 for selected values of parameters for AR(1) and MA(1) of selected lengths n.

As expected, the percentiles of Ω^* are larger than those of Ω_A and Ω_I. The percentiles differ depending on the length of the series and the value of the parameter. Also the critical values are invariant under error variance σ_a^2 (see Appendix A, Tables A1-A3 of AR(1) and A5-A7 of MA(1)). As anticipated (Fox, 1972; Bustos and Yohai 1986), the critical values and the performance of the

Table 4.1

Estimated Percentiles of Ω^{*}, Ω_A and Ω_I
($\sigma_a^2 = 1$, 5000 replications)

Series	%	n = 50			n = 100			n = 150		
		Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I
AR(1)	10	12.27	11.46	11.55	12.58	11.80	11.96	13.09	12.39	12.38
$\phi = 0.3$	5	14.12	13.18	13.43	14.22	13.40	13.64	14.65	13.80	13.86
AR(1)	10	12.57	11.04	11.57	12.93	11.68	11.86	13.32	12.09	12.34
$\phi = -0.6$	5	14.46	12.72	13.39	14.53	13.14	13.71	14.80	13.60	13.97
AR(1)	10	12.46	11.06	11.52	12.86	11.61	11.90	13.16	12.11	12.35
$\phi = 0.6$	5	14.36	12.83	13.18	14.62	13.19	13.39	14.69	13.45	13.66
AR(1)	10	12.68	11.00	11.49	12.95	11.41	12.00	13.18	11.97	12.30
$\phi = 0.9$	5	14.52	12.81	13.42	14.48	12.91	13.47	14.53	13.19	13.54
MA(1)	10	12.45	11.56	11.62	12.68	11.89	11.83	12.98	12.32	12.21
$\theta = -0.3$	5	14.49	13.5	13.49	14.35	13.44	13.47	14.41	13.72	13.67
MA(1)	10	13.76	12.61	11.75	13.45	12.31	11.94	13.47	12.50	12.25
$\theta = -0.6$	5	16.23	15.03	13.66	15.39	14.21	13.67	15.19	14.03	13.85
MA(1)	10	13.88	12.64	11.67	13.54	12.24	12.23	13.78	12.46	12.56
$\theta = 0.6$	5	16.35	15.27	13.86	15.37	14.14	13.94	15.60	14.13	14.29
MA(1)	10	16.15	15.28	12.06	16.53	15.73	12.35	17.02	16.37	12.86
$\theta = -0.9$	5	18.54	17.74	14.18	19.30	18.86	14.19	20.44	20.01	14.71

procedure do not depend on the sign of the time series parameter, particularly for AR(1) and MA(1). In the next two sections we report the performance evaluation for positive values of ϕ for AR(1) and negative values of θ for MA(1) since in case of MA(1) the negative values yield positive lag-one correlation.

Alternatively, Ledolter (1990) suggested that rather than comparing with an upper percentile of the reference distribution, a warning value can be considered for outlier detection. For instance, for AR(1) of length 100, a warning value of 14.5 seems suitable as a 95% percentile. The warning value can be used to indicate a particular observation which needs to be scrutinized.

We adapt the suggestion of Ledolter to use warning line or warning limit in ADV plots where both $ADV_{A,i}$ and $ADV_{I,i}$ are plotted against i to get an initial idea about the type and level of contamination of the given series. The software STDS which is presented along has the ADV plot as one of the menus. The ADV plots are used for the data analysis presented in Chapter 5.

To estimate significance level of the proposed ADV procedure using a general reference value, we computed simulation based estimates of the significant level of Ω^*. The empirical estimates are based on 1000 replications using computer program CP-5 in attached CD and cited in the list of Appendix C. The two series AR(1) and MA(1) of length n = 50(50)150 were generated. For both the series the values of parameter considered are –0.9, –0.6, –0.3, 0.3, 0.6 and 0.9. For AR(1), the warning value is taken to be 14.5, and for MA(1), it is 15.0. The empirical estimates are presented in Table 4.2.

As can be expected from the percentile values presented in Table 4.1, the estimated significance levels of AR(1) do not vary much with the value of parameters whereas for MA(1) the estimated significant level increases with increase in the length of the series, analogous to that of likelihood ratio test proposed by Chang et al. (1988). Based on the Table 4.2, it is clear that a common warning value for different MA(1) series with different parameter values is not satisfactory.

Table 4.2

Estimated Level of Significance of Ω^* for a Series
($\sigma_a^2 = 1$; 1000 replications)

Parameter	AR(1) Series using C = 14.5			MA(1) Series using C = 15.0		
	n = 50	n = 100	n = 150	n = 50	n = 100	n = 150
−0.9	4.2	5.6	6.3	7.4	9.7	11.9
−0.6	5.7	4.7	5.1	5.7	6.4	4.7
−0.3	4.2	4.6	6.6	4.2	3.3	4.7
0.3	4.3	4.1	5.7	4.2	3.7	3.6
0.6	4.8	6.2	5.3	6.8	6.3	5.3
0.9	4.8	4.4	5.2	6.5	10.3	11.6

4.3 Performance Evaluation of Adjustment Diagnostic Using MLE

In this and the next section, we present a detailed empirical study of the performance of the proposed ADV procedure in the presence of a single outlier in time series. The procedure for multiple outliers and its evaluation in the presence of multiple outliers is postponed to Sections 4.5 to 4.7. We consider contaminated

AR(1) and MA(1) series with a single outlier of either type for various values of time series parameters and series length n = 50, 100(25)300.

In this section, the performance of the proposed procedure is evaluated using the ML estimates of the time series parameters β based on the contaminated series. As mentioned in Section 2.4, following Box et al. (1994) these are nonlinear least squares (LS) estimates and for the computations presented here, IMSL subroutine NSLSE is used to obtain the estimates based on a given series.

Alternatively robust estimate of the time series parameter β can be used for the evaluation. In Section 4.4, we present the evaluation of the procedure using robust estimate of β proposed by Bustos and Yohai (1986) for the contaminated series. In addition, the comparison of the performance using LS estimates and robust estimates is also presented.

The iterative procedure used for evaluation of the performance in the presence of a single outlier is as follows.

1. At the initial stage, using the contaminated series, the estimate of β based on iterative nonlinear LS procedure (IMSL subroutine NSLSE) and the estimate the error variance (σ^2) is obtained. It is not possible to drop any time series observation at the initial stage and estimate β since the dropped observation needs to be imputed which in turn requires an estimate of β.

2. For $i = 1, 2, \ldots, n$,

a) assume that the series is contaminated at a fixed time point i, and estimate the outlier parameter (ω) in the presence of AO as well as IO using the estimators derived in Section 2.4;

b) the series is appropriately adjusted using $\hat{\beta}$ and $\hat{\omega}_{A,i}$ and $\hat{\omega}_{I,i}$ under the assumption of AO and IO type of outlier respectively. The adjusted series $\{\ddot{Y}_{t(i),A}\}$ and $\{\ddot{Y}_{t(i),I}\}$ (eqns (4.3) and (4.4)) are obtained at this stage depending on AO or IO adjustment respectively;

c) estimate the parameters β and σ_a^2 based on the series adjusted at i using iterative IMSL subroutine NSLSE. These are the estimates $\hat{\beta}_{(i),A}$ ($\hat{\beta}_{(i),I}$) and $\hat{\sigma}^2_{e(i),A}$ ($\hat{\sigma}^2_{e(i),I}$) corresponding to the adjusted series which is adjusted under the assumption of an AO (IO) outlier at i^{th} position;

d) compute the test statistics $ADV_{I,i}$ and $ADV_{A,i}$.

3. Using $ADV_{I,i}$ and $ADV_{A,i}$ for all $i = 1, 2 \ldots, n$, compute Ω_I and Ω_A.

4. Further compute Ω^* and carry out the outlier detection procedure.

Thus for the detection of a single outlier, the procedure uses n +1 estimates of β based on the original contaminated series and n adjusted series. Since the evaluation presented in this section is only for the single outlier case, for every simulated series with one outlier, the procedure is terminated at step 4 and the performance of the procedure is evaluated. For k outliers, steps 1 to 4 can be repeated k times with an appropriate modification in step 1. The steps are presented in details in Section 4.5.

We report the performance evaluation for AR(1) and MA(1) series here. The performance evaluation presented here is based on 1000 replications of each contaminated series for values of outlier parameter ω = 0, 2(1)7. In order to evaluate the performance, it is important to investigate the proportion of times the procedure identifies correct outlier type and the proportion of times it indicates the correct outlier position. Hence, while reporting the performance of Ω^*, we also report the percentage of times the correct outlier type is diagnosed and the percentage of times the correct outlier position is identified.

It is crucial to compare the performance of the proposed adjustment diagnostic procedure with that of the deletion diagnostic procedures available in the literature which are based on estimates of error variance. The available procedures are those by Peña (1987), Bruce and Martin (1989) and Ledolter (1990), which are of similar nature. Ledolter presented a detailed evaluation of the performance of the procedure which is same as the deletion diagnostic for error variance proposed by Bruce and Martin. Hence, we present the comparison of the performance of the proposed procedure with that of Ledolter.

All the simulations and computations presented here are carried out using the computer programs CP-6 and CP-7 in the attached CD and given in the programs' list of Appendix C.

outperform Ledolter's procedure in the presence of IO, we separately report the performance of the procedures based on Ω_A and Ω_I in Tables 4.3 and 4.5 respectively and report the comprehensive performance of Ω^* in Table 4.6.

First, an additive outlier is introduced at t = 50 in the generated series. Table 4.3 reports the percentage of times the adjustment diagnostic and deletion diagnostic indicates the presence of outlier and identifies the correct position.

Columns 2 and 3 are based on performance of Ω_A using the fact that the type of outlier is known and hence are equal to the columns 6 and 7 based on deletion diagnostic. The values are close to the values reported by Ledolter (1990) which are reproduced in Table 4.4.

Table 4.4

Proportion of Outlier Detection and Position Identification
for Deletion Diagnostics: AR(1) with an AO at t = 50
($C = 13, n = 100, \phi = 0.5, \sigma_a^2 = 1$; 1000 replications)

ω	Detection	Position Identification
0	0.042	-
1	0.061	0.011
2	0.130	0.093
3	0.446	0.426
4	0.799	0.791
5	0.967	0.966

Source: Ledolter (1990).

The small differences in the values are possibly due to different estimation method of procedures and sampling variation. Columns 4 and 5 of Table 4.3 report the performance of Ω_I in the presence of an AO and as expected, the performance based on Ω_A is better. The correct picture of performance evaluation will emerge from the use of Ω^* which is presented later.

Next, an innovational outlier was introduced at t = 50 in the generated AR(1) series and the performance evaluation is presented in Table 4.5. The proportion of times Ω_I identified the IO type outliers and the proportion of times it identified correct position is reported in columns 4 and 5 respectively. The corresponding quantities based on Ω_A are reported in columns 2 and 3 which are equivalent to those based on the available deletion diagnostic in columns 6 and 7 of the table.

Table 4.5

Proportion of Outlier Detection and Position Identification for Adjustment and Deletion Diagnostics: AR(1) with an IO at t = 50
($C = 13$, $n = 100$, $\phi = 0.5$, $\sigma_a^2 = 1$; 1000 replications)

ω	Adjustment Diagnostic				Deletion Diagnostic	
	Ω_A		Ω_I			
	Detection $\Omega_A>C$	Position Identification	Detection $\Omega_I>C$	Position Identification	Detection	Position Identification
0	0.048	-	0.053	-	0.048	-
1	0.052	0.002	0.062	0.008	0.052	0.002
2	0.090	0.048	0.131	0.080	0.090	0.048
3	0.238	0.196	0.321	0.294	0.238	0.196
4	0.498	0.470	0.669	0.655	0.498	0.470
5	0.800	0.768	0.909	0.906	0.800	0.768

It is clear from the comparison of columns 2 and 3 with columns 4 and 5 respectively that the proposed adjustment diagnostic procedure outperforms the deletion diagnostic procedure when the outlier present in the series is of IO type. For instance, for $\omega = 4$, the deletion diagnostic method fails to detect the presence of an outlier more than 50% of the time. It supports the claim made in Chapter 2 that the deletion diagnostic method fails to satisfactorily handle the presence of IO type of outliers.

We now present the performance of the proposed ADV based diagnostic procedure which uses Ω^* given by (4.11). The procedure was carried out on two contaminated AR(1) series, contaminated with an AO and an IO at $t = 51$. In Tables 4.6 and 4.7, we present the performance of Ω^* in the presence of outlier of AO and IO type respectively. The warning value is retained at 13 for these tables and range of outlier parameter is increased to $\omega = 0, 2(1)7$. Note that the value of time series parameter ϕ is changed to 0.6 for the computations.

Columns 2 and 3 of Table 4.6 give the proportion of times Ω^* identifies the outlier and proportion of times Ω^* identifies the outlier at correct position respectively. As mentioned earlier, it is not surprising that the proportions obtained for $\omega = 0$ are much larger than 0.05 for $C = 13$, since Table 4.2 indicates that a value close to 14.5 will be more appropriate. Column 4 presents the proportion of times out of its corresponding proportions in column 2 the procedure identifies the correct outlier type.

Table 4.6

Performance Analysis of Adjustment Diagnostic Procedure:
AR(1) with an AO at t=51
($C = 13$, $n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

ω	Ω^*		AO Type ($\Omega^* = \Omega_A$)		IO Type ($\Omega^* = \Omega_I$)	
	Detection $\Omega^*>C$	Position Identification	Detection	Position Identification	Detection	Position Identification
0	0.080	-	0.525	-	0.475	-
2	0.197	0.137	0.670	0.715	0.330	0.285
3	0.532	0.475	0.791	0.838	0.209	0.162
4	0.865	0.845	0.879	0.889	0.121	0.111
5	0.988	0.984	0.889	0.892	0.111	0.108
6	1.000	1.000	0.936	0.936	0.064	0.064
7	1.000	1.000	0.966	0.966	0.034	0.034

In this table, the figures in column 4 (5) look inflated compared to the figures in column 2 (3) of Table 4.3 since they report the proportion out of the total proportion given in column 2 (3). For instance, for $\omega = 4$ out of 86.5% of times Ω^* identifies an outlier, 87.9% of times it correctly identifies the type as AO. This is because the proportion of times $\Omega^* > C$ and $\Omega^* = \Omega_A$ out of 1000 replications is 0.76. As anticipated, this proportion is less than the proportion of times $\Omega_A > C$ reported in Table 4.3. Plots of these actual proportions are presented in Figure 4.1 later.

Column 5 gives proportion of times the AO is identified at correct position out of the total proportion presented in column 3. For instance, for $\omega = 4$ out of 84.5% of times Ω^* identifies an outlier at correct position, 88.9% of times it identifies correct type at correct position. Columns 6 and 7 give the corresponding proportions in case of wrong identification of the type of outlier.

Table 4.7 presents the analogous figures in case of a series contaminated with an IO.

Table 4.7

Performance Analysis of Adjustment Diagnostic Procedure:
AR(1) with an IO at t=51
(C = 13, n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

ω	$\Omega^* > C$		AO Type ($\Omega^* = \Omega_A$)		IO Type ($\Omega^* = \Omega_I$)	
	Detection	Position Identification	Detection	Position Identification	Detection	Position Identification
0	0.100	-	0.580	-	0.420	-
2	0.159	0.091	0.308	0.209	0.692	0.791
3	0.351	0.307	0.211	0.153	0.789	0.847
4	0.714	0.679	0.140	0.110	0.860	0.890
5	0.934	0.924	0.081	0.074	0.919	0.926
6	0.990	0.985	0.057	0.053	0.943	0.947
7	0.999	0.996	0.023	0.021	0.977	0.979

It is clear from Tables 4.6 and 4.7 that the identification of the correct outlier type and the correct position are all significantly high for the proposed procedure based on adjustment diagnostics in case of AR(1).

To get a clearer picture of the comparative performance of ADV in the presence of AO and IO type of outlier, in Figure 4.1, the plots of percentage of correct type identification at correct position are presented. Out of 1000 replications, the proportion of times $\Omega^* > C$, $\Omega^* = \Omega_A$ and the outlier position is 51 when the series is contaminated with an AO is plotted against values of outlier parameter ω in Figure 4.1 (a) by solid line (—). The dotted line (-----) indicates the proportion of times $\Omega^* > C$, $\Omega^* = \Omega_I$ and the outlier position is 51 for the same series with an AO. The same percentages in case of the series with an IO are presented in Figure 4.1 (b).

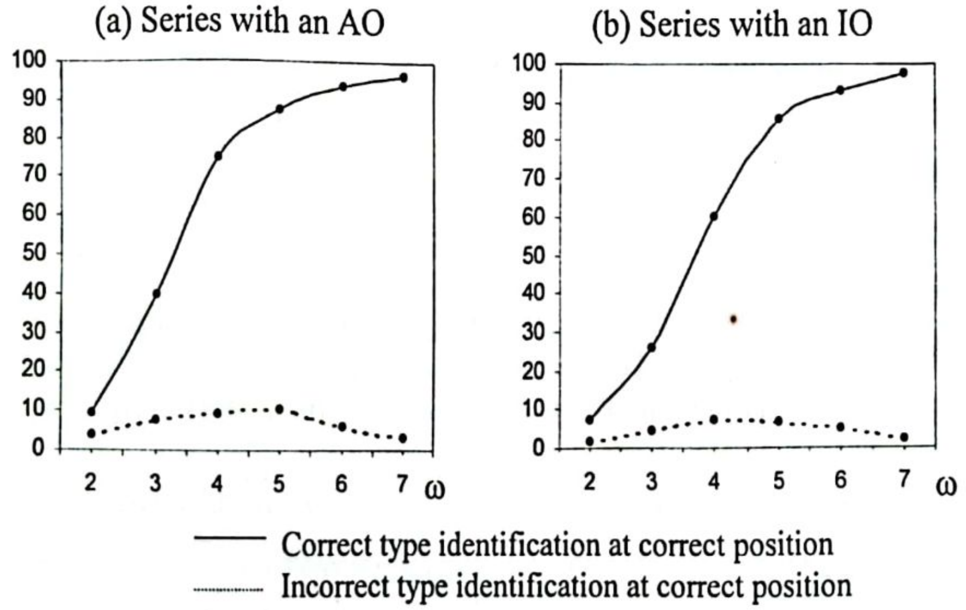


Figure 4.1: Percentage of Correct Identification of Type and Position: AR(1) with an Outlier at t = 51 (C=13, n=100, ϕ= 0.6, σ_a^2=1 ; 1000 replications)

As can be seen from Figure 4.1, the percentages of times the correct types are identified at true position for both type of series are significantly higher than those for the incorrect type identification. The percentages of incorrect type identification decreases after certain value of ω, whereas for small values of ω the errors in type and position identification can be higher, as is expected. The percentage of times in correct type identification at correct position significantly increases for values of ω larger than 3. For AR(1), the correct identification rate for AO type is higher than that for IO type, particularly for small values of ω. This is not surprising, considering the effect of AO on the estimates of error variance (Table 2.1) in comparison with that of IO on the same estimates (Table 2.2).

We now present the performance of Ω^{*} using the warning value C = 14.5 based on two contaminated AR(1) series, contaminated by an AO and IO at t = 51. The performance in the presence of AO and IO is presented together in Table 4.8.

Table 4.8

Performance Analysis of Adjustment Diagnostic Procedure:
AR(1) with an Outlier at t=51
(C = 14.5, n=100, ϕ=0.6, σ_a^2=1; 1000 replications)

	Series with an AO				Series with an IO			
	$\Omega^{*} > C$		AO Type ($\Omega^{*}=\Omega_A$)		$\Omega^{*} > C$		IO Type ($\Omega^{*}=\Omega_I$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.062	-	0.468	-	0.065	-	0.554	-
2	0.134	0.091	0.627	0.692	0.094	0.055	0.638	0.764
3	0.442	0.412	0.799	0.820	0.284	0.255	0.845	0.875
4	0.805	0.796	0.870	0.873	0.613	0.592	0.874	0.892
5	0.976	0.973	0.899	0.899	0.902	0.895	0.915	0.918
6	1.000	1.000	0.937	0.937	0.983	0.983	0.953	0.953
7	1.000	1.000	0.955	0.955	0.998	0.998	0.974	0.974

It can be seen that for $\omega = 0$ the proportion is closer to 0.05 than that for C = 13, though high. The figures in Table 4.8 can be interpreted analogous to figures in Table 4.6 and 4.7. For instance, in case of an AR(1) contaminated by an IO at t = 51 for $\omega = 4$, Ω^* indicates an outlier at correct position 59.2% of times, out of which 89.2% times correct type is identified at correct position.

We present the comparative performance of Ω^* in the presence of AO and IO in Figure 4.2 by plotting percentage of times $\Omega^* > C$ against the value of ω.

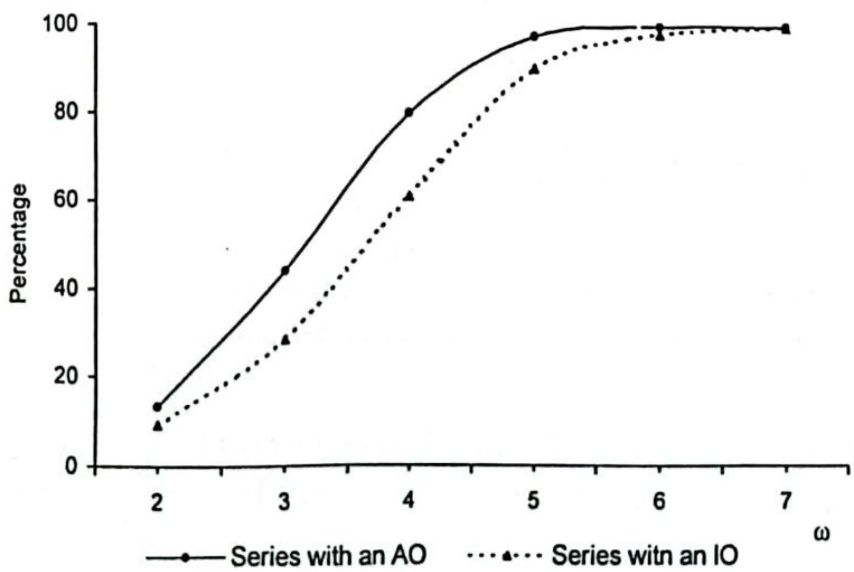


Figure 4.2: Plot of Percentage of Performance: AR(1) with an Outlier at t = 51 ($C = 14.5$, $n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

It is clear from the figure that the performance of the Ω^* is better when the outlier is of AO type than when it is of IO type. This is as anticipated since the procedure is based on estimated error variance and it is shown in Tables 2.1 and 2.2 that the AO type of outlier affects the estimates of error variance more than IO type of outlier.

Analogous to Figure 4.1(a) and (b) we present the plot of percentage of times the detected outlier is of correct type at correct position (reported by solid line) and that of incorrect type at correct position (reported by dotted line) in Figure 4.3(a) and (b) for AR(1) series with an AO and IO type of outlier respectively, when the warning value C = 14.5.

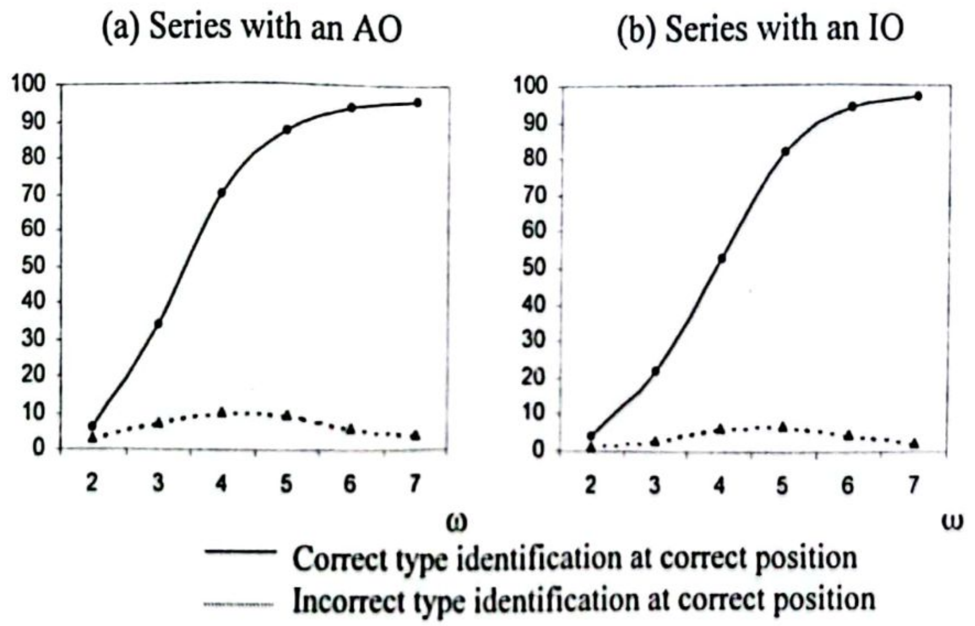


Figure 4.3: Percentage of Correct Identification of Type and Position: AR(1) with an Outlier at t = 51 (C=14.5, n=100, ϕ=0.6, σ_a^2=1 ; 1000 replications)

It can be seen that the conclusion similar to those in case of Figure 4.1 can be drawn in this case also.

From the discussion presented so far, it can be concluded that the proposed adjustment diagnostic satisfactorily identifies the presence of outlier, the type of outlier and also the position of outlier, particularly for large magnitudes of outlier parameter in case of AR(1) series. In addition, the adjustment diagnostic method is more comprehensive than the deletion diagnostic since the deletion diagnostic does not work well for an IO type of outlier and does not identify the type of the outlier.

The performance evaluation presented so far is for a fixed value of AR(1) parameter $\phi = 0.6$. In order to evaluate the performance for different values of ϕ, the evaluation was carried out for $\phi = -0.9, -0.6, -0.3, 0.3, 0.6, 0.9$ for series length n =100 on identical lines for C = 14.5. The findings are similar to those in Table 4.8. The detailed performance for each value of ϕ is not reported here. Instead, we report the dependence of the performance of ADV on the time series parameter ϕ. In particular, we present plots of percentage of times $\Omega^* > C$ against the value of parameter ϕ for values of outlier parameter ω = 3, 4, 5. The warning value of C is fixed at 14.5 for this evaluation. Figure 4.4 (a) and (b) give these plots in the presence of AO and IO type of outlier at t = 51 respectively.

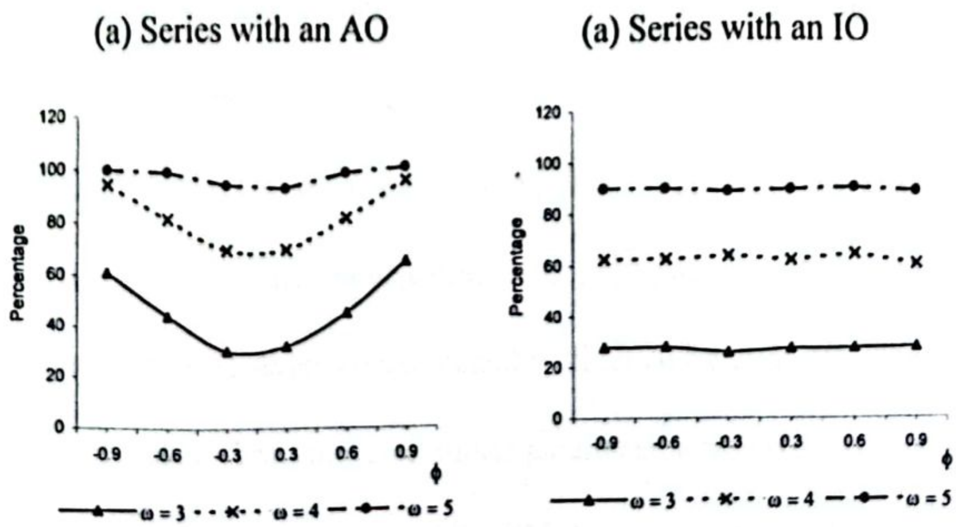


Figure 4.4: Plot of Percentage of Performance: AR(1) with different values of ϕ (C = 14.5, n =100, σ_a^2=1; 1000 replications)

It can be seen that the performance of the diagnostic procedure depends on the time series parameter and which significantly improve with increases in the value of $|\phi|$ when an AO type of outlier is present for moderate values of outlier parameter ω. For large values of ω the performance of procedure is almost 100%. Surprisingly, the performance of the procedure in the presence of an IO type of outlier does not depend on the time series parameter ϕ.

From the two plots, it is clear that the estimate of error variance in the presence of AO type of outlier depends on the time series parameter ϕ in case of AR(1). That does not seem to be the case in the presence of IO. The reason behind this phenomenon is not clear and needs further investigations. However, it not been carried out in this work.

MA(1) with One Outlier

We investigated the performance of the proposed procedure on a MA(1) series with a single outlier an analogous lines. Initially MA(1) series with parameter $\theta = -0.6$ and $\sigma_\varepsilon^2 = 1$ and of length $n = 100$ was considered. As in the case of AR(1), the simulation was replicated 1000 times and from each series two contaminated MA(1) series contaminated with an AO and an IO at time point $t = 51$ were generated. The values of outlier parameter were taken to be $\omega = 0, 2(1)7$ as before. Considering the values for MA(1) presented in Table 4.2, a warning value of $C = 15$ was selected subjectively.

On the lines of Table 4.8, Table 4.9 presents the performance of the proposed ADV procedure in case of an MA(1) when the series is contaminated with an AO and an IO. From the proportions reported for $\omega = 0$, it is clear that a smaller value of C would have been more appropriate. The figures in Table 4.9 can be interpreted on the same lines as those in Table 4.8.

Table 4.9

Performance Analysis of Adjustment Diagnostic Procedure:
MA(1) with an Outlier at t=51
($C=15, n=100, \theta = -0.6, \sigma_a^2=1$; 1000 replications)

	Series with an AO				Series with an IO			
	$\Omega^* > C$		AO Type ($\Omega^*=\Omega_A$)		$\Omega^* > C$		IO Type ($\Omega^*=\Omega_I$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.034	-	0.529	-	0.058	-	0.552	
2	0.140	0.096	0.629	0.740	0.110	0.053	0.636	0.755
3	0.459	0.421	0.813	0.841	0.261	0.226	0.808	0.872
4	0.825	0.805	0.893	0.901	0.597	0.574	0.901	0.918
5	0.975	0.963	0.929	0.931	0.866	0.854	0.940	0.951
6	0.999	0.997	0.953	0.954	0.973	0.970	0.967	0.969
7	1.000	1.000	0.957	0.957	0.996	0.996	0.985	0.985

For instance, in case an AO is present in the series for $\omega = 4$, the percentage of times Ω^* indicates an outlier is 82.5% out of which 89.3% times the procedure correctly identifies the outlier type as AO. Further, it correctly identifies the outlier type and position 90.1% of times out of 80.5% of times Ω^* identifies the outlier position correctly. In the presence of an IO, the percentage identification of an outlier based on Ω^{*} for $\omega = 4$ is 59.7%, much less than that in the presence of AO. The percentage of correct identification of outlier type, however, is 90.1% close to that in the presence of AO.

Analogous to Figure 4.2, we present the comparative performance of Ω^{*} for MA(1) series in the presence of AO and IO type of outlier in Figure 4.5.

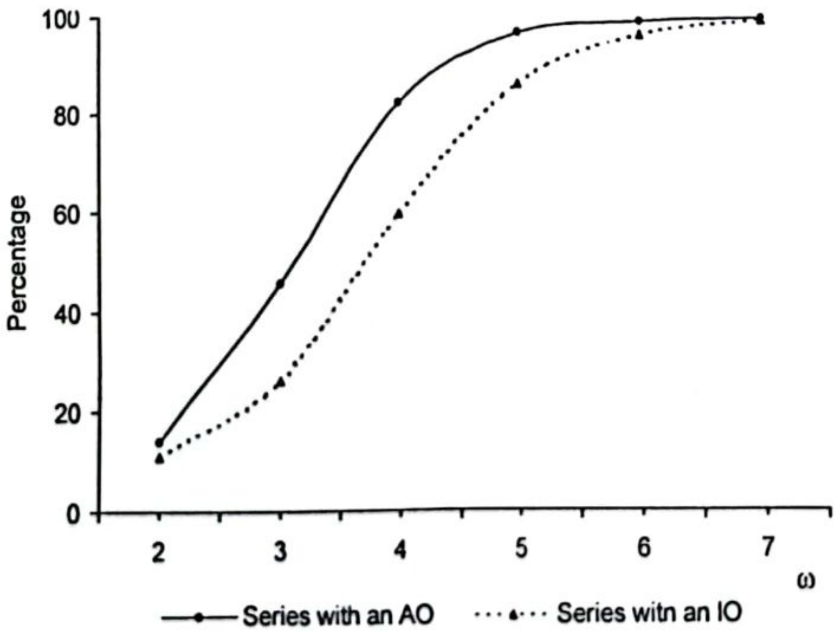


Figure 4.5: Plot of Percentage of Performance: MA(1) with an Outlier at t = 51 ($C = 15$, $n = 100$, $\theta = -0.6$, $\sigma_a^2 = 1$; 1000 replications)

The figure clearly shows that the procedure has significantly better performance in the presence of AO than in the presence of IO. As in case of AR(1), this is as anticipated in view of Tables 2.3 and 2.4 where it is shown that presence of an AO type of outlier affects the estimate of error variance more than the presence of an IO type.

Analogous to Figure 4.3 (a) and (b), we present the comparative performance of Ω^* in the presence of AO and IO type of outlier in Figure 4.6 (a) and (b) below.

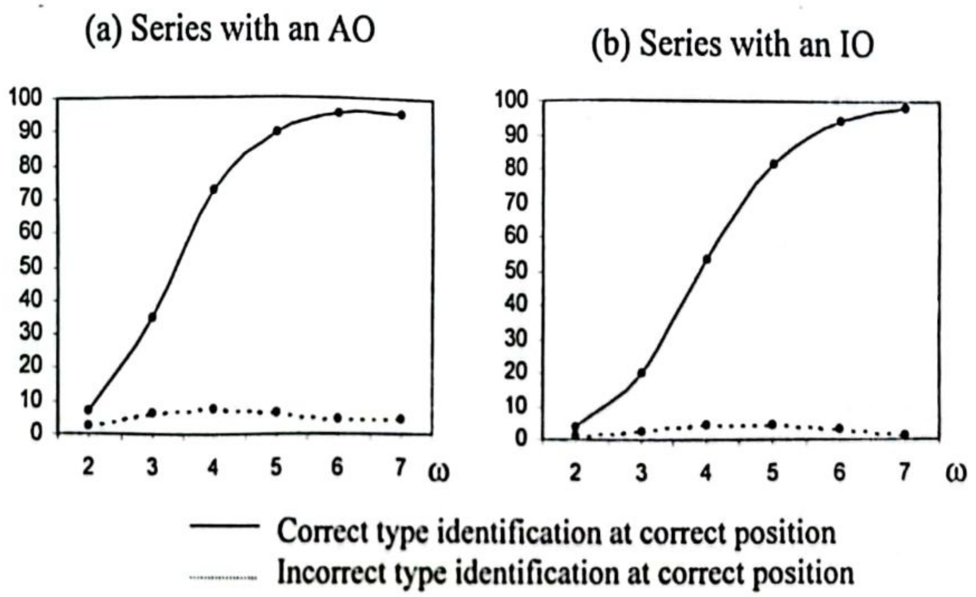
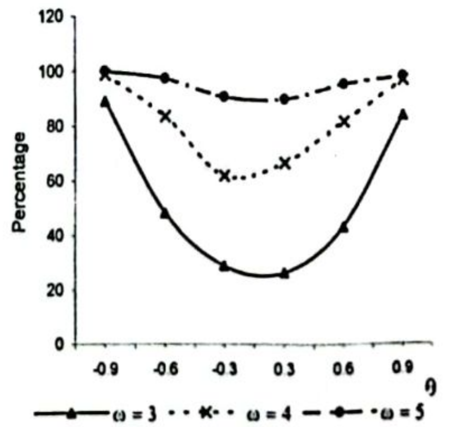


Figure 4.6: Percentage of Type Identification at Correct Position: MA(1) with an Outlier at t=51 ($C = 15, n = 100, \theta = -0.6, \sigma_a^2=1$; 1000 replications)

The plots are of percentages of type identification at correct position against the value of outlier parameter ω. The two plots are comparable to the plots presented in Figure 4.1 for AR(1) and the conclusions for MA(1) based on them are similar to those of AR(1). Comparing Figure 4.6 with Figure 4.3 of AR(1), the detection of a wrong type of outlier or wrong position of an outlier seems slightly less in a MA(1) series than an AR(1) series. In the presence of an IO in the series, the chances of wrong identification of the outlier type reduce considerably for a moderately large value of ω.

We now present the performance of Ω^* in case of MA(1) series when the series parameter θ takes different values. The chosen values of θ are $\theta = -0.9$, -0.6, -0.3, 0.3, 0.6, 0.9, keeping the values of other factors same. The plots of performance of Ω^* against the values of θ is shown in Figure 4.7 below.

(a) Series with an AO



(a) Series with an IO

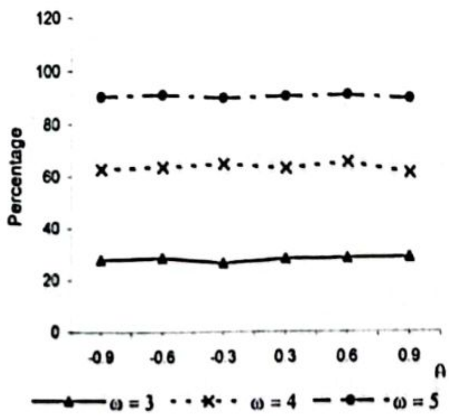


Figure 4.7: Plot of Percentage of Performance: MA(1) with different values of θ ($C = 14.5$, $n = 100$, $\sigma_a^2 = 1$; 1000 replications)

The plots can be interpreted analogous to those in Figure 4.4 for AR(1). It can be seen that in the presence of an AO, the performance of Ω^* depends on the values of θ and increases with $|\theta|$ for moderate values of ω. As in case of AR(1), here too, the performance of the procedure in the presence of an IO type of outlier does not depend on the time series parameter θ. Further investigations are needed to understand the reasons behind it.

In conclusion, the adjustment diagnostic based procedure (ADV) outperforms the deletion diagnostic based procedure since it works comprehensively in the presence of both AO and IO type of outliers unlike deletion diagnostics which satisfactorily handles only the AO type of outliers. In addition, ADV satisfactorily handles both the problem of identification of outlier type and outlier position. The evaluation here is based on ML estimator of β and in Section 4.4 we present the performance evaluation using robust estimator of β.

Applications of ADV to simulated data sets and some numerical examples available in the literature are presented in Chapter 5.

4.4 Performance Evaluation of Adjustment Diagnostic Using Robust Estimator

We evaluate the performance of the proposed procedure in the presence of a single outlier in an AR(1) and MA(1) series using robust estimator of β in this section. Further, we compare the performance with that presented so far using ML estimator.

As mentioned earlier, the ML estimate of β in the simulation study is obtained with the help of IMSL subroutine NSLSE. This uses an iterative nonlinear least squares (LS) estimator (Box et al., 1994, Chapter 7) and we refer it by BJ (Box-Jenkins) estimator in this section (Bustos and Yohai, 1986).

It is well known that the LS estimators are asymptotically efficient when the errors are i.i.d. normal with mean 0 and variance σ_a^2 (Box et al., 1994, Chapter 7). On the other hand, these are not necessarily robust when the series has a few outliers or abnormal observations (Barnett and Lewis, 1994; Maddala and Yin, 1997). The fact that the LS estimators are not robust in the presence of AO type of outliers has been well documented in the literature (Denby and Martin, 1979; Barnett and Lewis, 1994, p. 404) and can also be seen from the discussion presented in Section 2.3 for AR(1) and MA(1) (Tables 2.1, 2.3 and Figures 2.4, 2.7) for BJ estimators.

The LS estimators are consistent in the presence of IO outliers under the assumption of Gaussian errors (Li, 2004, Chapter 4). Further, based on Monte Carlo results and asymptotic variance, Denby and Martin (Concluding Comments, 1979) claim that for AR(1) "... no great loss will be suffered in case of innovations outlier" using LS. This claim is further supported by the study presented in Bustos and Yohai (1986) and Martin and Yohai (1985) (also see Barnett and Lewis, 1994, pp. 403-405). The simulation study of BJ estimators presented in Section 2.3 in the presence of one IO outlier (Tables 2.2, 2.4 and Figures 2.5 and 2.8) shows that BJ estimators are fairly robust in the presence of IO for AR(1) as well as MA(1).

Some of the robust estimators for time series parameters proposed in the literature are M-estimators, GM-estimators (Denby and Martin, 1979), RA (Residual Autocovariances) and TRA (Truncated Residual Auto-covariances) estimators proposed by Bustos and Yohai (1986).

Bustos and Yohai (1986) show that RA estimator is better than M, GM and LS estimators in AR(1) as well as MA(1) with an AO type of outlier. In particular, based on Monte Carlo studies, they claim that "RA ...behave robustly in terms of efficiency for the MA(1) model". They also report that in the presence of AO, the

RA-estimates are qualitatively robust for autoregressive process and behave robustly in terms of efficiency when the order q of MA model is greater than 0. Goodness-of-fit tests for ARMA(p,q) series using RA estimates are also proposed in the literature (Li, 2004, Chapter 4).

Here, we consider the Mallows type RA-estimator proposed by Bustos and Yohai (1986) with $\eta(u,v) = \psi(u)\ \psi(v)$, where $\psi(.)$ is the redescending function from a bisquare family proposed by Beaton and Tukey (1974), given by

$$\psi_{B,c}(u) = \begin{cases} u(1-u^2/c^2)^2 & \text{for} |u| \le c \\ 0 & \text{for} |u| > c \end{cases} \tag{4.12}$$

where c is a tuning constant.

Following the Monte Carlo study of asymptotic relative efficiency presented in Bustos and Yohai (1986) for the above mentioned Mallow's Bisquare type RA estimator (RAMB), the tuning constant c is set at 5.58. This choice of tuning constant achieved 95% asymptotic relative efficiency with respect to the LS estimator under Gaussian errors in the simulation study presented by Bustos and Yohai (1986).

The iterative algorithm used for RA estimates for a stationary and invertible ARMA(p,q) is as follows (Martin and Yohai, 1985).

Suppose we have the LS estimates of $\beta = (\phi', \theta')'$ and scale estimate of σ given by $\hat{\beta}^{(i)}$ and $\hat{\sigma}^{(i)}$ respectively for the i^{th} iteration. For the estimates for the $(i+1)^{th}$ iteration,

(i) Compute the residuals $r_t(\hat{\beta}^{(i)})$ for $p+1 \le t \le n$ where $r_t(\hat{\beta}) = \theta^{-1}(B)\,\phi(B)Y_t$.

(ii) Modify the residuals by applying the ψ-function,

$$r_t^{\bullet} = \hat{\sigma}^{(i)}\psi(r_t(\hat{\beta}^{(i)})/\hat{\sigma}^{(i)}) \text{ where } \psi(.) \text{ is given by (4.12)}.$$

(iii) Calculate a new "pseudo-observations" process $\{Y_t^{\bullet}\}$ using $\hat{\beta}^{(i)}$ and $r_t^{\bullet}$,

given by

$$Y_t^{\bullet} = \hat{\phi}^{(i)-1}(B)\ \hat{\theta}^{(i)}(B)r_t^{\bullet}.$$

(iv) Compute $\hat{\beta}^{(i+1)}$ as the least squares estimate of β for $Y_t^{\bullet}$.

(v) Compute the scale estimate $\hat{\sigma}^{(i+1)}$.

In the computations presented, we consider the median of the absolute values of the residuals divided by 0.6745 as a robust scale estimate of residuals in step (v). The computer program CP-7 used for the computations of RA is provided in the attached CD and listed in Appendix C.

As in case of Section 4.3, the performance evaluation was carried out for AR(1) and MA(1) with series length $n = 100$ for various values of time series parameters. The evaluation is based on 1000 replications of each contaminated series with outlier parameter $\omega = 0, 2(1)7$ for AO and IO type. We report the performance of the proposed procedure using both BJ and RA estimates of β for comparison.

AR(1) with One Outlier

Initially, we report the performance of the proposed procedure for AR(1) with ϕ and σ_a^2 same as in Section 4.3 ($\phi = 0.6$, $\sigma_a^2=1$). The critical value is taken same as that in Tables 4.8 (C = 14.5). We first consider the situation when the series is contaminated with an AO at the same position t = 51.

Table 4.10

Performance Evaluation Using BJ and Robust Estimates:
AR(1) with an AO at t=51
($C = 14.5$, $n=100$, $\phi=0.6$, $\sigma_a^2=1$; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		AO Type ($\Omega^*=\Omega_A$)		$\Omega^* > C$		AO Type ($\Omega^*=\Omega_A$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.043	-	0.349	-	0.043	-	0.349	-
2	0.123	0.099	0.691	0.747	0.119	0.096	0.748	0.802
3	0.446	0.428	0.787	0.794	0.445	0.429	0.818	0.823
4	0.812	0.811	0.853	0.852	0.811	0.810	0.882	0.880
5	0.978	0.977	0.908	0.906	0.978	0.977	0.929	0.927
6	0.999	0.999	0.952	0.952	0.999	0.999	0.969	0.969
7	1.000	1.000	0.956	0.956	1.000	1.000	0.981	0.981

Analogous to tables in Section 4.3, columns 2 and 3 (6 and 7) of Table 4.10 give the proportion of times Ω^* detects the outlier and the proportion of times Ω^* indicates the outlier at correct position using BJ (RA) estimate. Column 4(8) gives the proportion of times the procedure identifies the correct outlier type AO out of its corresponding proportion in column 2(6) using BJ (RA) estimate. Further, column 5(9) gives the proportion of times the AO is detected at the correct position using BJ (RA) estimate.

It can be seen from the table (columns 2,3 and 6,7) that the percentages of outlier detection and position identification based on Ω^* using BJ and RA estimates of ϕ are almost identical. The procedure using RA estimate of ϕ however performs marginally better than that using BJ estimate for identification of correct outlier type. For instance for $\omega = 5$, out of 97.8% of times Ω^* detects an outlier, the procedure correctly identifies the type as AO 92.9% of times using RA estimate whereas the procedure identifies the type as AO 90.8% of times using BJ estimate. Based on the discussion presented at the beginning of this section, the procedure using RA estimate is expected to perform better than LS estimator in the presence of an AO. However, based on the simulation study presented here, it can be seen that on the whole only the percentage performance of correct type identification and position detection of the correct type marginally improves by about 2 to 3 units for $\phi = 0.6$. The performance of outlier detection and position identification remains the same.

However, the simulation study in the presence of an IO type of outlier showed that the performance of the proposed procedure using BJ estimator of ϕ in AR(1) is marginally better than that using RA estimator which is discussed below.

In Table 4.11 we report the performance evaluation of the proposed procedure using BJ and RA estimates of time series parameters of an AR(1) with ϕ=0.6, which is contaminated by an IO at t =51.

Table 4.11

Performance Evaluation Using BJ and Robust Estimates:
AR(1) with an IO at t=51
(C = 14.5, n=100, ϕ=0.6, σ_a^2=1; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		IO Type ($\Omega^* = \Omega_I$)		$\Omega^* > C$		IO Type ($\Omega^* = \Omega_I$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.059	0.003	0.492	0.333	0.058	0.002	0.483	0.000
2	0.094	0.054	0.702	0.796	0.091	0.053	0.670	0.755
3	0.270	0.249	0.793	0.811	0.266	0.244	0.771	0.795
4	0.628	0.616	0.876	0.875	0.621	0.610	0.857	0.857
5	0.894	0.890	0.924	0.925	0.888	0.883	0.896	0.897
6	0.989	0.989	0.958	0.958	0.988	0.988	0.943	0.943
7	0.999	0.999	0.980	0.980	0.999	0.999	0.963	0.963

The figures in Table 4.11 can be interpreted on the same lines as those in Table 4.10 above. As can be seen from the table, Ω^* performs almost identically in detection of outlier and identification of correct position of outlier whether BJ estimate or RA estimate of ϕ is used. For instance, for ω = 5 it detects the outlier 89.4 % (88.8%) of times and identifies the correct position 89.0% (88.3%) of times using BJ (RA) estimate of ϕ. However, there is difference between the performances of the procedure using BJ and RA estimates in identification of correct outlier type (columns 4, 5 and 8, 9). For instance for $\omega = 5$, the proposed procedure identified the correct outlier type 92.4% (89.6%) of times and identified the correct position 92.5% (89.7%) of times using BJ (RA) estimate of ϕ.

Based on the simulation study it can be claimed that the procedure using BJ estimate performs marginally better than that using RA estimate in correctly identifying the type of outlier for moderate values of ω when the outlier is of IO type. For large values of ω the procedure achieves high accuracy irrespective of the choice of either of the estimates. The difference in the percentage performance of two procedures however is small, about 2 to 3 as in the case of AO, thought the procedure based on BJ performs better in the presence of IO.

As illustrated in Section 4.3, the performance of the proposed procedure using BJ estimate depends on the time series parameter ϕ and improves with increase in the value of $|\phi|$ in the presence of an AO type of outlier. A study of the performance using BJ and RA estimates for various values of ϕ was carried out in the presence of AO and IO outlier in a contaminated AR(1) and similar phenomenon was observed.

Table 4.12 presents the performance evaluation using BJ and RA estimates for $\phi = 0.9$ in the presence of an AO where the figures can be interpreted analogous to those in Table 4.10.

Table 4.12

Performance Evaluation Using BJ and Robust Estimates:
AR(1) with an AO at t=51
($C = 14.5$, n=100, ϕ=0.9, σ_a^2=1; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		AO Type ($\Omega^*=\Omega_A$)		$\Omega^* > C$		AO Type ($\Omega^*=\Omega_A$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.058	-	0.517	-	0.057	-	0.526	-
2	0.192	0.149	0.729	0.772	0.192	0.150	0.745	0.787
3	0.614	0.590	0.889	0.897	0.612	0.589	0.904	0.908
4	0.943	0.938	0.954	0.952	0.943	0.938	0.965	0.963
5	0.995	0.995	0.980	0.980	0.995	0.995	0.986	0.986
6	1.000	1.000	0.996	0.996	1.000	1.000	0.998	0.998
7	1.000	1.000	0.997	0.997	1.000	1.000	0.999	0.999

It can be seen that the performance of the procedure to detect the outlier and identify its position is same (columns 2,3 and 6,7), whether BJ or RA estimate is used in the presence of AO. Further the difference between the percentage performances of the procedure to identify the outlier type and its correct position using BJ and RA estimates is reduced to less than 1.

For the values of ϕ used for simulations, it was seen that the performances to detect the outlier and its correct position was same using BJ and RA estimators of ϕ. For small values of ϕ, the procedure using RA estimator identified the type and its correct position with marginally better accuracy than BJ estimator in the presence of AO. For instance, for $\phi = 0.3$ the difference between columns 4 and 8 is about 5 to 7.

Analogous to the AO case, the performance of the proposed procedure was evaluated for various values of ϕ in the presence of IO. We present the evaluation based on 1000 simulations for AR(1) with $\phi = 0.9$ contaminated at t = 51 with an IO in Table 4.13.

Table 4.13

Performance Evaluation Using BJ and Robust Estimates:
AR(1) with an IO at t=51
($C = 14.5$, $n=100$, $\phi=0.9$, $\sigma_a^2=1$; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		IO Type ($\Omega^*=\Omega_I$)		$\Omega^* > C$		IO Type ($\Omega^*=\Omega_I$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.047	0.001	0.532	-	0.045	0.001	0.511	-
2	0.108	0.060	0.630	0.733	0.106	0.060	0.623	0.733
3	0.271	0.250	0.838	0.852	0.257	0.239	0.825	0.841
4	0.623	0.606	0.904	0.916	0.610	0.593	0.892	0.906
5	0.888	0.885	0.956	0.957	0.884	0.881	0.948	0.949
6	0.989	0.988	0.982	0.983	0.988	0.988	0.977	0.977
7	0.999	0.999	0.996	0.996	0.999	0.999	0.990	0.990

It can be seen from the table that the difference in the performances of the proposed procedure using BJ and RA estimates of ϕ is much less for $\phi = 0.9$ and is almost negligible for large values of ω.

Based on the simulation study it can be claimed that the performance of the procedure to detect the outlier and identify its position remained same using the two difference estimators, but the performance to identify the type and its position was marginally better using BJ estimator in the presence of IO. For small values of ϕ the difference is slightly larger, for instance, for $\phi = 0.3$ the difference in the percentages is about 5 to 7 and the procedure using BJ performs better when IO is present.

In conclusion, for a contaminated AR(1) series, it can be claimed that no loss in accuracy of detection and identification of outlier is incurred by using BJ estimate in the proposed procedure based on Ω^* irrespective of whether the outlier is an AO or IO in an AR(1) series. Further, the percentage identification of correct type of outlier is marginally less using BJ estimate as against a robust RA estimate when the outlier is of AO type. A reverse trend is observed when the outlier is of IO type and the percentage identification of the correct type is again marginally better using BJ estimate as against RA estimate. This is not surprising considering the discussion and references presented at the beginning of this section claiming that LS estimators are not very sensitive to the presence of IO type of outlier.

MA(1) with One Outlier

We now present the performance evaluation of the procedure for a contaminated MA(1) series of length 100 with parameters $\theta = -0.6$ and $\sigma^2 = 1$.

The critical value is taken to be the same as in Table 4.9, namely C = 15. Table 4.14 reports the performance evaluation when the series is contaminated with a single AO at t = 51.

Table 4.14

Performance Evaluation Using BJ and Robust Estimates:
MA(1) with an AO at t = 51
(C = 15, n = 100, $\theta = -0.6$, $\sigma_a^2 = 1$; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		AO Type ($\Omega^* = \Omega_A$)		$\Omega^* > C$		AO Type ($\Omega^* = \Omega_A$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.037	-	0.541	-	0.037	-	0.568	-
2	0.130	0.099	0.746	0.747	0.127	0.097	0.787	0.784
3	0.476	0.451	0.847	0.854	0.470	0.448	0.872	0.875
4	0.849	0.846	0.921	0.911	0.849	0.846	0.947	0.936
5	0.972	0.971	0.942	0.940	0.974	0.973	0.961	0.959
6	1.000	0.999	0.966	0.966	1.000	0.999	0.981	0.981
7	1.000	1.000	0.961	0.961	1.000	1.000	0.988	0.988

The figures in the table can be interpreted on the lines of Tables 4.10-4.13 presented earlier. For instance, for ω = 5, 97.2% (97.4%) of times the proposed procedure based on Ω^* using BJ (RA) estimate of the time series parameter θ detects an outlier, out of which it detects the correct position of the outlier 97.1% (97.3%) of times. Further, the procedure using BJ estimate identifies the correct outlier (AO) 94.2% of times out of which 94.0% of times it correctly identifies its position. The procedure using RA estimate identifies the correct outlier (AO) 96.1% of times out of which 95.9% of times it correctly identifies its position.

It can be seen from the table presented above that analogous to the study for AR(1), the performances of the procedure based on Ω^* to correctly identify the outlier and its position are almost the same in the presence of AO, irrespective of using BJ or RA estimate of θ. The correct identification of outlier type, however, is marginally better using RA estimate of θ, for moderate to large values of ω. The percentage difference in the performances is however not more than 3. Also, the procedure using BJ estimate itself gives highly accurate results, for instance, the correct type and position identification is 91.1% of times out of total number of times the procedure detects an outlier in the series for $\omega = 4$.

Next we present the performance evaluation of the proposed procedure using BJ and RA estimates for MA(1) with a single IO type of outlier. Table 4.15 reports the simulations based result for $\theta = -0.6$.

It can be seen from Table 4.15 that the columns 2(3) and 6(7) are very close. Thus, analogous to the AR(1) study, the performance of the procedure to detect the outlier and its position does not depend on which of the BJ and RA estimates is used. Also, as in case of AR(1) with an IO, the percentage performance of the proposed procedure to identify the outlier type and detect its position correctly improves by about 2 to 3 when BJ estimate is used instead of RA estimate.

Table 4.15

Performance Evaluation Using BJ and Robust Estimates:
MA(1) with an IO at t = 51
(C = 15, n = 100, $\theta = -0.6$, $\sigma_a^2 = 1$; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		IO Type ($\Omega^* = \Omega_I$)		$\Omega^* > C$		IO Type ($\Omega^* = \Omega_I$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.050	-	0.500	-	0.050	-	0.500	-
2	0.085	0.048	0.635	0.750	0.082	0.047	0.610	0.723
3	0.252	0.230	0.829	0.843	0.245	0.224	0.804	0.821
4	0.592	0.581	0.900	0.905	0.584	0.574	0.887	0.890
5	0.892	0.890	0.960	0.960	0.891	0.889	0.941	0.940
6	0.985	0.985	0.982	0.982	0.985	0.985	0.971	0.971
7	1.000	1.000	0.992	0.992	1.000	1.000	0.981	0.981

Simulation study was carried out for various values of MA(1) parameter θ in the presence of AO and IO type of outlier. The conclusions based on the simulation study were similar to those for AR(1) and for the case $\theta = -0.6$ reported above in Tables 4.14 and 4.15.

In Table 4.16 we report the evaluation for MA(1) with $\theta = -0.9$ in the presence of AO at position t = 51.

The observations based on this table are similar to those based on Table 4.14. It is clear that the difference between performances of the procedure using BJ and RA estimates reduces for larger absolute values of θ. For all the values of θ for which the simulations were carried out, it was observed that the performance

Table 4.16

Performance Evaluation Using BJ and Robust Estimates:
MA(1) with an AO at t = 51
(C = 15, n = 100, $\theta = -0.9$, $\sigma_a^2 = 1$; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		AO Type ($\Omega^* = \Omega_A$)		$\Omega^* > C$		AO Type ($\Omega^* = \Omega_A$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.090	0.001	0.744	-	0.089	0.001	0.753	-
2	0.465	0.343	0.923	0.889	0.463	0.338	0.940	0.911
3	0.890	0.793	0.988	0.953	0.892	0.794	0.993	0.965
4	0.992	0.957	0.995	0.977	0.993	0.955	0.998	0.982
5	0.999	0.991	0.998	0.993	1.000	0.992	0.999	0.995
6	1.000	1.000	0.997	0.994	1.000	1.000	0.999	0.996
7	1.000	1.000	0.997	0.997	1.000	1.000	1.000	1.000

of the procedure to detect the outlier and identify its position is satisfactory and does not depend on which of the BJ or RA estimate of θ is used. The procedure using RA estimate performed slightly better in identifying the outlier type AO. For instance the difference in the percentage of identification of the outlier type was about 4 to 5 for $\theta = -0.3$ and $\omega = 5$.

In Table 4.17, we report the simulation study for $\theta = -0.9$ in the presence of IO. It can once again be seen that there is no difference in the performances using BJ and RA estimates when outlier detection and position identification is attempted. The procedure using BJ performs marginally better than that using RA to identify the outlier type as IO and detect its correct position.

Table 4.17

Performance Evaluation Using BJ and Robust Estimates:
MA(1) with an IO at t = 51
(C = 15, n = 100, $\theta = -0.9$, $\sigma_a^2 = 1$; 1000 replications)

	Using BJ estimate				Using RA estimate			
	$\Omega^* > C$		IO Type ($\Omega^*=\Omega_I$)		$\Omega^* > C$		IO Type ($\Omega^*=\Omega_I$)	
ω	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion	Detect-ion	Position Identifica-tion
0	0.079	-	0.278	-	0.080	0.001	0.263	-
2	0.116	0.049	0.440	0.694	0.113	0.047	0.398	0.681
3	0.274	0.239	0.807	0.870	0.260	0.225	0.796	0.876
4	0.611	0.597	0.938	0.951	0.598	0.583	0.935	0.949
5	0.878	0.874	0.985	0.987	0.867	0.864	0.987	0.988
6	0.979	0.977	0.994	0.996	0.979	0.977	0.994	0.996
7	0.997	0.997	0.998	0.998	0.997	0.997	0.998	0.998

Similar phenomenon was observed for various other values of θ. The difference in the percentage performances using BJ and RA estimates to identify the outlier type when it is IO was more for smaller values of θ. For instance, for $\theta = -0.3$ and $\omega = 5$, the procedure using BJ estimate identified the correct outlier type 79% of times out of 88% of times an outlier was detected. The percentages for the procedure using RA were 75.63% and 87.8% respectively.

Remarks:

Based on the simulation study presented in this section, following remarks can be made.

a) The performance of the proposed procedure based on Ω^* to detect the outlier and identify its correct position in AR(1) or MA(1) is same whether BJ or RA estimate of time series parameter is used. This is irrespective of whether AO or IO outlier type is present in the series. A wide range of values of time series parameters ($\phi = 0.1(0.1)0.9$ and $\theta = -0.1(-0.1)-0.9$) was used for simulations and the claim holds for all the values considered.

b) There is marginal difference in the performances using BJ and RA estimates of time series parameter to identify the outlier type and its correct position.

c) The performance of the procedure to identify the correct type of outlier and its correct position improves in the presence of AO when RA estimate is used. The difference between the performances, however is marginal even for small values of time series parameter and is almost negligible for large values of time series parameter.

d) The performance of the procedure to identify the correct type of outlier and its correct position improves in the presence of IO when BJ estimate is used. The difference between the performances is again not significantly large even for small values of time series parameter and is almost negligible for large values of time series parameter.

e) The simulation study using RA estimator took longer than that using BJ estimator even in the simplest possible case of AR(1), thus making use of RA estimate computationally more expensive.

In conclusion, the performance of the procedure using RA estimate cannot be claimed to be uniformly better than that using BJ estimate when the outlier type is unknown. The use of robust estimator of time series parameter entails only a marginal gain in efficiency of the proposed procedure in the presence of AO. Also the gain is in case of the correct detection and position identification only if the unknown outlier type is an AO at the expense of heavier computations. If the unknown type of outlier is IO, the procedure using BJ estimate performs better than that using RA estimate in identifying the outlier type.

Often in a contaminated time series both AO and IO type of outliers are present (Chapter 5, Section 5.4). If the type of outlier is unknown, the adjustment diagnostic procedure using BJ estimates seems to be a reasonable way for outlier detection as well as type identification, except that it may miss the correct type identification of AO outlier by a small margin in comparison with the procedure using RA estimator in some cases. Hence for the remaining discussion, only the BJ estimator of time series parameters is used in the proposed procedure.

4.5 Diagnostic for Multiple Outliers

In practice, the number of outliers which might be present in an observed series is rarely known and a procedure which detects the presence of multiple outliers is needed. Identification of multiple outliers is a challenging problem, particularly due to the masking and swamping effects (Barnett and Lewis, 1994).

In many practical situations, however, most of the jointly influential observations are detected by employing the single case diagnostic procedures iteratively (Chatterjee and Hadi, 1988, pp. 185-186). In time series analysis as well, most of the existing multiple outliers detection procedures are based on iterative use of the single outlier detection procedure (Chang and Tiao, 1983; Abraham, 1987; Chang et al. 1988).

We propose the iterative procedure based on ADV for identification of multiple outliers here. The performance evaluation and the drawbacks of the proposed procedure are postponed to next sections.

Step 1.

Compute the maximum likelihood estimates of the model parameters ϕ, θ and error variance based on the series where it is assumed to be outlier free.

Hence, the estimates are $\hat{\beta} = \left(\hat{\phi}', \hat{\theta}'\right)'$ and

$$\hat{\sigma}_e^2 = \frac{1}{n}\sum_{t=1}^{n}\hat{e}_t^2$$

The ψ weights and π weights are recursively computed as,

$$\hat{\psi}_j = \hat{\phi}_1\hat{\psi}_{j-1} - \hat{\phi}_2\hat{\psi}_{j-2} - \cdots - \hat{\phi}_p\hat{\psi}_{j-p} - \hat{\theta}_j \qquad \text{for } j>0$$

where $\hat{\psi}_0 = 1$, $\hat{\psi}_j = 0$ for $j<0$, and $\hat{\theta}_j = 0$ for $j>q$, and

$$\hat{\pi}_j = \hat{\theta}_1\hat{\pi}_{j-1} + \hat{\theta}_2\hat{\pi}_{j-2} + \cdots + \hat{\theta}_q\hat{\pi}_{j-q} + \hat{\phi}_j \qquad \text{for } j>0$$

where $\hat{\pi}_0 = -1$, $\hat{\pi}_j = 0$ for $j<0$, and $\hat{\phi}_j = 0$ for $j>p$ respectively.

Step 2.

For i=1,2, …, n in turn, calculate the estimated outlier parameters at 'i',

$$\hat{\omega}_{A,i} = \frac{\hat{\pi}(F)\hat{e}_i}{\sum_{j=0}^{n-i}\hat{\pi}_j^2},$$

$$\hat{\omega}_{I,i} = \hat{e}_i,$$

and adjust the series for all $t\in\tau$ by using the estimated observations given by

$$Y_{t(i),A} = Y_t - \hat{\omega}_{A,i}\,\xi_t^{(i)}$$

$$Y_{t(i),I} = Y_t - \hat{\omega}_{I,i}\,\hat{\psi}(B)\,\xi_t^{(i)}.$$

Compute the adjustment diagnostic measure,

$$ADV_{S,i} = n\left(\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{e(i),S}} - 1\right) \qquad \text{for } S = A \text{ and } I$$

where $\hat{\sigma}^2_{e(i),S} = \frac{1}{n}\sum_{t=1}^{n}\hat{e}^2_{t(i),S}$ is obtained based on adjusted series $Y_{t(i),S}$, S is A or I,

and $\hat{e}_{t(i),S}$ is the residual using the estimated parameters $\hat{\beta}_{(i),S} = ((\hat{\phi}'_{(i),S}, \hat{\theta}'_{(i),S})'$ for the adjusted series $\{Y_{t(i),S},\ t\in\tau\}$.

Step 3.

Define $\Omega^* = \max(\Omega_A, \Omega_I)$ where $\Omega_A = ADV_{A,T} = \max_i ADV_{A,i}$ and $\Omega_I = ADV_{I,T} = \max_i ADV_{I,i}$, T is the time point where the maximum occurs.

If $\Omega^* \le C$ where C is the predefined positive value, go to Step 4.

If $\Omega^* > C$ and

if $\Omega^* = \Omega_A$, then there is an AO at time T with its effect $\hat{\omega}_{A,T}$.

Take the adjusted series with observation Y_t replaced by $Y_{t(T),A}$

for all $t \in \tau$; and go to Step 1.

if $\Omega^* = \Omega_I$, then there is an IO at time T with its effect, $\hat{\omega}_{I,T}$.

Take the adjusted series with observation Y_t replaced by $Y_{t(T),I}$

for all $t \in \tau$, and go to Step 1.

Step 4.

Suppose 'm' number of outliers are identified with the positions T_j, for j = 1, 2, …, m. The model now is

$$Y_t = \sum_{j=1}^{m} \omega_j D_j(B)\xi_t^{(Tj)} + \frac{\theta(B)}{\phi(B)} a_t \qquad t \in \tau.$$

where $D_j(B) = 1$ for AO type and $D_j(B) = \dfrac{\theta(B)}{\phi(B)}$ for IO type at $t = T_j$. The simultaneous estimation is carried out to get the final estimates for a set of parameters $\beta = (\omega', \phi', \theta')'$ where $\omega' = (\omega_1, \omega_2, \ldots, \omega_m)$ and error variance σ_a^2 using the maximum likelihood estimation procedure in Section 2.4.

In the procedure proposed above, a robust estimator of the time series parameter β can be used in Step 1 instead of the maximum likelihood estimator analogous to the single outlier procedure discussed in Section 4.4. However, from the performance evaluation presented therein, it can be seen that the performance of the proposed adjustment diagnostic procedure does not improve significantly or uniformly using a robust estimator. Also, as is well accepted, a robust estimation procedure is computationally more expensive. Hence only the BJ estimator of β is considered for the further discussion.

The multiple outliers in time series can occur in isolation or in patch which are called as isolated outliers or patch outliers respectively (Martin, 1979; Bruce and Martin, 1989). The evaluation of the proposed procedure in the presence of two isolated or patch outliers is presented in Section 4.6. Additional discussion is deferred to Chapter 5, particularly Sections 5.2 and 5.3.

4.6 Performance Evaluation of Adjustment Diagnostic for Multiple Outliers

The performance of the diagnostic procedure for multiple outliers proposed in the earlier section is presented here. We investigate the performance in the presence of two outliers which may occur in isolation or in patch. In particular, we consider contaminated AR(1) and MA(1) series, contaminated with two outliers which are

i) both of AO type (2 AOs)

ii) both of IO type (2 IOs)

iii) an AO type followed by an IO type (AO IO)

iv) an IO type followed by an AO type (IO AO)

occuring at isolated time points and at consecutive time points. The outlier parameter ω is considered same for both the outliers and we consider the values of

ω in the range $\omega = 2(1)7$. The study presented here is based on AR(1) with $\phi = 0.6$ and $\sigma_a^2 = 1$ and MA(1) with $\theta = -0.6$ and $\sigma_a^2 = 1$.

We carried out 1000 simulations for each series with n = 100. The cut off points considered are same as in case of single outlier (Section 4.4), namely C = 14.5 for AR(1) and C = 15 for MA(1). The program CP-8 is given in the attached CD and is listed in Appendix C. We consider the case of isolated outliers, followed by the patch outliers.

(a) Isolated Outliers

For each series of length n = 100, the two isolated outliers are assumed to be present at the well separated points chosen to be t = 34 and 67 which we refer by "1st outlier" for t = 34 and "2nd outlier" for t = 67. Tables 4.18 and 4.19 give the performance evaluation for AR(1) and MA(1) respectively. The figures in the tables give the frequency (out of 1000) of detection of correct position of both outliers, 1st outlier and 2nd outlier respectively, based on Ω^*. The numbers within the parentheses under each figure give the percentage of correct identification of outlier types out of correct position identified. For instance, in Table 4.18 in the presence of 2 AOs with $\omega = 4$, 65.1% of times Ω^* detects both the outliers at the correct position, 75% of which correctly identifies both as AOs. Further, 79.9% of times it correctly detects the position of the 1st outlier out of which 86% of times the procedure detects the correct type AO.

Table 4.18

Frequency of Correct Detection of Outlier(s) position (percentage of correct identification of type): AR(1) with Two Isolated Outliers at t = 34 and 67 (C = 14.5, n = 100, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

Outliers' Type	ω					
	2	3	4	5	6	7
2 AO'S	11	166	651	948	999	1000
	(0.55)	(0.70)	(0.75)	(0.81)	(0.81)	(0.86)
1st Outlier	79	347	799	970	1000	1000
	(0.73)	(0.83)	(0.86)	(0.90)	(0.90)	(0.92)
2nd Outlier	67	351	785	973	999	1000
	(0.81)	(0.84)	(0.86)	(0.90)	(0.90)	(0.94)
2 IO'S	4	75	412	795	974	995
	(0.00)	(0.73)	(0.75)	(0.85)	(0.90)	(0.95)
1st Outlier	60	232	591	885	990	997
	(0.68)	(0.85)	(0.86)	(0.91)	(0.95)	(0.97)
2nd Outlier	49	238	613	883	983	998
	(0.78)	(0.79)	(0.88)	(0.93)	(0.95)	(0.98)
AO IO	3	125	492	869	983	999
	(0.33)	(0.72)	(0.76)	(0.86)	(0.89)	(0.95)
1st Outlier	92	361	774	969	997	1000
	(0.77)	(0.83)	(0.89)	(0.92)	(0.94)	(0.97)
2nd Outlier	46	248	584	890	986	999
	(0.80)	(0.85)	(0.88)	(0.93)	(0.95)	(0.98)
IO AO	10	122	481	873	990	997
	(0.40)	(0.75)	(0.79)	(0.85)	(0.90)	(0.94)
1st Outlier	49	242	579	885	992	998
	(0.84)	(0.87)	(0.90)	(0.93)	(0.95)	(0.98)
2nd Outlier	89	380	771	981	998	999
	(0.69)	(0.83)	(0.88)	(0.91)	(0.94)	(0.97)

Based on Tables 4.18 and 4.19, it can be seen that the proposed ADV procedure performs satisfactorily for AR(1) as well as MA(1) in the presence of two outliers, particularly for large values of outlier parameter ω. Also, on comparing

Table 4.19

Frequency of Correct Detection of Outlier(s) Position (percentage of correct identification of types): MA(1) with Two Isolated Outliers at $t = 34$ and 67 ($C = 15$, $n = 100$, $\theta = -0.6$, $\sigma_a^2 = 1$; 1000 replications)

Outliers' Type	ω					
	2	3	4	5	6	7
2 AO'S	14	157	657	942	995	1000
	(0.57)	(0.74)	(0.79)	(0.81)	(0.82)	(0.84)
1st Outlier	68	330	757	964	996	1000
	(0.68)	(0.87)	(0.87)	(0.91)	(0.90)	(0.93)
2nd Outlier	84	333	768	971	998	1000
	(0.79)	(0.84)	(0.88)	(0.89)	(0.91)	(0.91)
2 IO'S	2	46	329	768	959	999
	(0.50)	(0.76)	(0.86)	(0.91)	(0.96)	(0.97)
1st Outlier	40	174	559	870	978	999
	(0.80)	(0.89)	(0.92)	(0.95)	(0.97)	(0.99)
2nd Outlier	45	187	525	860	979	1000
	(0.87)	(0.92)	(0.93)	(0.96)	(0.99)	(0.99)
AO IO	8	104	475	858	980	997
	(0.50)	(0.71)	(0.82)	(0.92)	(0.94)	(0.96)
1st Outlier	90	408	783	976	998	999
	(0.81)	(0.89)	(0.91)	(0.96)	(0.96)	(0.97)
2nd Outlier	31	186	554	874	982	998
	(0.87)	(0.87)	(0.91)	(0.96)	(0.98)	(1.00)
IO AO	4	99	459	842	974	999
	(0.50)	(0.76)	(0.84)	(0.91)	(0.95)	(0.97)
1st Outlier	27	180	539	866	977	999
	(0.78)	(0.88)	(0.93)	(0.97)	(0.98)	(0.99)
2nd Outlier	101	391	787	969	997	1000
	(0.78)	(0.85)	(0.89)	(0.94)	(0.97)	(0.98)

Tables 4.18 and 4.19 with Tables 4.8 and 4.9 respectively, it can be seen that the performance of the procedure in case of two isolated outliers is comparable with that in case of single outlier, irrespective of the outlier types.

(b) Patch Outliers

For each series of length n=100, a patch of two outliers occurring at consecutive time points t = 51 and 52 is considered for this study. As before the outlier at t = 51 is referred by "1st outlier" and that at t =52 by "2nd outlier".

Table 4.20

Frequency of Correct Detection of Outlier(s) Position (percentage of correct identification of types): AR(1) with a Patch of Two Outliers at t = 51 ($C = 14.5$, $n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$; 1000 replications)

Outliers' Type	ω					
	2	3	4	5	6	7
2 AO'S	1	10	80	234	472	680
	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.00)
1st Outlier	55	229	542	859	974	998
	(0.05)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)
2nd Outlier	7	31	113	253	479	682
	(0.86)	(0.97)	(0.99)	(0.99)	(1.00)	(1.00)
2 IO'S	3	35	301	727	942	993
	(1.00)	(0.86)	(0.83)	(0.87)	(0.86)	(0.91)
1st Outlier	36	213	570	872	983	998
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
2nd Outlier	44	187	499	830	958	995
	(0.75)	(0.79)	(0.84)	(0.88)	(0.87)	(0.91)
AO IO	0	3	8	30	69	93
	-	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
1st Outlier	47	246	590	879	989	999
	(0.04)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)
2nd Outlier	3	7	14	38	69	93
	(0.67)	(0.86)	(0.57)	(0.68)	(0.59)	(0.61)
IO AO	1	102	488	870	982	999
	(1.00)	(0.89)	(0.89)	(0.92)	(0.97)	(0.97)
1st Outlier	33	208	592	888	984	999
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
2nd Outlier	105	388	784	972	998	1000
	(0.72)	(0.84)	(0.89)	(0.92)	(0.97)	(0.97)

Table 4.21

Frequency of Correct Detection of Outlier(s) Position (percentage of correct identification of types): MA(1) with a Patch of Two Outliers at t = 51 ($C = 15$, $n = 100$, $\theta = -0.6$, $\sigma_a^2 = 1$; 1000 replications)

Outliers' Type	ω					
	2	3	4	5	6	7
2 AO'S	0	0	17	38	93	135
	-	-	(0.00)	(0.00)	(0.00)	(0.00)
1st Outlier	36	222	553	875	981	998
	(0.03)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)
2nd Outlier	2	9	28	48	94	135
	(0.00)	(0.89)	(0.75)	(0.81)	(0.79)	(0.82)
2 IO'S	1	39	250	615	894	973
	(1.00)	(0.87)	(0.95)	(0.96)	(0.98)	(0.99)
1st Outlier	33	235	577	859	977	998
	(0.97)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
2nd Outlier	46	177	427	720	909	975
	(0.80)	(0.80)	(0.92)	(0.96)	(0.98)	(0.99)
AO IO	0	1	7	12	57	92
	-	(0.00)	(0.00)	(0.08)	(0.11)	(0.09)
1st Outlier	40	201	551	829	973	997
	(0.25)	(0.08)	(0.08)	(0.03)	(0.02)	(0.01)
2nd Outlier	3	3	16	16	57	92
	(0.67)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
IO AO	2	85	462	838	969	999
	(0.50)	(0.93)	(0.93)	(0.96)	(0.98)	(0.99)
1st Outlier	52	205	584	875	972	1000
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
2nd Outlier	82	425	805	961	997	999
	(0.84)	(0.91)	(0.95)	(0.96)	(0.98)	(0.99)

Tables 4.20 and 4.21 give the performance evaluation for AR(1) and MA(1) respectively. The figures in the tables can be interpreted analogous to those in Tables 4.18 and 4.19. For instance, in Table 4.20, in the presence of a patch of two AO outliers with $\omega = 4$ in an AR(1), only 8.0% of times the procedure

correctly detects and identifies the position of both the outliers and only 1.0% of times it correctly identifies both as AOs. It identifies the 1st outlier 54.2% of times but fails to identify it as an AO (0.0%) whereas the 2nd is detected 11.3% of times, out of which it is correctly identified as an AO 99% of times.

It is clear that the performance of the proposed procedure is better in the presence of isolated outliers as against patch outliers, a fact already pointed out in the literature (Chen and Liu, 1993; Justel et al., 2001). Also, it can be seen from the tables that the presence of the patch of two consecutive AOs misleads the procedure into identifying the 1st outlier as an IO and hence not identifying the second outlier at all. Similar situation occurs when the 1st outlier is an AO. The fact that in case of a patch the procedure is highly likely to identify the 1st outlier as an IO type can be seen from its performance in the presence of two consecutive IOs or when the 1st outlier is an IO in the presented tables.

More importantly, it can be seen from the tables that irrespective of the type, in all four situations, the outlier detection of first outlier is quite satisfactory and increases to 100% as ω increases. However, the identification of the 1st outlier when it is an AO is significantly low. In any case, the proposed procedure detects the 1st outlier with percentage which is comparable to that of single outlier detection though it often fails to identify the outlier type when the 1st outlier is an AO. Thus in case of patch outliers, the presence of 2nd outlier seems to be 'masking' the effect of 1st outlier when it is an AO irrespective of the type of 2nd outlier. Significantly, the phenomenon seems to be restricted to 2 AOs and AO IO situations only. The procedure performs with high accuracy in the presence of 2 IOs or IO AO occurring in a patch.

We carried out further study to investigate the reasons behind the performance of the proposed procedure in case of multiple outliers which is presented in the next section.

4.7 Critical Evaluation of Diagnostic for Multiple Outliers

In this section, we aim to investigate the reasons behind the performance of multiple outlier detection procedure. A theoretical understanding of effect of multiple outliers on the series analysis seems intractable. Since the presence of outliers affects the error variance which also depends on the order of the model, type of outlier and whether they occur in isolation or in patch, simulation study was undertaken to evaluate the effect on the estimates of time series parameter and error variance. We considered AR(1) and MA(1) models with 'm' number of outliers, $n = 100$, $\sigma_a^2 = 1$ and $\omega = 5$. The number of outliers 'm' is varied from 2 to 8.

In this analysis, we assume that the isolated outliers occur in the series at positions which are well separated, to avoid the masking effect. The patch outliers are also assumed to be of the same type, i.e. either all AOs or all IOs. Firstly, 'm' isolated multiple outliers are introduced in each series at rounded integer value of $k \times \dfrac{n}{m+1}$ where n is the series length, m is the total number of outliers, and $k = 2$, 3, ..., m. The average of estimated parameters and estimated error variances for 1000 replications for each 'm' is calculated using the Computer Program CP-9 in the attached CD and in the programs' list of Appendix C. The results are shown in Table 4.22 and 4.23 for AR(1) and MA(1) series respectively.

Table 4.22

Average of $\hat{\phi}$ and $\hat{\sigma}_a^2$ for an AR(1) Series with 'm' Isolated Outliers
($n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$, $\omega = 5$; 1000 replications)

No of Outliers	Series with Aos		Series with IOs	
m	Avg. of $\hat{\phi}$	Avg. of $\hat{\sigma}_a^2$	Avg. of $\hat{\phi}$	Avg. of $\hat{\sigma}_a^2$
2	0.4395	1.6249	0.5893	1.4879
3	0.3906	1.9276	0.5913	1.7380
4	0.3490	2.2047	0.5911	2.0001
5	0.3248	2.4891	0.5863	2.2470
6	0.2936	2.7699	0.5941	2.4900
7	0.2712	3.0287	0.5920	2.7626
8	0.2547	3.3292	0.5955	2.9982

From Tables 4.22 and 4.23, we observe that the estimated error variance increases when the number of isolated outliers increases irrespective of series considered and type of outliers. Also, consistent with single outlier situation, the effect of multiple isolated AO outliers on the estimate of error variance is more than those of IO outliers for both models. The estimates of time series parameter do not show such a marked effect, irrespective of type of outliers.

Looking at the increase in the estimate of error variance, it does not seem surprising that the proposed procedure works satisfactorily in case of multiple isolated outliers. However it may be borne in mind that the number of isolated outliers should not be too many to bring them closer to each other which could provide the masking effect as in case of patch outliers, since "... subtler type of masking occurs when moderate outliers lie close to one another" (Bruce and Martin, 1989), thus rendering the usual outlier detection method ineffective (Justel, et al., 2001).

Table 4.23

Average of $\hat{\theta}$ and $\hat{\sigma}_a^2$ for an MA(1) Series with 'm' Isolated Outliers
($n = 100$, $\theta = -0.6$, $\sigma_a^2 = 1$, $\omega = 5$; 1000 replications)

No of Outliers m	Series with Aos		Series with IOs	
	Avg. of $\hat{\theta}$	Avg. of $\hat{\sigma}_a^2$	Avg. of $\hat{\theta}$	Avg. of $\hat{\sigma}_a^2$
2	-0.3581	1.6487	-0.6106	1.5280
3	-0.3063	1.9228	-0.6081	1.7793
4	-0.2598	2.1898	-0.6025	2.0214
5	-0.2369	2.4864	-0.6031	2.2846
6	-0.2123	2.7379	-0.6046	2.5237
7	-0.1934	2.9985	-0.6022	2.7887
8	-0.1736	3.2664	-0.6141	3.0245

Since most of the available multiple outliers detection procedures do not handle patch outliers satisfactorily, we investigated the change in the estimated error variance due to the presence of multiple outliers in patch for the outlier parameter $\omega = 5$ with an outlier patch starting at $t = 51$ for various patch lengths 2(1)8. Using the computer program CP-10 as listed in Appendix C, we calculated the estimated parameters and the estimated error variances when the outliers are ignored. The results are presented in Tables 4.24 and 4.25 for AR(1) and MA(1) respectively.

Table 4.24

Average of $\hat{\phi}$ and $\hat{\sigma}_a^2$ for an AR(1) Series with a Patch Outliers at $t = 51$ ($n = 100$, $\phi = 0.6$, $\sigma_a^2 = 1$, $\omega = 5$; 1000 replications)

Patch Length	Series with Aos		Series with IOs	
	Avg. of $\hat{\phi}$	Avg. of $\hat{\sigma}_a^2$	Avg. of $\hat{\phi}$	Avg. of $\hat{\sigma}_a^2$
2	0.5623	1.3738	0.6845	1.4756
3	0.6133	1.4035	0.7621	1.6607
4	0.6459	1.4572	0.8149	1.7448
5	0.6805	1.4740	0.8504	1.8355
6	0.7074	1.4929	0.8763	1.8764
7	0.7300	1.5183	0.8949	1.9105
8	0.7462	1.5332	0.9097	1.9271

As can be seen from Table 4.24 and 4.25, the estimated error variance shows consistent increase in its value for both models and for both outlier types when the length of a patch for IOs increases.

It shows that the bias amount in estimated error variance is approximately proportional to the length of patch outliers. At the same time, comparing Table 4.22 with Table 4.24, and Table 4.23 with Table 4.25, it is clear that for both AO and IO type of outliers the over all effects of 'm' patch outliers on estimate of error variance is much less compared to that of isolated outliers of equal numbers. This indicates the difficulty in handling patch outliers. As a result, any procedure based on estimate of error variance will not be as effective in identifying patch outliers as isolated outliers. Since the effect of patch of IO outliers is higher on the estimate, a patch of IO outliers seems "more identifiable" using the proposed procedure.

Table 4.25

Average of $\hat{\theta}$ and $\hat{\sigma}_a^2$ for an MA(1) Series with a Patch Outliers at t = 51
(n = 100, $\theta = -0.6$, $\sigma_a^2 = 1$, $\omega = 5$; 1000 replications)

Patch Length	Series with an AO		Series with an IO	
	Avg. of $\hat{\theta}$	Avg. of $\hat{\sigma}_a^2$	Avg. of $\hat{\theta}$	Avg. of $\hat{\sigma}_a^2$
2	-0.6600	1.3309	-0.7188	1.5412
3	-0.5721	1.7252	-0.7295	1.8159
4	-0.6807	1.6232	-0.7725	2.0021
5	-0.6401	2.0505	-0.7726	2.2699
6	-0.6963	1.8364	-0.8037	2.4127
7	-0.6714	2.1950	-0.8059	2.6438
8	-0.7098	2.0533	-0.8218	2.8112

However, this nature does not show in the case of patch for AOs. Especially in MA(1), the change of estimated error variance does not depend on the length of patch. Hence, a patch of AOs can reduce the performance of many outlier detection methods as well as the identification of their types.

We discuss patch outliers heuristically using a patch length 2 for both types. Consider presence of patch outliers of length 2 which occurs in the observed series $\{Y_t, t \in \tau\}$ at time point $t = T$. Then the two outlier models can be expressed as

$$Y_t = Z_t + \omega \{\xi_t^{(T)} + \xi_t^{(T+1)}\}, \qquad t \in \tau \quad \text{for AO} \tag{4.13}$$

$$Y_t = Z_t + \omega\, \psi(B) \{\xi_t^{(T)} + \xi_t^{(T+1)}\}, \qquad t \in \tau \quad \text{for IO} \tag{4.14}$$

More clearly, the models can be written as follows.

For AO patch model,

$$Y_t = \begin{cases} Z_t & \text{for } t < T, \\ Z_t + \omega & \text{for } t = T, \\ Z_t + \omega & \text{for } t = T+1, \\ Z_t & \text{for } t > T+1. \end{cases} \tag{4.15}$$

We can see that the patch of two AOs may look more like an IO at T. Hence, an IO can be misidentified instead of AO at T point.

For IO patch model,

$$Y_t = \begin{cases} Z_t & \text{for } t < T, \\ Z_t + \omega & \text{for } t = T, \\ Z_t + \omega \{1+\psi_1\} & \text{for } t = T+1, \\ Z_t + \omega \{\psi_{t-T+1} + \psi_{t-T}\} & \text{for } t > T+1. \end{cases} \tag{4.16}$$

It shows that the effect on Z_{T+1} is larger than Z_T when the sign of ω and ψ_1 is positive. Provided ω is sufficiently large and there are no high fluctuation in the original outlier free series, the proposed method is likely to indicate the presence of IO at T+1 initially. Then, the IO at T may be detected as an outlier at the next iteration. The misidentification of the outlier type may be a rare instance for a patch of IOs.

We now consider the series adjustment analogous to Section 2.6 under the assumption that the time series parameters are known.

Series Adjusted for AO

Suppose the series is adjusted using the estimated value of outlier parameter ω based on an AO at 'i', giving the adjusted series for $t,i \in \tau$

$$\ddot{Y}_{t(i),A} = [\, Z_t + \omega\, \{\xi_t^{(T)} + \xi_t^{(T+1)}\}\,] - \hat{\omega}_{A,i}\, \xi_t^{(T)}, \qquad \text{for AO} \qquad (4.17)$$

$$\ddot{Y}_{t(i),A} = [\, Z_t + \omega\, \psi(B)\, \{\xi_t^{(T)} + \xi_t^{(T+1)}\}\,] - \hat{\omega}_{A,i}\, \xi_t^{(T)}, \qquad \text{for IO} \qquad (4.18)$$

where T and T+1 are the correct outlier positions and 'i' is the adjusted outlier position.

We can see that the adjustment leads to a reduction of effect ω for AO patch in (4.17) when the adjusted outlier position is either i = T or T+1 and the $\hat{\omega}_{A,i}$ for i = T or T+1 is close to the outlier parameter ω. One outlier among two still remains. But, in practice, the estimator $\hat{\omega}_{A,i}$ may not be close to the outlier parameter ω because the effect of a neighboring outlier influences this estimate. We also know that the AO adjustment replaces the value of the observed series at 'i' by its predicted value using least squares estimate. Consequently, the adjusted value of the observed series at $i = T$ or $T+1$ is affected by the neighboring outlier leading to the bias in estimated error variance.

Similarly, the AO adjustment for IO patch model in (4.18) can reduce the effect of outlier at time position either $i = T$ or $T+1$ only. One of these two positions is still influenced by contaminants and carry-over effects of outlier remains in the adjusted series. Hence the AO adjustment may not work well for the presence of patch outliers for either types in the observed series.

Series Adjusted for IO

We now consider the IO adjustment using the estimated outlier parameter at i, $\hat{\omega}_{I,i}$ for both models in (4.13) and (4.14) for $t,i \in \tau$, which gives

$$\ddot{Y}_{t(i),I} = [Z_t + \omega\, \{\xi_t^{(T)} + \xi_t^{(T+1)}\}] - \hat{\omega}_{I,i}\, \psi(B)\, \xi_t^{(T)}, \qquad \text{for AO} \qquad (4.19)$$

$$\ddot{Y}_{t(i),I} = [Z_t + \omega\, \psi(B)\, \{\xi_t^{(T)} + \xi_t^{(T+1)}\}] - \hat{\omega}_{I,i}\psi(B)\xi_t^{(T)}, \quad \text{for IO} \qquad (4.20)$$

When $i = T$, the adjusted series of (4.19) becomes

$$\ddot{Y}_{t(i),I} = Z_t + (\omega - \hat{\omega}_{I,T})\, \xi_t^{(T)} + (\omega - \hat{\omega}_{I,T}\psi_1)\, \xi_t^{(T+1)} - \hat{\omega}_{I,T} \sum_{j=2}^{n-T} \psi_j \xi_{t-j}^{(T)} .$$

We also know that $\hat{\omega}_{I,T} = \hat{e}_T$ does not get affected by outlier ω at $T+1$. Thus the adjustment replaces the observed value at T by its MMSE forecast which also does not get affected by outlier ω at $T+1$. Hence the IO adjustment leads to reduction of the effect of ω on the observed series at T. The value of ω at $T+1$ may be changed to $(\omega - \hat{\omega}_{I,T}\, \psi_1)$. Hence the IO adjustment has better chances of detection of a patch outlier position at T and can reduce the effect of ω when the sign of ψ_1 is positive. The outlier at T+1 may look more like an AO itself with effect (ω - $\hat{\omega}_{I,T}\psi_1$) and it may be identified by the AO adjustment.

When i = T+1, the estimated parameter is $\hat{\omega}_{I,T+1} = \hat{e}_{T+1}$ which is affected by the outlier ω at T. Hence, the IO adjustment for i=T+1 does not lead to reduction of outlier effect from the series at time point T in the presence of AO patch outliers.

As a result, heuristically, an AO patch of two outliers starting at T is likely to get indicated as an IO at T and an AO at T+1 with a smaller estimate of outlier parameter. One should carefully analyze the behavior of observed values and estimated errors at these positions.

In (4.20), the IO adjustment for i = T or T+1 leads to reduction of the effect of ω from the IO patch series. Moreover, (4.16) shows that the outlier at T+1 can be indicated as the first outlier, depending on the value of time series parameter. Hence the IO adjustment might work well for both AO and IO patch outliers with patch length 2. The diagnostic plots and identification of patch outliers in numerical study are presented in Sections 5.2 and 5.3 of Chapter 5.

To overcome the drawback of patch outliers, various alternative procedures can be suggested, which are not investigated in details in the present work. For instance, the adjustment method can be extended to "block adjustment" where the series can be modified by adjusting k consecutive observations, that is, the estimate of outlier parameter ω_i at 'i' is used for adjustment of observations at time points i, i+1, ..., i+k-1 where k is a block size. It can be expressed for both outliers AO and IO as

$$\ddot{Y}_{t(i),A} = Y_t - \hat{\omega}_{A,i} \sum_{j=0}^{k-1} \xi_{t-j}^{(i)}, \qquad t,i \in \tau \quad \text{for AO block adjustment} \qquad (4.21)$$

$$\ddot{Y}_{t(i),I} = Y_t - \hat{\omega}_{I,i}\, \psi(B) \sum_{j=0}^{k-1} \xi_{t-j}^{(i)}, \qquad t,i \in \tau \quad \text{for IO block adjustment} \qquad (4.22)$$

Suppose that the observed series $\{Y_t, t \in \tau\}$ follows the outlier models in (4.13) and (4.14) for AO and IO respectively and suppose the block size k is 2. The block adjustment for i = T of same outlier type gives the adjusted series as follows.

The proposed AO block adjustment for AO patch Model is

$$\ddot{Y}_{t(T),A} = [\, Z_t + \omega\, (\, \xi_t^{(T)} + \xi_t^{(T+1)})\,] - \hat{\omega}_{A,T} \sum_{j=0}^{1} \xi_{t-j}^{(T)}, \qquad t,i \in \tau$$

$$= Z_t + (\omega - \hat{\omega}_{A,T})\, \xi_t^{(T)} + \omega\, \xi_t^{(T+1)} - \hat{\omega}_{A,T}\, \xi_{t-1}^{(T)}$$

Since $\xi_t^{(T+1)} = \xi_{t-1}^{(T)}$, we get

$$\ddot{Y}_{t(T),A} = Z_t + (\omega - \hat{\omega}_{A,T})\, [\, \xi_t^{(T)} + \xi_t^{(T+1)}] \qquad (4.23)$$

From (4.23), analogous to (2.38) it can be seen that the block adjustment leads to reduction in contamination of the observed series when the estimated outlier parameter $\hat{\omega}_{A,T}$ is close to the true parameter value ω.

The IO block adjustment for IO patch Model is

$$\ddot{Y}_{t(T),I} = [\, Z_t + \omega\, \psi(B)\, (\, \xi_t^{(T)} + \xi_t^{(T+1)}\,)\,] - \hat{\omega}_{I,T}\, \psi(B) \sum_{j=0}^{1} \xi_{t-j}^{(T)}, \qquad t,i \in \tau$$

Since $\xi_t^{(T+1)} = \xi_{t-1}^{(T)}$, we get

$$\ddot{Y}_{t(T),I} = Z_t + (\omega - \hat{\omega}_{I,T})\, \psi(B)\, [\, \xi_t^{(T)} + \xi_t^{(T+1)}]$$

which shows that the IO block adjustment reduces the effect of outlier in the observed series when the estimated outlier parameter $\hat{\omega}_{I,T}$ is close to ω.

But the determination of the block size may be a problem in a case of patch outliers, especially for AO patch outliers. One can use the estimated error variance to decide the block size for IO patch because Tables 4.24 and 4.25 show that the change in the estimated error variance is directly proportional to the patch length of outliers. The proposed procedure needs to be appropriately modified for satisfactorily handling the presence of patch outliers starting with AO. An alternative way to handle multiple outliers is to use the diagnostics plots that are developed in STDS which can help us to guess the position of outliers and nature of the patch outliers.

Chapter 5

Data Analysis Using Adjustment Diagnostic

5.1 Introduction

Understanding outliers is an important issue in time series data analysis. In this chapter we illustrate how the proposed procedure helps us to diagnose the outliers in simulated time series data sets as well as real life data sets. The data sets considered are various types of AR(1) and MA(1) series with outliers of different types at different positions and some real life data sets available in the literature.

The analysis presented here is carried out using the Statistical Time Series Diagnostic Software (STDS) developed as a part of this work. The software is supplied along and apart from the analysis mentioned above, also provides a few additional things such as basic statistics for time series, various diagnostic plots such as residual plots and adjusted AIC etc, and the proposed iterative diagnostic procedure ADV. The content page of STDS manual is presented in Appendix B.

Initially, the proposed outlier diagnostic procedure is illustrated using various data sets consisting of single and multiple outliers of two types considered so far. For a particular data set, it is suggested that initially the plot of residuals and the ADV plot introduced in Section 4.2 be drawn. The plots give a fairly good idea about the positions of the outliers and also their respective types.

In Section 5.2, we discuss in details the ADV plots in case of AR(1) and MA(1) in the presence of single and multiple outliers of either type. The value of outlier parameter chosen is $\omega = 4$. The performance of ADV procedure for these series has already been presented in Section 4.3. Section 5.3 presents the performance of the proposed diagnostic procedure where we consider various series with different types of outliers at different positions.

Finally, in Section 5.4, we apply the proposed procedure to some real life data sets available in the literature. The data sets considered are Truck Defects Series by Wei (1990, p. 446) and Box and Jenkins' Series C, Series A, Series D and Series J (Box et al, 1994, pp.541-549). The proposed procedure is carried out on all these series and the performance of the procedure is compared with the analysis available in the literature. It is observed that there are differences in the parameter estimates reported in the literature and the parameter estimates given by STDS due to difference in the estimation procedures. In order to compare the reduction in the estimate of error variance, we computed the estimate of error variances using SPSS software for Windows. Improved analysis of outliers was observed for those series which indicated the presence of IO type of outliers. The parsimony of the model was also taken into account while comparing.

5.2 Adjustment Diagnostic of Variance (ADV) Plots

In this section, we present the diagnostic procedure based on ADV (Adjustment Diagnostic based on Variance) plot which is the plot of estimated error variance obtained on adjusting the series at all time points in turn. The plots are expected to indicate the presence of outliers along with the number of outliers and their position in the series. We present the study for AR(1) and MA(1) series separately in the present section.

AR(1) Model:

We considered an AR(1) model and generated the series for $n = 100$, $\phi = 0.6$ and $\sigma_a^2 = 1$.

AR(1) with one outlier

Using the generated AR(1) series, an outlier of an AO and an IO type is introduced at t=51 for the value of outlier parameter $\omega = 4$. Before analyzing the data using the adjustment diagnostic, the residuals of AO and IO contaminated series are plotted in Figures 5.1 and 5.2 respectively.

Figure 5.1 shows that the residual value at t=51 is 3.5121. Since it is slightly higher than others, the observation at t = 51 can be a possible outlier in generated AR(1) series. Apart from this, the graph indicates that the possible outliers can be expected at t = 69, 83, 90 and 91 and their residuals are –2.29, 2.64, 2.09, and –2.16 respectively.

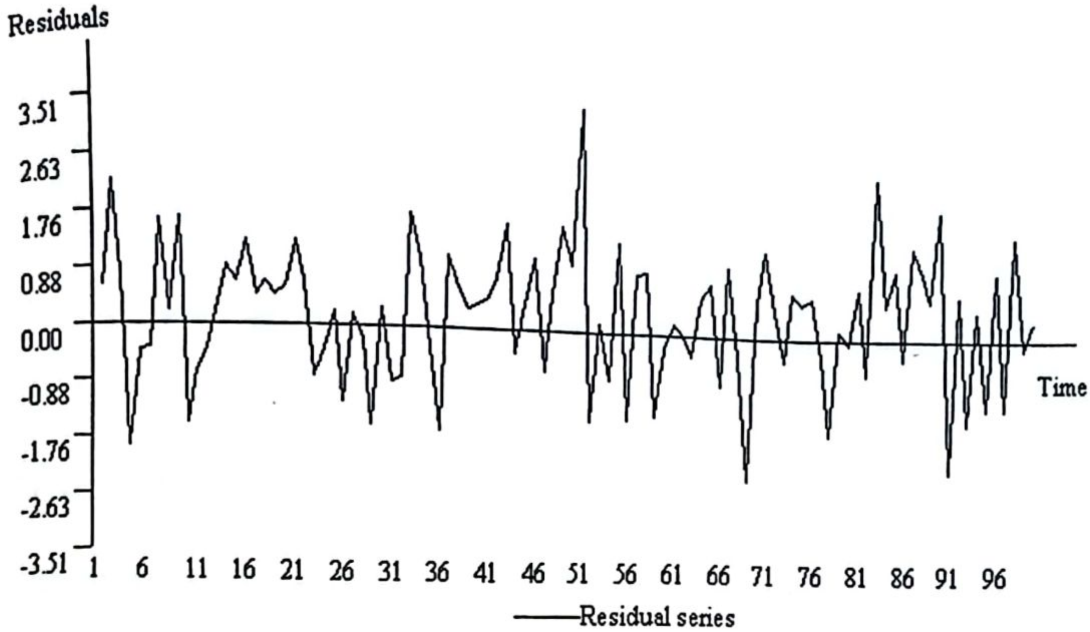


Figure 5.1: Plot of Residuals: AR(1) with an AO at t = 51
(n = 100, ϕ = 0.6, σ_a^2 = 1, ω = 4)

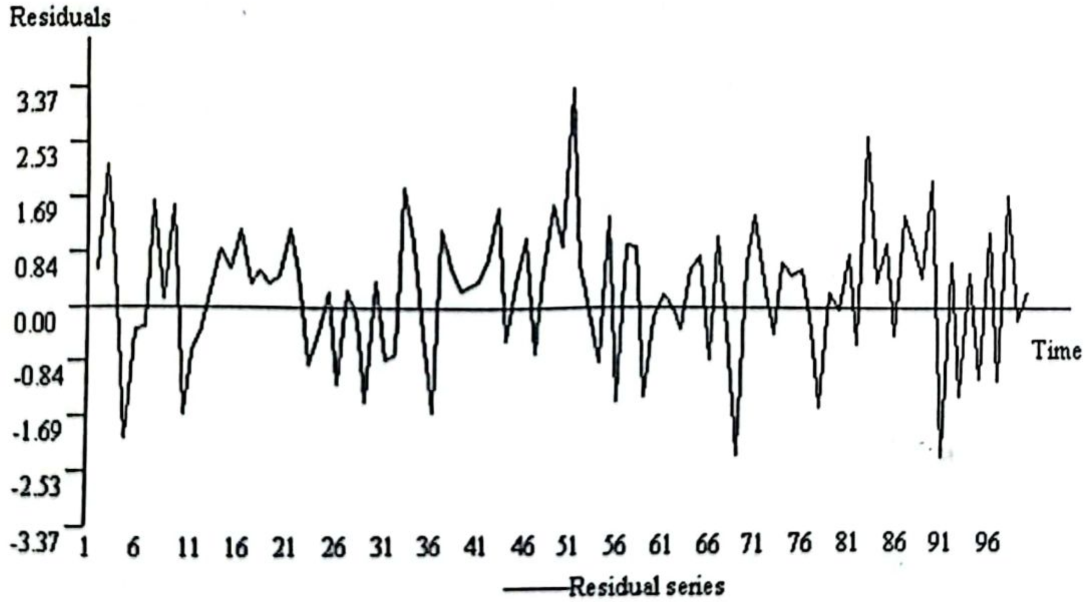


Figure 5.2: Plot of Residuals: AR(1) with an IO at t = 51
(n = 100, ϕ = 0.6, σ_a^2 = 1, ω = 4)

Figure 5.2 also shows the same pattern in residuals series except at the time $t = 51$ and some succeeding values. The residual at $t = 51$ is 3.37. It is clear that the residuals plot cannot effectively show the outlier position when the outlier parameter is not sufficiently large.

We also diagnose these series using the proposed ADV plot. The values, $ADV_{A,i}$ and $ADV_{I,i}$; $i = 1, 2, \ldots, 100$ are calculated and plotted against i for series with each type of outlier. The results are shown in Figures 5.3 and 5.4 for series with AO and IO at $t = 51$ respectively.

In Figure 5.3, the values of $ADV_{A,51} = 13.71$ and $ADV_{I,51} = 11.81$ are higher than those at other time points. The largest value is 13.71 which is given by AO adjustment at $t = 51$. It indicates the possible presence of an AO outlier at $t = 51$ for the given series.

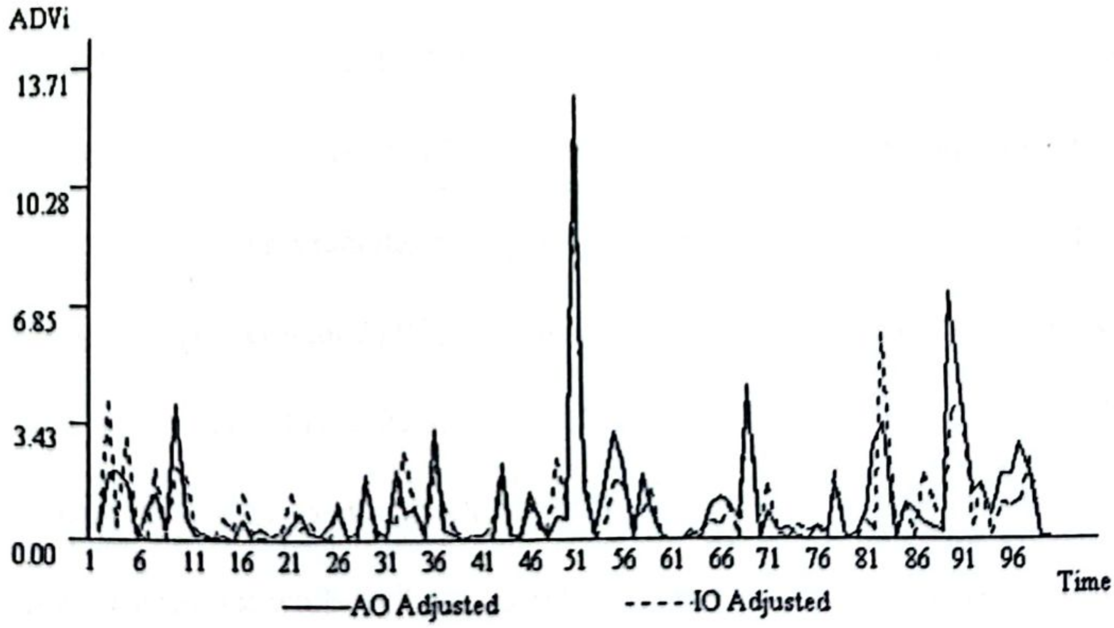


Figure 5.3: Plot of ADV : AR(1) with an AO at $t = 51$ ($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

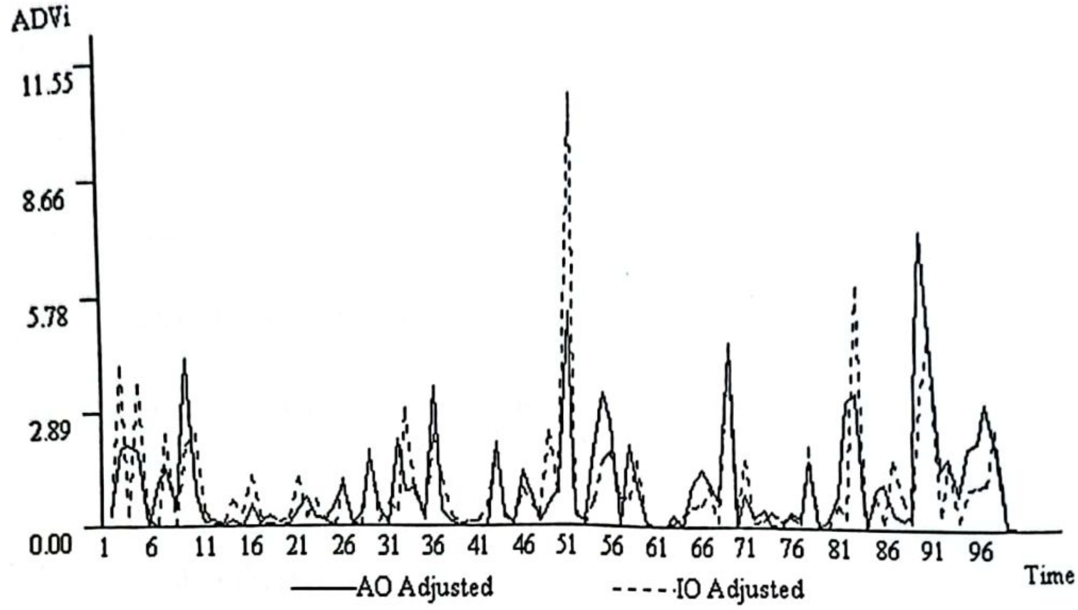


Figure 5.4: Plot of ADV : AR(1) with an IO at t=51
($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

In Figure 5.4, it can be seen that the value of $ADV_{I,51} = 11.55$ is largest value. Hence, we can suspect that an IO outlier occurs at $t = 51$ for the given series.

In both Figures 5.3 and 5.4, we notice that the adjustment by correct outlier type gives the largest value of ADV than the incorrect outlier type adjustment at correct position. For example, in Figure 5.4, the $ADV_{I,51}$ of IO adjustment is 11.55 and it is greater than $ADV_{A,51} = 5.67$ of AO adjustment for IO outlier series at $t = 51$ point. Since it is an IO at $t = 51$, $ADV_{I,51}$ is the largest value. But the AO adjustment at $t = 51$ is not much larger than others. It shows that the AO adjustment does not work well for IO series. Consequently, we can say that the deletion diagnostic may not work well for series with an IO outlier since the series adjustment for AO outlier and the deletion diagnostics are the same.

AR(1) with multiple outliers

We now consider two outliers in AR(1) series and these outliers are introduced with outlier parameter $\omega = 4$ at isolated points and consecutive points in the series. We also consider the situation where both type of outliers are simultaneously present in the data at consecutive points to investigate the problem of patch outliers mentioned in Chapter 4.

(a) Isolated outliers

Suppose two isolated outliers, either both AO or both IO occur in AR(1) series for $\omega = 4$ at time points t = 34 and 67. The plot of ADV can be seen in Figures 5.5 and 5.6 for AOs and IOs respectively.

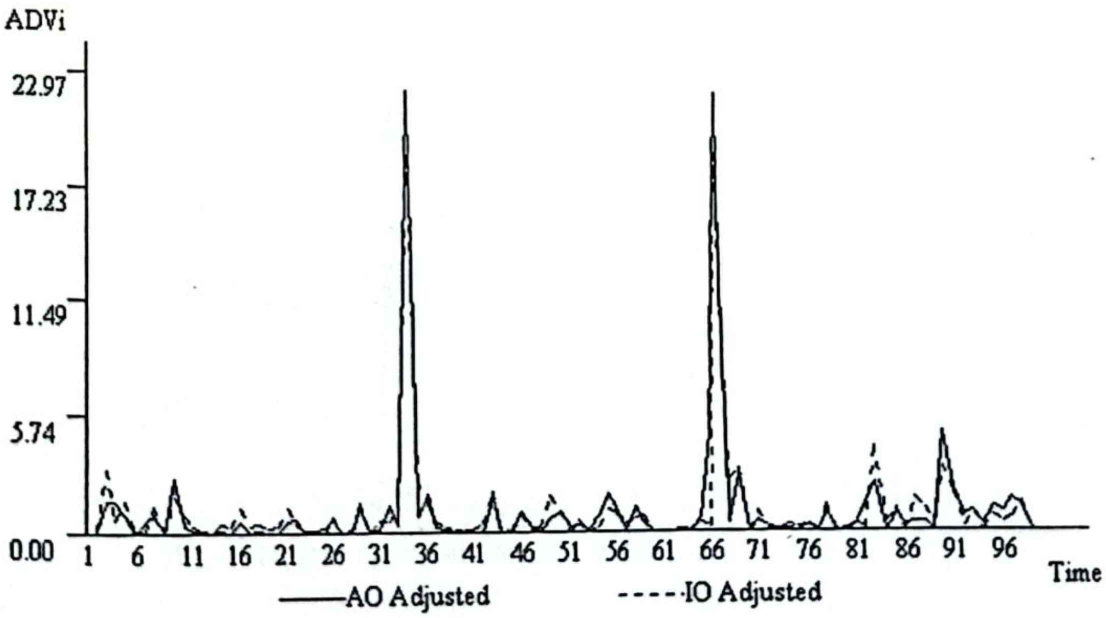


Figure 5.5: Plot of ADV : AR(1) with Two Isolated AOs at t = 34 and 67
($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

From Figure 5.5, the values $ADV_{A,34} = 22.97$, $ADV_{I,34} = 19.64$, $ADV_{A,67} = 22.87$, and $ADV_{I,67} = 19.71$ indicate the position of outliers in time series at t = 34 and 67. Both adjustments clearly show the position of outliers at t = 34 and t = 67.

But AO adjustment values are larger than IO adjustment values for each point t = 34 and 67, indicating the possibility of two AO outliers at t = 34 and 67. Thus, the adjustment diagnostic plot is helpful in guessing the positions of isolated multiple outliers in AR(1) series as well as distinguishing their types.

In Figure 5.6, the ADV plot for AR(1) with two isolated IO outliers is displayed. The values at t = 34 and 67 are larger than others for both types of adjustment. These are - $ADV_{A,34} = 15.25$, $ADV_{I,34} = 20.22$, $ADV_{A,67} = 15.52$ and $ADV_{I,67} = 20.62$. The large values of ADV show the position of outliers.

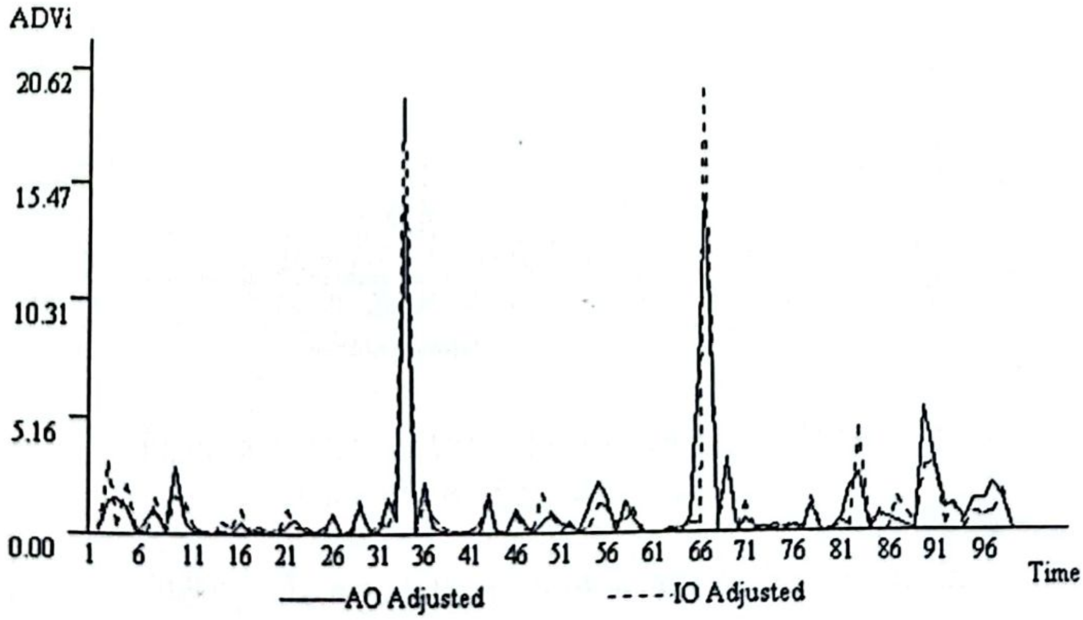


Figure 5.6: Plot of ADV : AR(1) with Two Isolated IOs at t = 34 and 67
($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

In addition, the IO adjustment values are larger than AO adjustment values for each of these two points. Based on Figure 5.6, the possibility of two IO outliers occurring in given time series at t = 34 and 67 can be explored.

(b) Patch outliers

Let us consider presence of a patch of outliers starting at t = 51 with patch length 2. The ADV plots of patch outliers are presented in Figures 5.7 and 5.8 for both outliers of the type AO and IO respectively.

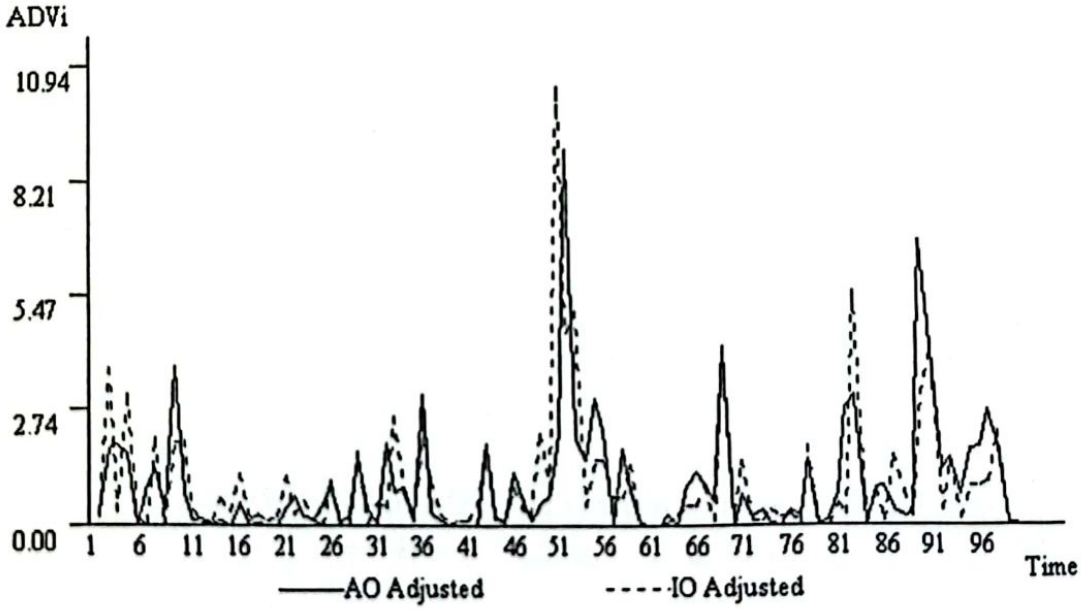


Figure 5.7: Plot of ADV : AR(1) with a Patch of AOs at t = 51
($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

In Figure 5.7, the $ADV_{I,51}$=10.94 is the highest value indicating the presence of an IO at t = 51. The second largest value $ADV_{A,52}$ = 9.40 shows that it can be an AO at t = 52. It does not show as the patch AO outliers at t = 51. This illustrates how ADV fails to identify a patch of two AO type of outliers, supporting some of the observations made in Section 4.7.

The ADV values for a patch IOs are plotted in Figure 5.8. It can be seen that the largest value is $ADV_{I,52} = 17.61$. The second largest value is $ADV_{A,52} = 13.26$. Both values are observed at the same time point t = 52. But $ADV_{I,52}$ is larger than $ADV_{A,52}$. It indicates that the IO outlier occurs at t=52.

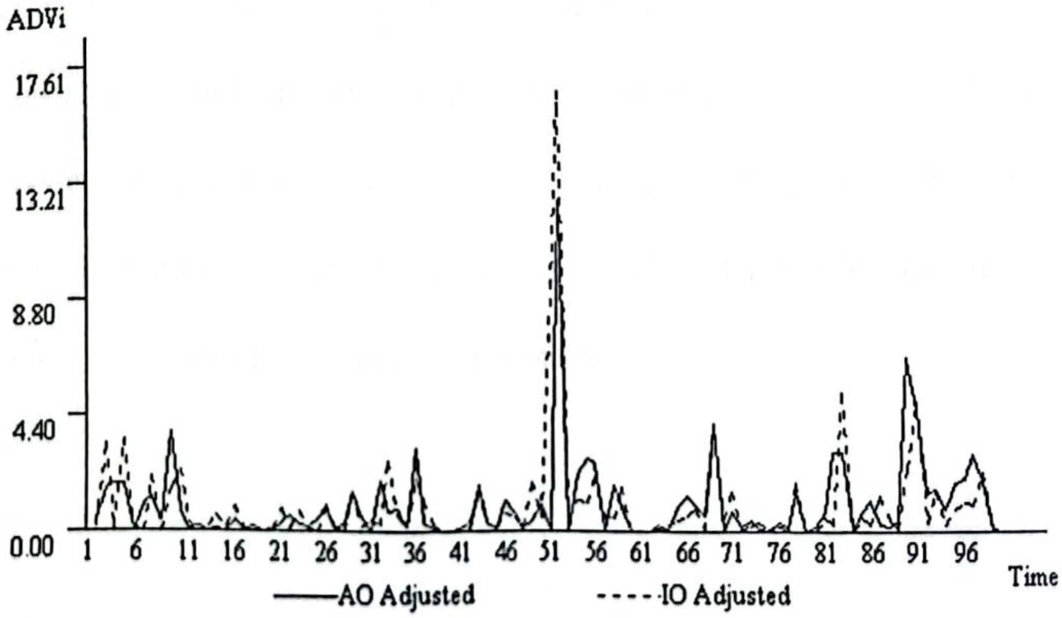


Figure 5.8: Plot of ADV : AR(1) with a Patch of IOs at t = 51
($n = 100, \phi = 0.6, \sigma_a^2 = 1, \omega = 4$)

Except the values of the position t = 52, the $ADV_{I,51} = 9.85$ is the largest value. It also indicates that the IO outlier occurs at t = 51. So, as expected, the ADV may indicate the IO at t= 52 first and IO at t=51 later. The plot helps in indicating that possibly one patch of IO outliers occurs at t=51 with patch length 2.

(c) Different Types of Outliers at Consecutive Time Points

Suppose two types of outliers, either an AO at t = 51 and an IO at t = 52 or an IO at t = 51 and an AO at t = 52, occur in AR(1) series for ω = 4. The plot of ADV can be seen in Figures 5.9 and 5.10 for the presence of consecutive "AO and IO" and "IO and AO" type of outliers respectively.

From Figure 5.9, the $ADV_{I,51}$ = 11 is the largest value. The values of ADV at t = 52 are $ADV_{A,52}$ = 3.23 and $ADV_{I,52}$ = 4.27. The plot indicates that only an IO occurs in the series at time point t = 51. It does not show the effects of both outlier types simultaneously. The AO type of outlier at t = 51 is indicated as an IO type and the plot fails to highlight the IO outlier at time point t = 52. The plot indicates that the ADV procedure may misidentify the types of multiple outliers in some cases, particularly if they occur in patches.

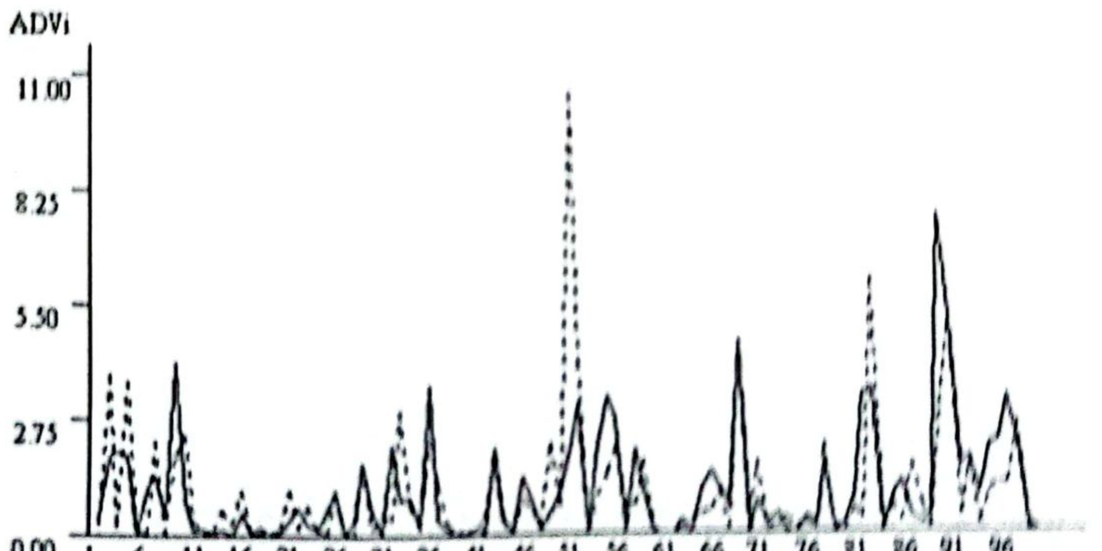


Figure 5.9: Plot of ADV : AR(1) with an AO at t = 51 and an IO at t = 52 (n = 100, ϕ = 0.6, σ_a^2 = 1, ω = 4)

<u>MA(1) with one outlier</u>

We introduced an outlier at t = 51 of AO and IO type in the generated MA(1) series, resulting in two contaminated MA(1) series with an AO and an IO. The plots of $ADV_{A,i}$ and $ADV_{I,i}$ against i =1, 2, ..., 100 are presented in Figures 5.11 and 5.12 for the MA(1) series with AO and IO outlier respectively.

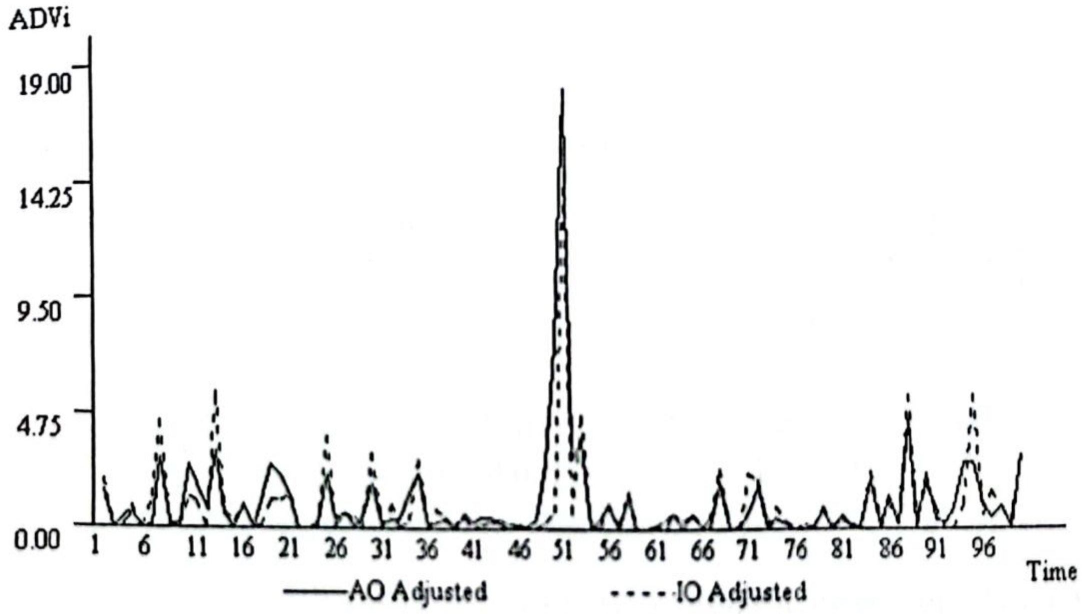


Figure 5.11: Plot of ADV : MA(1) with an AO t = 51
($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4$)

Figure 5.11 indicates that the AO outlier may have occurred at t=51 for AO generated series as the value of $ADV_{A,51} = 19$ is larger than $ADV_{I,51} = 14.78$.

From Figure 5.12, the value of $ADV_{I,51} = 15.28$ is higher than $ADV_{A,51} = 8.07$ for an IO contaminated series. The ADV value obtained after adjustment of the series under the presence of an IO at t = 51 is significantly larger than that at other time points as well as the values based on adjustment for an AO type at all time points, indicating the presence of an IO outlier at t = 51 for this series.

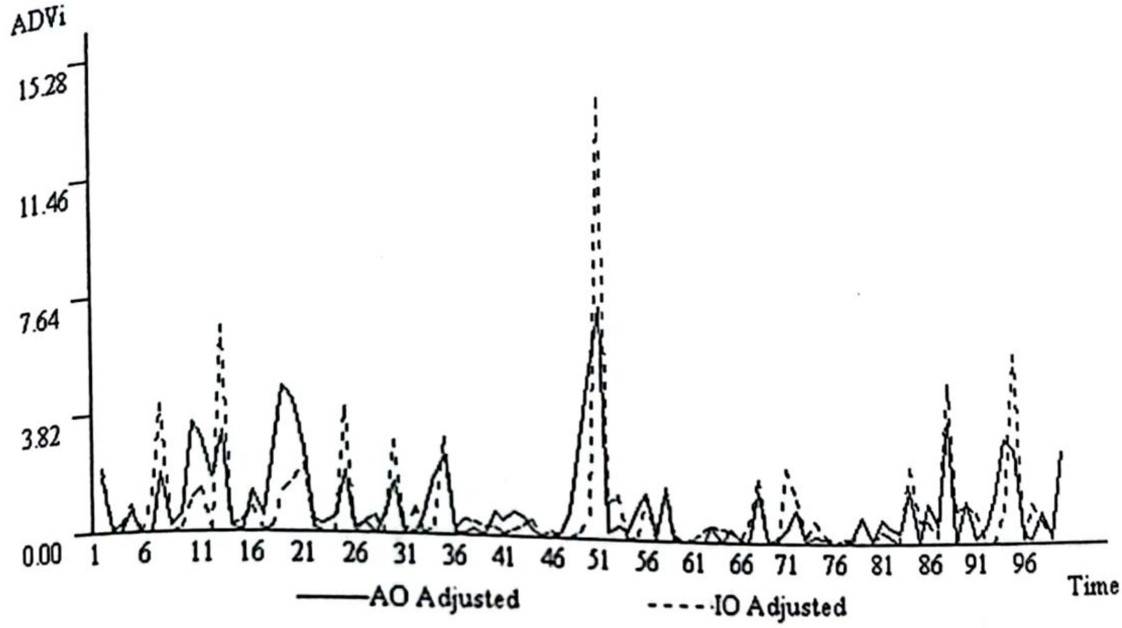


Figure 5.12: Plot of ADV : MA(1) with an IO at t = 51
($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4$)

In both Figures 5.11 and 5.12 the adjustment diagnostic shows highly significant values at time position t = 51. We can also see that the true outlier type adjustment gives the large value of ADV at true position.

MA(1) with multiple outliers

We now study the ADV plot in the presence of two outliers that are introduced in the generated MA(1) series with value of outlier parameter $\omega = 4$. As in previous section, we analyze the plots of isolated outliers and patch outliers separately.

(a) Isolated outliers

We consider the two isolated outliers, either two AOs or two IOs introduced at points t = 34 and 67. The ADV plot of two series can be seen in Figures 5.13 and 5.14 for two isolated AOs and IOs respectively.

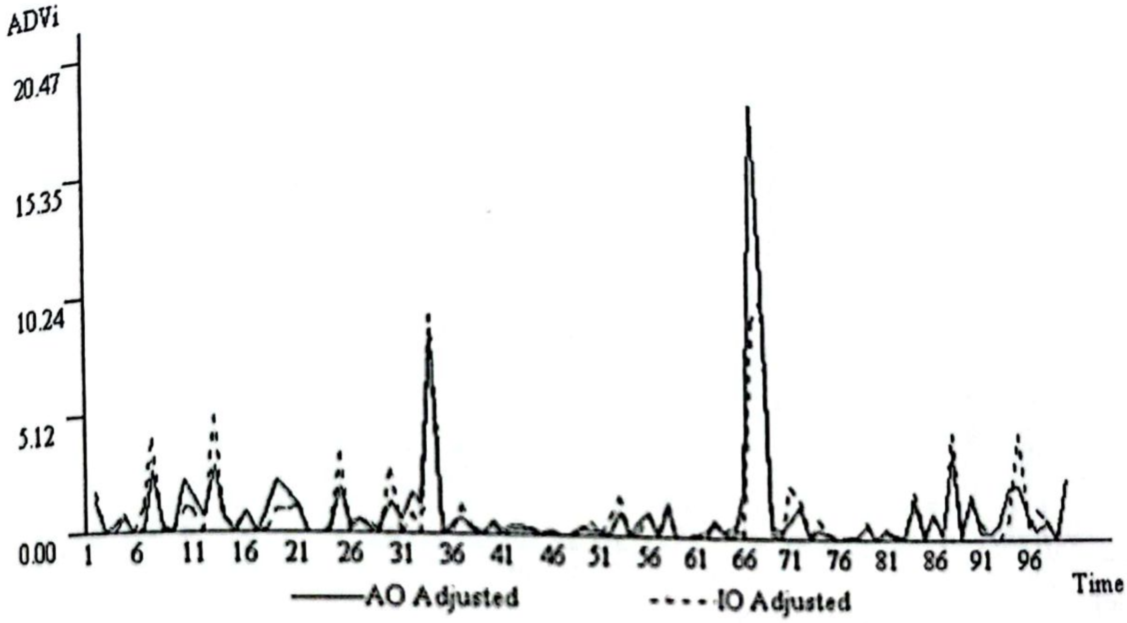


Figure 5.13: Plot of ADV : MA(1) with Two Isolated AOs at t=34 and 67
$(n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4)$

From Figure 5.13, it can be seen that the values at t = 34 and 67 are $ADV_{A,34} = 8.81$, $ADV_{I,34} = 10.02$, $ADV_{A,67} = 20.47$ and $ADV_{I,67} = 10.31$. It indicates the true positions of both outliers. The plot indicates presence of IO at t = 34 instead of the true type AO.

The ADV plot of series with two isolated IOs is presented in Figure 5.14. The plot clearly indicates the positions of two isolated outliers at t = 34 and 67.

Although the $ADV_{I,34} = 11.59$ indicates the IO outlier, the ADV at t = 67 shows the AO outlier instead of true type IO. The reason for this is the value of the generated series at time point t = 68 and the moderate value of outlier parameter ω.

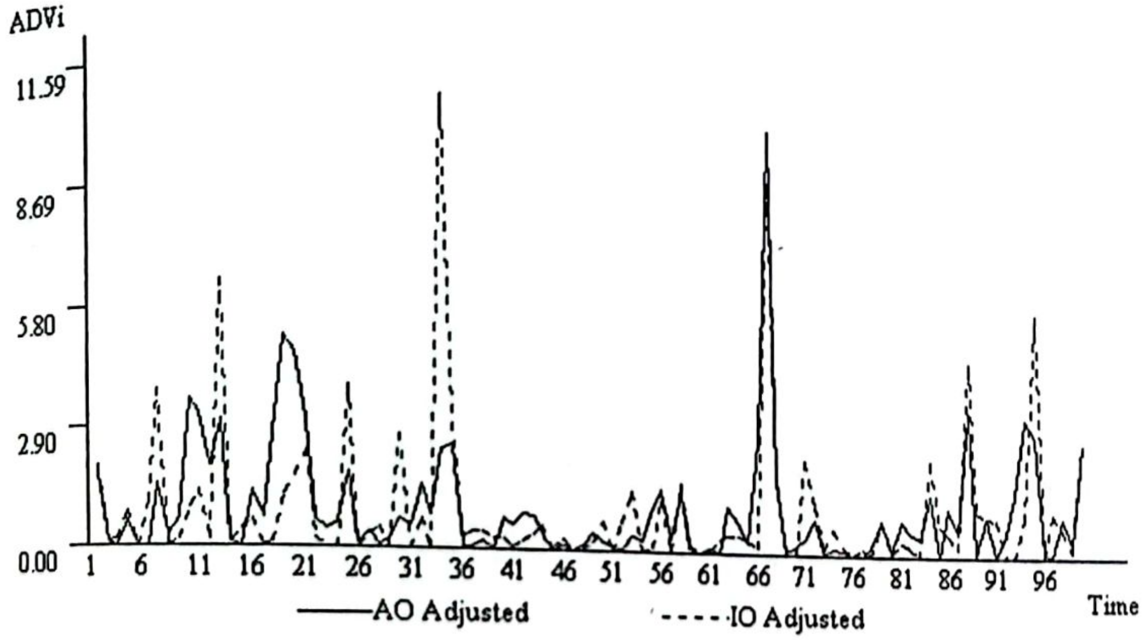


Figure 5.14: Plot of ADV : MA(1) with Two Isolated IOs at t= 34 and 67
($n = 100, \theta = -0.6, \sigma_a^2 = 1, \omega = 4$)

(b) Patch outliers

We introduced a patch of two outliers of same type in MA(1) at points t = 51 and 52 resulting in two series with an AO patch and an IO patch with patch length 2. The ADV plots for the two series are shown in Figures 5.15 and 5.16 for AO and IO respectively.

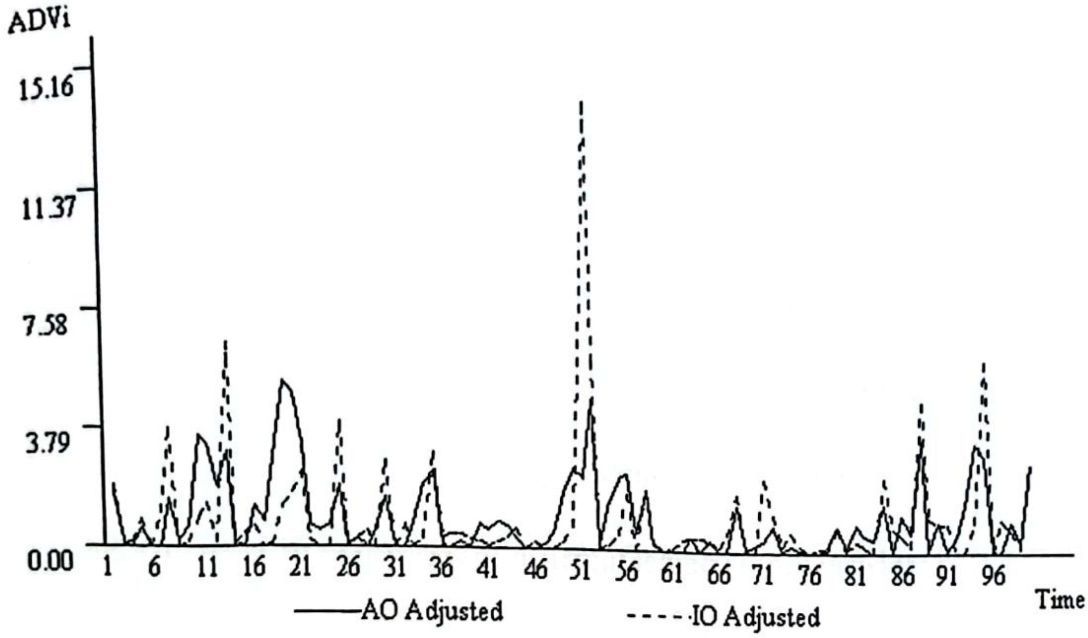


Figure 5.15: Plot of ADV : MA(1) with a Patch of AOs at t = 51
(n = 100, $\theta = -0.6$, $\sigma_a^2 = 1$, $\omega = 4$)

Figure 5.15 indicates a patch of two IO outliers at t = 51 instead of the actual patch of two AO since the largest value is $ADV_{I,51} = 15.16$ and the second largest value is $ADV_{I,52} = 6.41$.

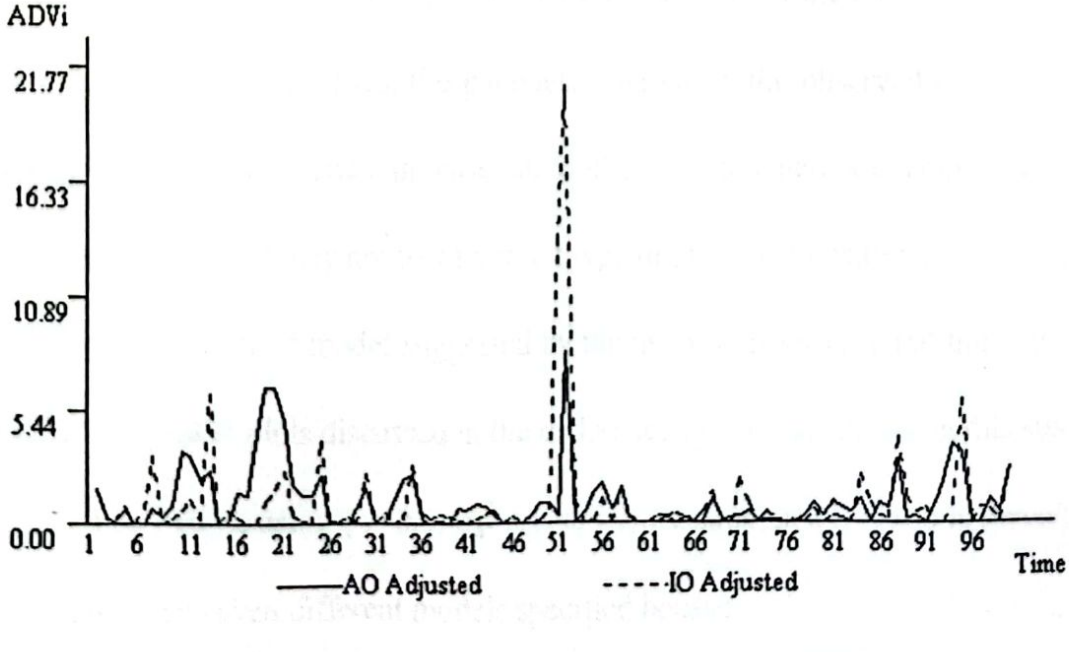


Figure 5.16: Plot of ADV : MA(1) with a Patch of IOs at t = 51
(n = 100, $\theta = -0.6$, $\sigma_a^2 = 1$, $\omega = 4$)

From Figure 5.16, it can be seen that two largest values are $ADV_{I,51} = 14.03$ and $ADV_{I,52} = 21.77$ and the plot indicates the presence of a patch of two IO outliers at $t = 51$.

In conclusion, the ADV plots give a fair idea about the total number of outliers, their types and positions. In addition, the ADV plot can also help in selecting the warning line suggested by Ledolter (1990), since getting exact cut-off points for all types of time series of different lengths is not possible.

5.3 Analysis of Simulated Data with Outliers

In this section we study the diagnosis of outliers in simulated data using the proposed adjustment diagnostic measure ADV introduced in Section 4.2. For various generated series, to be specified later, at most two outliers are introduced and ADV is applied. For all types of outliers, the value of the outlier parameter ω is kept at 4. For each series, the parameters based on the observed contaminated series and adjusted series in case an outlier is indicated are computed. The indication of outlier may not be of correct type or at correct position and hence, for all the series the final model suggested by the analysis is specified at the end. The outlier diagnostic plots discussed in the earlier section are not shown in this study.

For AR(1), using $\phi = 0.6$, $\sigma_a^2 = 1$, $\omega = 4$, we simulated $n = 100$ observations on each of the seven different models specified below:

1. AR(1) Model without outlier

$$Y_t = \frac{1}{1-\phi B} a_t$$

2. AR(1) Model with AO at t = 51

$$Y_t = \frac{1}{1-\phi B} a_t + \omega \xi_t^{(51)}$$

3. AR(1) Model with two AOs at t = 34 and 67

$$Y_t = \frac{1}{1-\phi B} a_t + \{\omega \xi_t^{(34)} + \omega \xi_t^{(67)} \}$$

4. AR(1) Model with two AOs at t = 51 and 52

$$Y_t = \frac{1}{1-\phi B} a_t + \{\omega \xi_t^{(51)} + \omega \xi_t^{(52)} \}$$

5. AR(1) Model with IO at t = 51

$$Y_t = \frac{1}{1-\phi B} \{ \omega \xi_t^{(51)} + a_t \}$$

6. AR(1) Model with two IOs at t = 34 and 67

$$Y_t = \frac{1}{1-\phi B} \{ \omega \xi_t^{(34)} + \omega \xi_t^{(67)} + a_t \}$$

7. AR(1) Model with two IOs at t = 51 and 52

$$Y_t = \frac{1}{1-\phi B} \{\omega \xi_t^{(51)} +\omega \xi_t^{(52)} + a_t \}$$

Using the adjustment diagnostic method, various outlier types and positions were indicated. The estimated parameters and the estimates after deleting the indicated outliers are summarized in Table 5.1 for the seven series specified above.

Table 5.1

A Summary Analysis of AR(1) Series

Model	Outlier & Position	Est. Value (Before Detection)	Indicated Outlier	Est. Value (After Detection)	Reduce % of $\hat{\sigma}_a^2$
1	No outlier	$\hat{\phi}$ = 0.5681 $\hat{\sigma}_a^2$ = 1.0389	–	–	–
2	1 AO at 51	$\hat{\phi}$ = 0.5790 $\hat{\sigma}_a^2$ = 1.1730	$\hat{\omega}_{AO,51}$ = 3.2529	$\hat{\phi}$ = 0.5805 $\hat{\sigma}_a^2$ = 1.0316	−12.06%
3	2 AO at 34 & 67	$\hat{\phi}$ = 0.4544 $\hat{\sigma}_a^2$ = 1.6370	$\hat{\omega}_{AO,34}$ = 4.9557 $\hat{\omega}_{AO,67}$ = 4.9916	$\hat{\phi}$ = 0.5719 $\hat{\sigma}_a^2$ = 1.0360	− 36.71%
4	2 AO at 51 & 52	$\hat{\phi}$ = 0.6162 $\hat{\sigma}_a^2$ = 1.2638	$\hat{\omega}_{IO,51}$ = 3.5673 * $\hat{\omega}_{AO,52}$ = 2.9461	$\hat{\phi}$ = 0.5544 $\hat{\sigma}_a^2$ = 1.0379	− 17.88%
5	1 IO at 51	$\hat{\phi}$ = 0.6433 $\hat{\sigma}_a^2$ = 1.1543	$\hat{\omega}_{IO,51}$ = 3.4653	$\hat{\phi}$ = 0.5994 $\hat{\sigma}_a^2$ = 1.0405	− 9.86%
6	2 IO at 34 & 67	$\hat{\phi}$ = 0.5592 $\hat{\sigma}_a^2$ = 1.5272	$\hat{\omega}_{IO,34}$ = 5.0787 $\hat{\omega}_{IO,67}$ = 5.0925	$\hat{\phi}$ = 0.5435 $\hat{\sigma}_a^2$ = 1.0109	− 33.81%
7	2 IO at 51 & 52	$\hat{\phi}$ = 0.7002 $\hat{\sigma}_a^2$ =1.3590	$\hat{\omega}_{IO,51}$ = 3.5082 $\hat{\omega}_{IO,52}$ = 4.9489	$\hat{\phi}$ = 0.5813 $\hat{\sigma}_a^2$ = 1.0292	− 24.27%

The first row of Table 5.1 gives the estimated values $\hat{\phi}$ = 0.5679 and $\hat{\sigma}_a^2$ = 1.0389 which are the maximum likelihood estimates for the parameter and error variance of outlier free AR(1) series which is generated from $\phi = 0.6$, $\sigma_a^2 = 1$ and n = 100. We can see that the estimated values are close to the true value. When we introduced the outlier(s) in the series, the estimated results mostly depart from the estimated values of the outlier free series due to the effects of outliers and their positions. From the table, the series with outliers overestimate the error variance.

Using the ADV, the outliers are mostly indicated by correct types of outlier at their correct positions except for one patch of AO outlier with patch length 2 (indicated by * in Table 5.1). In this case, the diagnostic indicates presence of the IO and AO outliers instead of two AOs at t = 51 and 52. It can be possible in practice because the first outlier of consecutive AO outliers may have an effect more like an IO. However, the estimated error variance $\hat{\sigma}_a^2 = 1.0379$ is close to the estimated error variance based on the outlier free series.

The estimated values for time series parameter and error variance after deleting the suspected observation are approximately the same as the estimated values of outlier free series. The last column of the table shows the reduced percentages of the estimated error variances. The reduced percentages are high especially for two isolated outliers because the presence of isolated outliers significantly increases the variance (see Table 4.22 and 4.23).

At the end, the corresponding fitted AR(1) models with the estimated parameters are as follows:

1. $$Y_t = \frac{1}{1-0.57B}\, a_t$$

2. $$Y_t = \frac{1}{1-0.58B}\, a_t + 3.25\, \xi_t^{(51)}$$

3. $$Y_t = \frac{1}{1-0.57B}\, a_t + 4.96\, \xi_t^{(34)} + 4.99\, \xi_t^{(67)}$$

4. $$Y_t = \frac{1}{1-0.55B}\, \{3.57\, \xi_t^{(51)} + a_t\} + 2.95\, \xi_t^{(52)}$$

5. $$Y_t = \frac{1}{1-0.60B}\{3.47\,\xi_t^{(51)} + a_t\}$$

6. $$Y_t = \frac{1}{1-0.54B}\{5.08\,\xi_t^{(34)} + 5.09\,\xi_t^{(67)} + a_t\}$$

7. $$Y_t = \frac{1}{1-0.58B}\{3.51\,\xi_t^{(51)} + 4.95\,\xi_t^{(52)} + a_t\}$$

Similarly for MA(1) series, using $\theta = -0.6$, $\sigma_a^2 = 1$, and $\omega = 4$, we generated $n = 100$ observations on each of the following models:

1. MA(1) Model without outlier

$$Y_t = (1-\theta B)a_t$$

2. MA(1) Model with AO at $t = 51$

$$Y_t = (1-\theta B)a_t + \omega\,\xi_t^{(51)}$$

3. MA(1) Model with two AOs at $t = 34$ and 67

$$Y_t = (1-\theta B)a_t + \{\omega\,\xi_t^{(34)} + \omega\,\xi_t^{(67)}\}$$

4. MA(1) Model with two AOs at $t = 51$ and 52

$$Y_t = (1-\theta B)a_t + \{\omega\,\xi_t^{(51)} + \omega\,\xi_t^{(52)}\}$$

5. MA(1) Model with IO at $t = 51$

$$Y_t = (1-\theta B)\{\omega\,\xi_t^{(51)} + a_t\}$$

6. MA(1) Model with two IOs at $t = 34$ and 67

$$Y_t = (1-\theta B)\{\omega\,\xi_t^{(34)} + \omega\,\xi_t^{(67)} + a_t\}$$

7. MA(1) Model with two IOs at $t = 51$ and 52

$$Y_t = (1-\theta B)\{\omega\,\xi_t^{(51)} + \omega\,\xi_t^{(52)} + a_t\}$$

Table 5.2 presents consolidated results on analysing these series using ADV, analogous to Table 5.1.

Table 5.2

A Summary Analysis of MA(1) Series

Model	Outlier & Position	Est. Value (Before Detection)	Indicated Outlier	Est. Value (After Detection)	Reduce % of $\hat{\sigma}_a^2$
1	No outlier	$\hat{\theta} = -0.5915$ $\hat{\sigma}_a^2 = 0.9342$	–	–	–
2	1 AO at 51	$\hat{\theta} = -0.4388$ $\hat{\sigma}_a^2 = 1.1107$	$\hat{\omega}_{AO,51} = 3.6938$	$\hat{\theta} = -0.5795$ $\hat{\sigma}_a^2 = 0.9566$	−13.87%
3	2 AO at 34 & 67	$\hat{\theta} = -0.4222$ $\hat{\sigma}_a^2 = 1.2398$	$\hat{\omega}_{AO,34} = 2.8029$ $\hat{\omega}_{AO,67} = 4.1331$	$\hat{\theta} = -0.6111$ $\hat{\sigma}_a^2 = 0.9116$	−26.47%
4	2 AO at 51 & 52	$\hat{\theta} = -0.6137$ $\hat{\sigma}_a^2 = 1.1313$	$\hat{\omega}_{IO,51} = 3.8460$ *	$\hat{\theta} = -0.6376$ $\hat{\sigma}_a^2 = 0.9913$	−12.38%
5	1 IO at 51	$\hat{\theta} = -0.5730$ $\hat{\sigma}_a^2 = 1.0777$	$\hat{\omega}_{IO,51} = 3.8048$	$\hat{\theta} = -0.5993$ $\hat{\sigma}_a^2 = 0.9393$	−12.84%
6	2 IO at 34 & 67	$\hat{\theta} = -0.6123$ $\hat{\sigma}_a^2 = 1.1406$	$\hat{\omega}_{IO,34} = 3.4104$ $\hat{\omega}_{AO,67} = 2.7101$ *	$\hat{\theta} = -0.6443$ $\hat{\sigma}_a^2 = 0.9198$	−19.36%
7	2 IO at 51 & 52	$\hat{\theta} = -0.6955$ $\hat{\sigma}_a^2 = 1.2959$	$\hat{\omega}_{IO,51} = 3.8026$ $\hat{\omega}_{IO,52} = 4.9466$	$\hat{\theta} = -0.5983$ $\hat{\sigma}_a^2 = 0.9251$	−28.61%

In Table 5.2, the estimated values of MA(1) model without outlier are $\hat{\theta}$ = −0.5915 and $\hat{\sigma}_a^2$ = 0.9342. The estimated values of outlier free series are approximately the same as the true values. When we introduced the outlier(s) in the series, the estimated parameters clearly get affected due to outliers and the error variance gets overestimated due to the presence of outliers. In particular, it can be seen that for multiple outlier models the minimum reduction percentage is

12%. In Model (4), the procedure diagnosed IO at t = 51 instead of true outliers 2AOs at t = 51 and 52 since a patch of two AOs is identified as a single IO. In Model (6), the IO at = 67 is indicated as an AO at the same position which is due to the natural fluctuations in the series as the magnitude of the outlier is not very large since ω is taken to be 4.

The corresponding seven fitted models at the end of analysis are

1. $Y_t = (1+0.59B)a_t$

2. $Y_t = (1+0.58B)a_t + 3.69\ \xi_t^{(51)}$

3. $Y_t = (1+0.61B)a_t + \{2.80\ \xi_t^{(34)} + 4.13\ \xi_t^{(67)}\}$

4. $Y_t = (1+0.64B)\{3.85\ \xi_t^{(51)} + a_t\}$

5. $Y_t = (1+0.60B)\{3.81\ \xi_t^{(51)} + a_t\}$

6. $Y_t = (1+0.64B)\{3.41\ \xi_t^{(34)} + a_t\} + 2.71\ \xi_t^{(67)}$

8. $Y_t = (10.60B)\ \{3.80\ \xi_t^{(51)} +4.95\ \xi_t^{(52)} + a_t\}$

Generally, the adjustment diagnostic method can indicate the correct type of outliers and their positions.

An attempt was made to study the effect of presence of different types of outliers in an ARMA(1,1) series on the lines of study of AR(1) and MA(1) presented in the thesis. The diagnostic plots as well as the procedure for outlier detection give similar results.

5.4 Analysis of Outliers in Real Life Data Sets

In this section, we consider some numerical examples selected from the available literature to illustrate the application of the proposed procedure ADV using the STDS software. The resulting analysis of possible outliers in the series is compared with the existing analysis for each of the data sets available in the literature. Since the estimates depend on the estimation procedure used, the estimated parameters using STDS may be different from those presented in the literature. In order to compare the reduction in the estimate of error variance after the identification of outliers, we use the percentage of reduction rate instead of the reduced estimates. In addition, we used the software SPSS for Windows to compare the two estimates of error variance for those series for which ADV based outlier analysis was significantly different from the available analysis. Apart from reduction in the estimate of error variance, parsimony was also considered to be an important criterion of analysis.

The data sets considered are

1. Daily Average Number of Truck Manufacturing Defects Series
   (Wei, 1990, p. 446)

2. Series C (Chemical Process Temperature Readings: Every Minute)
   (Box et al., 1994, p. 544)

3. Series A (Chemical Process Concentration Readings: Every 2 Hours)
   (Box et al., 1994, p. 542)

4. Series D (Chemical Process Viscosity Readings: Every Hour)
   (Box et al., 1994, p. 545)

5. Series J (Gas Furnace Data: 9 second intervals on input gas feed rate) (Box et al., 1994, p. 548)

1. Truck Defects Series

The first example considered is the Daily Average Number of Truck Manufacturing Defects series of n = 45 (Wei, 1990, p. 446). The series is shown in Figure 5.17.

In the figure, the values at t = 4, 7, 17 and 36 look like the possible outliers. But the question is whether all are the outliers or not and what are the types of outliers. It may be difficult to get the right answer by visualization from the time sequence plot and the detection of outliers is an important issue in such a case.

Figure 5.17: Plot of Daily Average Number of Truck Manufacturing Defects Series

The model suggested by Wei (1990) for this data is AR(1) specified by

$$(1 - \phi B)Z_t = \theta_0 + a_t \tag{5.1}$$

where θ_0 is the over all constant in the model. The fitted model is

$$(1 - 0.43B)Z_t = 0.89 + a_t \tag{5.2}$$

with $\hat{\sigma}_a^2 = 0.21$. Under the model (5.1), Wei (1990, p. 201) applied the likelihood ratio test (LRT) proposed by Chang et al. (1988) to detect the outliers. Based on the observed series, the software STDS gives us the results for the parameters as $\hat{\phi} = 0.43$ (0.14) and $\hat{\sigma}_a^2 = 0.21$. These values are same as those estimated by Wei.

Before carrying out the adjustment diagnostic on the series, we look at the residual plot (Figure 5.18) and the ADV plot (Figure 5.19). The residual plots shows unusual jumps at time points t = 4, 7 and 36 leading to the suspicion of presence of multiple outliers in the data.

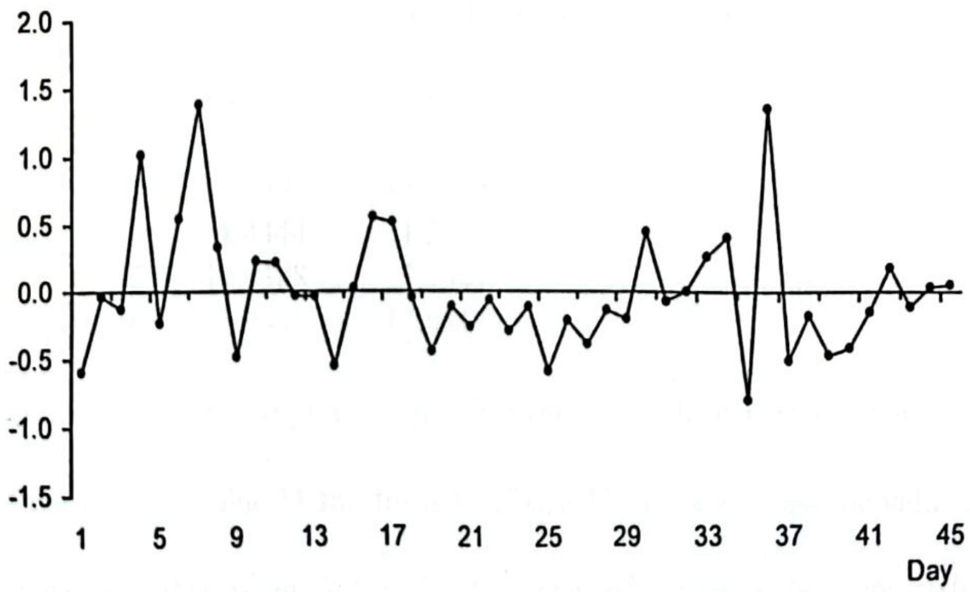


Figure 5.18: Plot of Residuals from AR(1) Model fitted to Daily Average Number of Truck Manufacturing Defects Series

The ADV plot in Figure 5.19 shows clear peaks at points t = 4, 7, 35, 36 indicating presence of outliers.

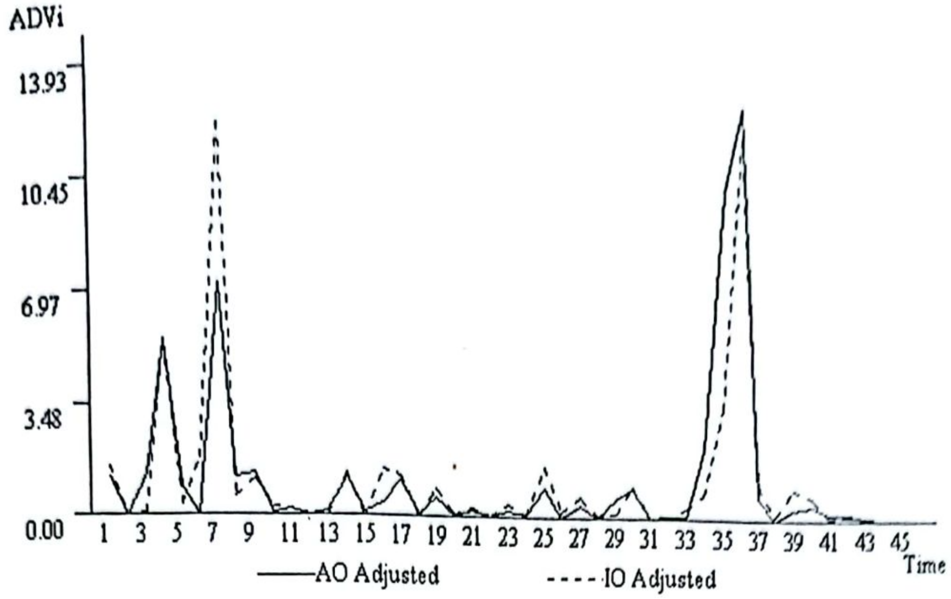


Figure 5.19: Plot of ADV from AR(1) Model fitted to Daily Average Number of Truck Manufacturing Defects Series

The suspected value at t = 17 in Figure 5.17 for the original series does not show any unusual behaviour in both these plots. The values of ADV statistic based on AO and IO adjustment at these peak points are as shown below.

t	$ADV_{A,t}$	$ADV_{I,t}$
4	5.7271	5.5824
7	7.5793	12.7433
9	1.4444	1.2332
35	11.0388	3.7541
36	13.9348	12.8517

The ADV plot also gives a guideline on selection of warning value. If we choose the warning value 11 for this data of size 45, it is clear that the value at t = 36 will be indicated as an AO at the first step of iteration. But, the value of $ADV_{A,35}$ also shows the possible outlier at t= 35 and it is possible that it affects the observation at t = 36. We now carry out the proposed iterative diagnostic procedure using STDS on the data with the reference value 11 which is obtained from ADV plot.

C is chosen. But in any case, the ADV procedure will not indicate the observation at $t = 9$ as an outlier.

Taking into account the outliers diagnosed by the two procedures, the following two outlier models emerge:

$$Z_t = \theta_0 + \omega_1 \xi_t^{(36)} + \omega_3 \xi_t^{(7)} + \frac{1}{(1-\phi B)} \left\{ \omega_2 \xi_t^{(9)} + \omega_4 \xi_t^{(4)} + a_t \right\} \quad \text{for LRT} \quad (5.3)$$

$$Z_t = \theta_0 + \omega_1 \xi_t^{(36)} + \omega_3 \xi_t^{(4)} + \frac{1}{(1-\phi B)} \left\{ \omega_2 \xi_t^{(7)} + a_t \right\} \quad \text{for ADV} \quad (5.4)$$

Using these models, the simultaneous estimation of parameters was carried out for both the models to obtain the final estimates. The estimated parameters and their standard error (SE) for corresponding models (5.3) and (5.4) are shown in Table 5.4.

Table 5.4

Estimated Parameters of AR(1) with outliers for Truck Defects Series

Parameter	LRT		ADV	
	Estimate	SE	Estimate	SE
θ_0	1.14	–	0.83	–
ϕ	0.28	0.11	0.51	0.11
ω_1	1.39	0.11	1.39	0.30
ω_2	-0.61	0.19	1.41	0.33
ω_3	0.99	0.37	0.99	0.29
ω_4	0.66	0.31	–	–
σ_a^2	0.11	–	0.10	–

In Table 5.4, the values of second and third columns are from Wei (1990), and the last two columns are given by ADV method. Both the models reduce the $\hat{\sigma}_a^2$ from 0.21 to 0.11 and 0.10 respectively. The estimated parameter $\hat{\phi}$ is reduced from 0.43 to 0.28 by LRT method whereas it is increased to 0.51 by ADV method. To compare the estimates using the two methods, the percentage reduction in estimated error variance is calculated as shown in Table 5.5.

Table 5.5

The Reduction Percentage in Error Variance for Truck Defects Series

Model	Estimated Variance	Reduction Percent	Method
AR(1) with 4 outliers	0.11 (*0.21*)	47.62 %	LRT
AR(1) with 3 outliers	0.10 (*0.21*)	52.38 %	ADV

Note: The italic values within the parentheses are the estimated error variances $\hat{\sigma}_a^2$ when the outliers are ignored.

From Table 5.5, it can be seen that the estimated error variances $\hat{\sigma}_a^2$ are 0.11 and 0.10 for 4 outliers model by LRT and 3 outliers model by ADV respectively. According to the reduction percentages of estimated residual variances, the ADV method gives less variance than LRT as well as less number of outliers. But the estimate $\hat{\sigma}_a^2$ depends on the estimation method. Hence we use the SPSS software to get the estimates $\hat{\sigma}_a^2$ for two models (5.3) and (5.4) proposed by LRT and ADV methods respectively.

To estimate the parameters of the outlier models using the available intervention analysis of SPSS, we need to define the independent variables (input variables) $x_{t,j}$, t=1,2, ..., 45 where j stands for the outlier. For model (5.3), the value $x_{t,j}$ are

$$x_{t,1} = 1 \text{ if } t = 36, \text{ and otherwise } 0,$$

$$x_{t,2} = \hat{\psi}_{t-9} \text{ if } t \geq 9, \text{ otherwise } 0,$$

$$x_{t,3} = 1 \text{ if } t = 7, \text{ and otherwise } 0,$$

$$x_{t,4} = \hat{\psi}_{t-4} \text{ if } t \geq 4, \text{ otherwise } 0.$$

whereas, for model (5.4) the values are

$$x_{t,1} = 1 \text{ if } t = 36, \text{ and otherwise } 0,$$

$$x_{t,2} = \hat{\psi}_{t-7} \text{ if } t \geq 7, \text{ and otherwise } 0, \text{ and}$$

$$x_{t,3} = 1 \text{ if } t = 4, \text{ otherwise } 0, \tag{5.5}$$

where $\hat{\psi}_j = \hat{\phi}^j = (0.4322)^j$ for j = 0, 1, ... and $\hat{\phi}$ is estimate based on the observed series. The estimated error variances obtained from SPSS are presented in Table 5.6.

Table 5.6

Comparison of the Reduction Percentage on $\hat{\sigma}_a^2$ for Truck Defects Series

Model	Estimated Variance	Reduction Percent	Method
AR(1) with 4 outliers	0.1136	48.76 %	LRT
AR(1) with 3 outliers	0.1055	52.41 %	ADV

Note: $\hat{\sigma}_a^2 = 0.2217$ and $\hat{\phi} = 0.43$ where the outliers are ignored.

It can be seen that the reduction percentages by LRT and ADV methods are 48.76% and 52.47% respectively and the proposed ADV method gives 3.65%

more reduction in the estimate of error variance. It also indicates less number of outliers in the series and the types of outliers indicated by ADV are justifiable. Thus ADV seems to be giving a more satisfactory analysis of the data than the LRT method.

2. Series C

The second example considered is Series C (Box et al., 1994, p. 544) which represents 226 uncontrolled temperature readings every minute in a chemical process. First 100 observations of the series ∇Z_t are plotted in Figure 5.20.

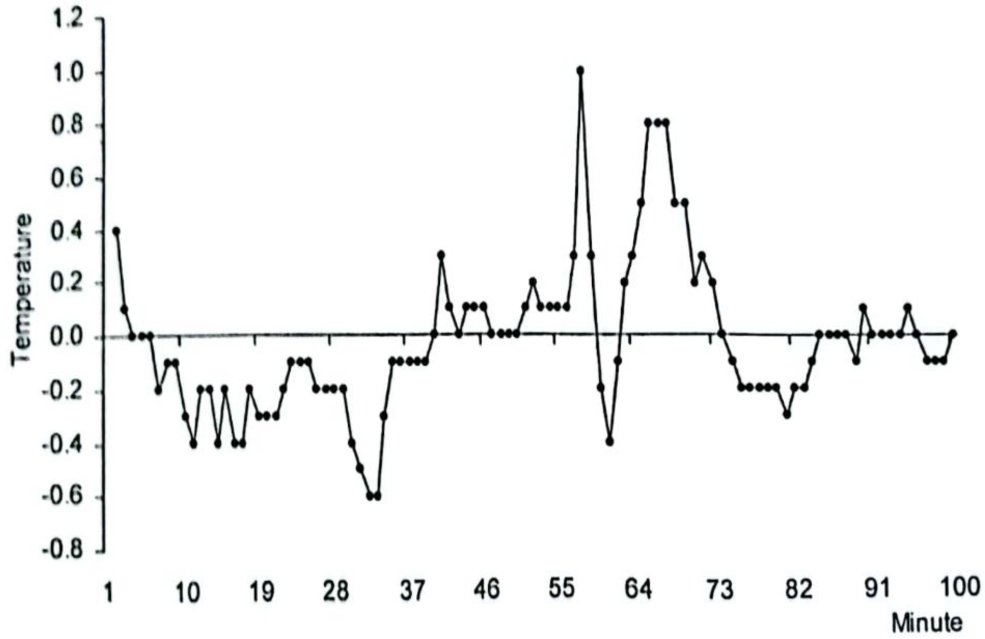


Figure 5.20: Plot of First 100 observations of series ∇Z_t for Series C

For Series C, the suggested model by Box and Jenkins (Box et al., 1994, p. 255) is ARIMA(1,1, 0) given by

$$(1 - \phi B)(1 - B) Z_t = a_t \tag{5.6}$$

and the fitted model is

$$(1-0.82B)(1-B)Z_t = a_t \tag{5.7}$$

with the variance $\hat{\sigma}_a^2 = 0.018$ using the unconditional least squares estimation procedure. Using the software STDS, we get the estimates $\hat{\phi} = 0.8072$ and $\hat{\sigma}_a^2 =$ 0.0185. The results are approximately same as those in (5.7) reported by Box et al. (1994).

Under the ARIMA(1,1,0) model, the analysis presented by Box et al. shows three innovational outliers at time points 58, 59, and 60 identified by the LRT method with the critical value 3.5 along with the conditional least squares estimation method. The suggested outlier model is

$$(1-B)Z_t = \frac{1}{1-\phi B}\left\{\omega_1\xi_t^{(58)} + \omega_2\xi_t^{(59)} + \omega_3\xi_t^{(60)} + a_t\right\}. \tag{5.8}$$

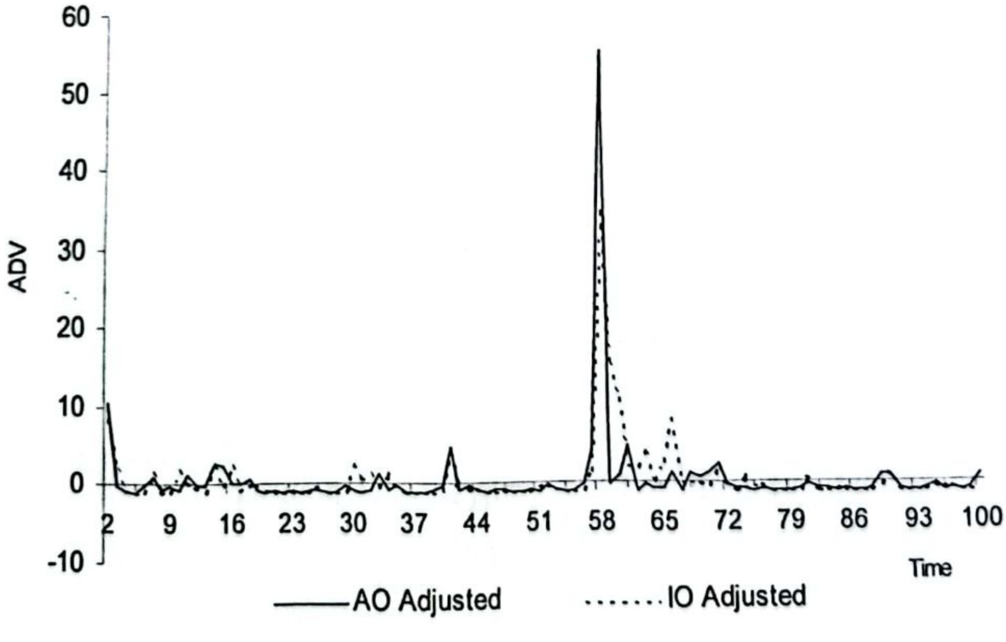


Figure 5.21: Plot of First 100 Values of ADV from ARIMA(1,1,0) Model fitted to Series C

We now apply the ADV plot to this series treating the first difference series as AR(1). The ADV plot for first 100 observations presented in Figure 5.21 shows significant peak at time point t = 58.

The values of the ADV statistics at time points t = 58, 59, 60 are

t	$ADV_{A,t}$	$ADV_{I,t}$
58	55.2538	34.7907
59	−0.2429	15.1374
60	1.0589	10.6395

The values and the ADV plot indicates that the observation at t = 58 is a possible outlier of an AO type. Though in Figure 5.20 the plot of the data set shows some pattern of observations between time points 66 to 68, the observations do not show any contamination in the analysis. Based on the ADV plot, we select the warning value to be 13. The results obtained on carrying out the proposed iterative procedure, and the results of the outlier detection analysis presented by Box et al. is summarized in Table 5.7 below.

Table 5.7

Outlier Detection for Series C

Iteration	LRT[1]		ADV	
	Position	Type	Position	Type
1	58	IO	58	AO
2	59	IO	60	IO
3	60	IO	–	–

Source: 1. Box et al., 1994, p. 473.

The proposed procedure diagnoses an AO at t = 58 and an IO at t = 60. Hence, the suggested model becomes

$$(1-B)Z_t = \omega_1\xi_t^{(58)} + \frac{1}{1-\phi B}\left\{\omega_2\xi_t^{(60)} + a_t\right\} \tag{5.9}$$

To investigate the reasons behind the difference between the two analyses, we scrutinize the first difference series plotted in Figure 5.20. The peak at t = 58 in Figure 5.20 indicates the presence of an AO at t =58 with positive value of ω. The observation at 59 does not look contaminated. Box et al. (1994, p. 474) claim that the presence of three innovational outliers at time points 58, 59 and 60 is apparent based on the residual plot of original observations, which is presented in Figure 5.22.

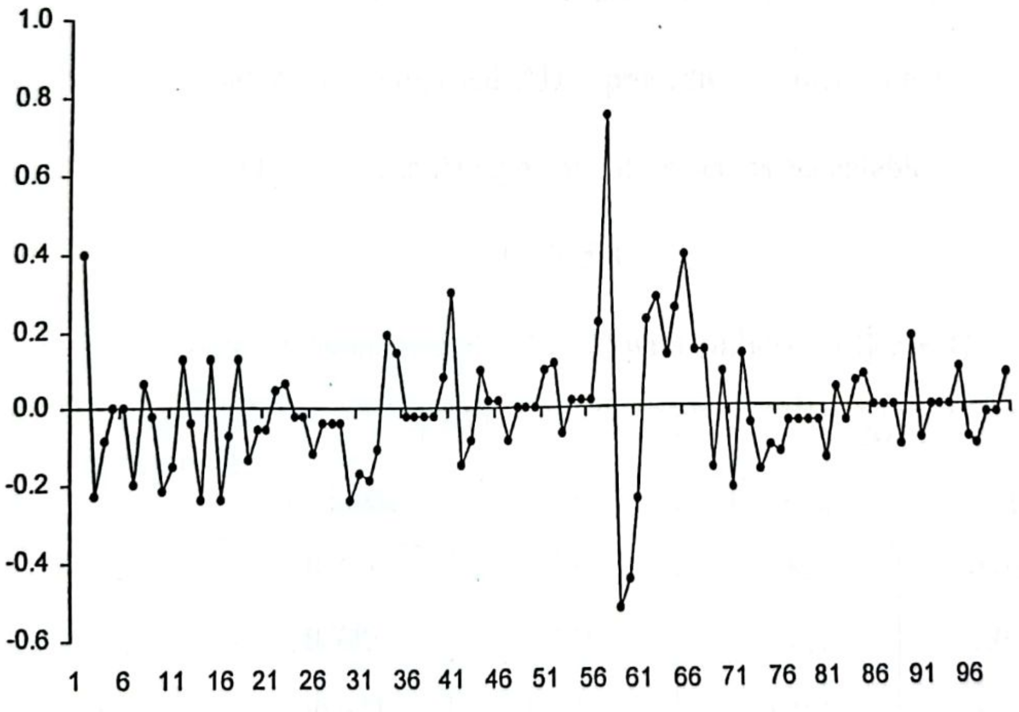


Figure 5.22: Plot of First 100 Residuals from ARIMA(1,1,0) Model fitted to Series C

In the presence of an IO at time point t = 58, the residuals at the successive time points t = 59, 60,... are not expected to show any unusual behaviour. However, in the presence of an AO type of outlier at t = 58, the impact on the residuals at successive time points $t \geq 59$ is expected to be of the type

$$a_t + \omega\pi(B)\xi_t^{(58)}, \qquad \text{for } t \geq 58.$$

(Box et al., 1994, p 471; also see Section 2.2). In particular, for this data set, we expect the residual at t = 59 to be of the type

$$e_{59} = a_{59} - \omega\pi_1$$

where $\hat{\omega} = 0.76$ and $\hat{\pi}_1 = 1.813$ are both positive based on the ARIMA(1,1,0) model fitted by Box et al. (1994, p. 473). This offers a more plausible explanation of the sudden drop in the residual plot at the time point t = 59. As a result, the analysis presented by the proposed ADV procedure, which concludes that the outlier at time point t = 58 is an AO type of outlier seems acceptable.

Table 5.8

Estimated Parameters of ARI(1,1) with outliers for Series C

Parameter	LRT[1]		ADV	
	Estimate	SE	Estimate	SE
ϕ	0.851	0.035	0.854	0.035
ω_1	0.745	0.116	0.705	0.092
ω_2	−0.551	0.120	−0.456	0.119
ω_3	−0.455	0.116	–	–
σ_a^2	0.0132	–	0.0139	–

Source: 1. Box, et al., 1994, p. 473.

The parameter estimation based on the models (5.8) and (5.9) is given in Table 5.8. From the last row of the table, though both the estimates $\hat{\sigma}_a^2$ based on the two models show reduction in comparison with the estimate obtained on ignoring the outliers, their values differ and model (5.8) gives a smaller estimate. The reduction percentages in error variances for Series C are shown in Table 5.9.

Table 5.9

The Reduction Percentage in Error Variance for Series C

Model	Estimated Variance	Reduction Percent	Method
ARI(1,1) with 3 outliers	0.0132 (*0.0179*)	26.26 %	LRT
ARI(1,1) with 2 outliers	0.0139 (*0.0185*)	24.86 %	ADV

Note: The italic values within the parentheses are the estimated variances $\hat{\sigma}_a^2$ when the outliers are ignored.

In Table 5.9, the difference between the reduction percentages is only 1.40%. Since the estimation methods are not the same, we use the SPSS software for the comparison of error variance. For SPSS, we define the independent variables (input variables) $x_{t,j}$, t=1,2, . . ., 226 analogous to (5.5). The estimated variances given by SPSS can be seen in Table 5.10.

Table 5.10

Comparison of The Reduction Percentage on $\hat{\sigma}_a^2$ for Series C

Model	Estimated Variance	Reduction Percent	Method
ARI(1,1) with 3 outliers	0.0135	25.82 %	LRT
ARI(1,1) with 2 outliers	0.0140	23.08 %	ADV

Note: $\hat{\sigma}_a^2 = 0.0182$ and $\hat{\phi} = 0.8202$ where the outliers are ignored.

In Table 5.10, the reduction percentages are calculated based on $\hat{\sigma}_a^2 = 0182$. It can be seen that the reduction percentages for LRT method and ADV method differ by about 2.74%. At the same time, by detecting less number of outliers, ADV procedure proposes a more parsimonious model.

3. Series A

The third data set considered is Series A (Uncontrolled concentration readings of a chemical process recorded at every two hour interval) from Box et al. (1994, p. 542). The first 70 observations of this series Z_t and first difference series ∇Z_t are shown in Figures 5.23 and 5.24 respectively.

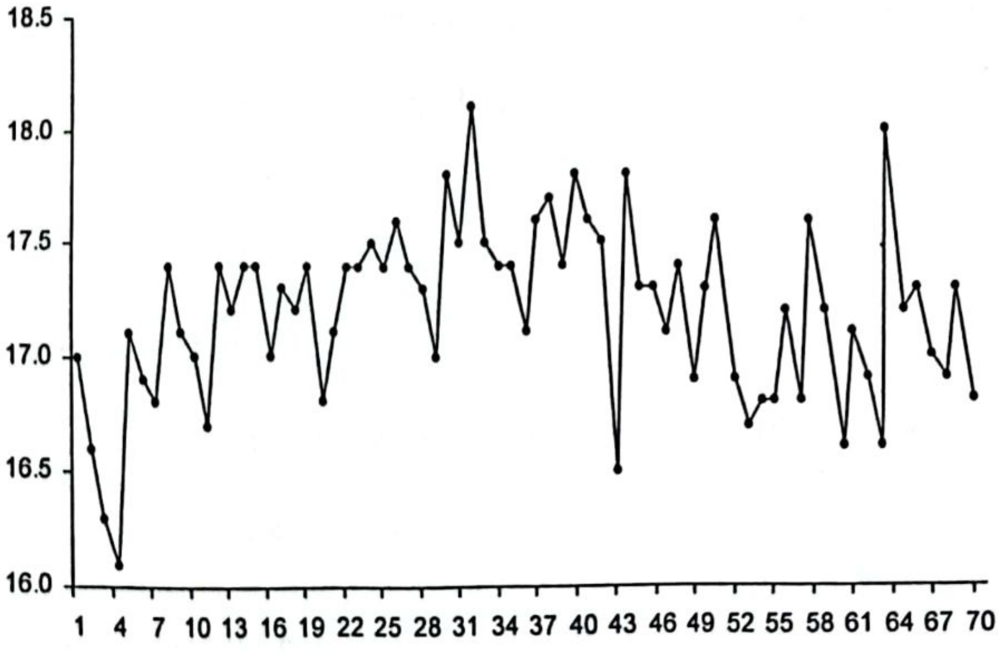


Figure 5.23: Plot of First 70 observations of Series A

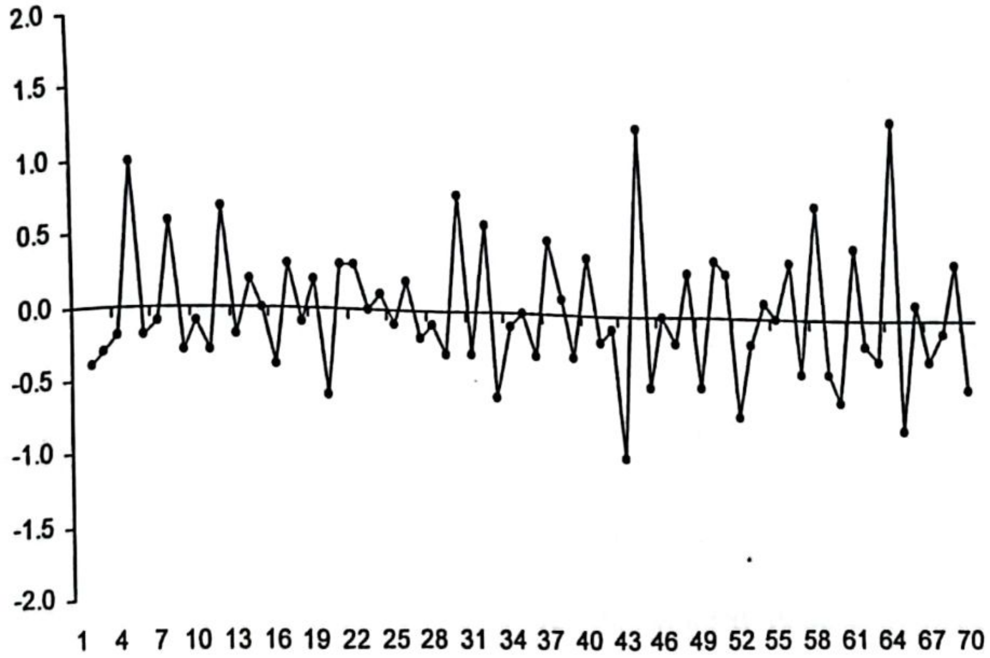


Figure 5.24: Plot of First 70 of first differences ∇Z_t for Series A

For this series, two models were suggested by Box and Jenkins, namely ARIMA (1,0,1) and ARIMA (0,1,1), which are

$$Z_t = \theta_0 + \frac{1-\theta B}{1-\phi B} a_t \ ,$$

$$(1-B)Z_t = (1-\theta B)a_t \ ,$$

respectively and the respective fitted models are (Box et al. 1994, p. 256),

$$Z_t - 0.92\, Z_{t-1} = 1.45 + a_t - 0.58\, a_{t-1} \quad \text{with } \hat{\sigma}_a^2 = 0.097,$$

$$(1-B)Z_t = a_t - 0.7\, a_{t-1} \qquad\qquad \text{with } \hat{\sigma}_a^2 = 0.101.$$

The LRT method with C = 3.5 was applied for the outlier detection by Chang et al., (1988) and it is reported that there are two outliers in this series, IO at t = 64 and AO at t = 43 for both the models. The two models with these outliers are respectively

$$Z_t = \theta_0 + \frac{1-\theta B}{1-\phi B}\left\{\omega_1 \xi_t^{(64)} + a_t\right\} + \omega_2 \xi_t^{(43)} , \tag{5.10}$$

$$(1-B)Z_t = (1-\theta B)\left\{\omega_1 \xi_t^{(64)} + a_t\right\} + \omega_2 \xi_t^{(43)}. \tag{5.11}$$

The proposed method ADV with reference value 11 diagnoses the same outliers and positions under ARIMA (1,0,1) in (5.10). But it indicates the different outlier type at position t = 64 under the assumed model ARIMA (0,1,1) in (5.11). The residuals plot and ADV plot are shown in Figures 5.25 and 5.26 respectively.

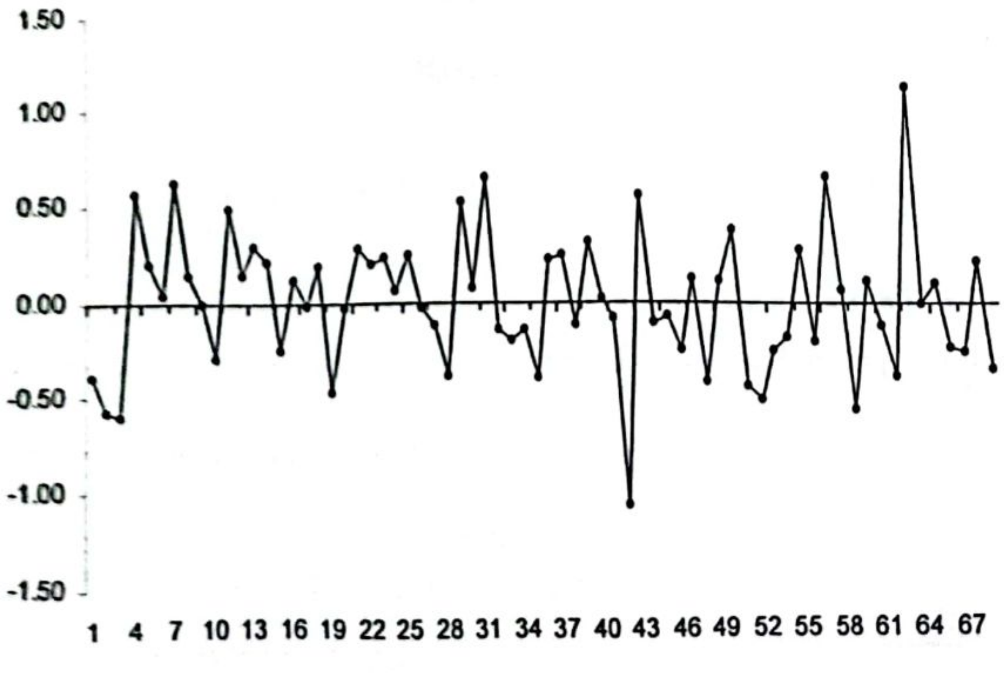


Figure 5.25: Plot of First 70 Residuals from ARIMA(0,1,1) Model fitted to Series A

Figures 5.25 and 5.26 clearly indicate that the two IO outliers occur at t = 43 and 64. Considering the possible impact of an IO on the time series, it is hard to accept the observation at t = 43 as an AO type of outlier and the existing analysis using LTR may have misidentified the outlier type.

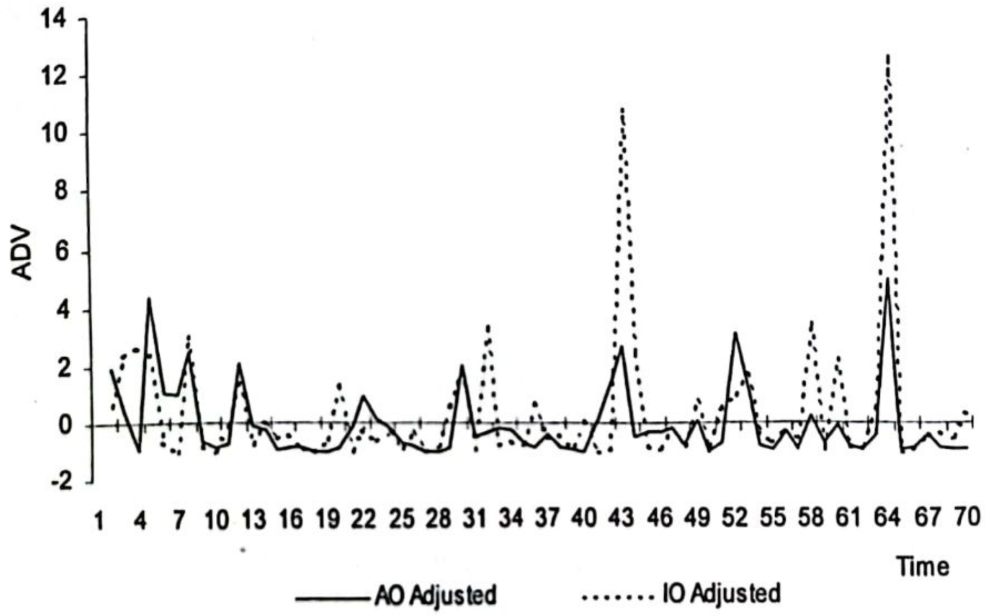


Figure 5.26: Plot of First 70 Values of ADV from ARIMA(0,1,1) Model fitted to Series A

Under the model ARIMA(0,1,1), the ADV method is applied for the diagnosis of outliers. The results are shown in Table 5.11.

Table 5.11
Outlier Detection for Series A

Iteration	LRT[1]		ADV	
	Position	Type	Position	Type
1	64	IO	64	IO
2	43	AO	43	IO

Source: 1. Chang et al, 1988.

Though the number of outliers diagnosed is same, there is difference in the identification of type of outlier. Looking at the first difference plot of observations presented in Figure 5.24, the observation at t = 44 is shown a sudden increase in its value. The diagnostic indicates presence of an IO at t =43 which is expected to affect the observation at t = 44 by an amount $\omega\psi_1$ (ref (2.1)). The estimates given by Chang et al. are $\hat{\omega} = -0.98$ and $\hat{\psi}_1 = 0.63$, which offers a possible explanation of sudden increase in the value of Y_{44}. Since the first difference process is MA(1), the observation at t = 45 does not get affected by the presence of IO at t = 43. Thus based on the ADV diagnostics, the suggested model is

$$(1-B)Z_t = (1-\theta B)\left\{\omega_1\xi_t^{(64)} + \omega_2\xi_t^{(43)} + a_t\right\}. \qquad (5.12)$$

The estimated parameters based on the models (5.11) and (5.12) are given in Table 5.12.

Table 5.12

Estimated Parameters of ARIMA(0,1,1) with outliers for Series A

Parameter	LRT[1]		ADV	
	Estimate	SE	Estimate	SE
θ	0.63	0.05	0.60	0.06
ω_1	1.13	–	1.18	0.30
ω_2	–0.98	–	–1.07	0.30
σ_a^2	0.0880	–	0.0910	–

Source: 1. Chang et al., 1988.

The reduction percentages in residual variances for Series A are shown in Tables 5.13 and 5.14.

Table 5.13

The Reduction Percentage in $\hat{\sigma}_a^2$ for Series A

Model	Estimated Variance	Reduction Percent	Method
IMA(1,1) with 2 outliers	0.0880 (*0.1007*)	12.61 %	LRT
IMA(1,1) with 2 outliers	0.0910 (*0.1009*)	9.81 %	ADV

Note: The italic values within the parentheses are the estimated error variance of the series assuming that there is no outlier.

Analogous to comparative analysis of earlier data sets, the reduction percentage in estimate of σ_a^2 using SPSS are presented in Table 5.14.

Table 5.14

Comparison of The Reduction Percentage in $\hat{\sigma}_a^2$ for Series A

Model	Estimated Variance	Reduction Percent	Method
IMA(1,1) with 2 outliers	0.0841	16.98 %	LRT
IMA(1,1) with 2 outliers	0.0844	16.68 %	ADV

Note: $\hat{\sigma}_a^2 = 0.1013$ and $\hat{\phi} = 0.6990$ where the outliers are ignored.

The reduction percentages are calculated based on $\hat{\sigma}_a^2 = 0.1013$. It can be seen that there is not much difference in the reduction percentages of LRT method and ADV method. However, the ADV method gives a more appropriate type of outlier at $t = 43$.

4. Series D

The example we consider here is Series D (Un-controlled viscosity readings every hour from a chemical process, Box et al., 1994, p. 545) for which an AR(1) model is suggested. The suggested model and fitted models are, respectively,

$$(1-\phi B)Z_t = \theta_0 + a_t$$

and

$$(1-0.862B)Z_t = 1.269 + a_t$$

with $\hat{\sigma}_a^2 = 0.089$ (Box et al, 1994, p. 473).

An IO outlier is identified at t=217 by likelihood ratio test (LRT) under this model. Using the adjustment diagnostic method ADV, an IO outlier is also identified at t=217. Thus the outlier analysis using the proposed method is in agreement with the existing outlier analysis of the data.

Table 5.15

Estimated Parameters of AR(1) with an outlier for Series D

Parameter	LRT[1]		ADV	
	Estimate	SE	Estimate	SE
θ	1.181	–	1.160	–
ω_1	0.872	0.027	0.871	0.028
ω_2	-1.296	0.292	-1.272	0.298
σ_a^2	0.0841	–	0.0881	–

Source: 1. Box et al., 1994, p. 473.

In conclusion, the fitted outlier model is

$$Z_t = \frac{1}{(1-\phi B)}\left\{\theta_0 + \omega\xi_t^{(127)} + a_t\right\}$$

giving the same estimates of parameters, except for the slight difference due to difference in estimation procedure. The estimates are presented in Table 5.15.

5. Series J

We now apply the proposed procedure to the gas furnace data for n = 296 which is taken at 9 seconds intervals on input gas feed rate (Box et al., 1994, p. 548,549). Tiao (1985) suggested ARMA(2,3) model for this series using the extended autocorrelation function (ESACF). The suggested model and fitted model (Tiao, 1985) respectively are

$$(1-\phi_1 B-\phi_2 B^2)\, Z_t = \theta_0 + (1-\theta_1 B-\theta_2 B^2-\theta_3 B^3)\, a_t$$

and

$$(1-1.29B+0.43B^2)\, Z_t = -0.0082 + (1+0.63B+0.50\,B^2+0.36B^3)\, a_t$$

with $\hat{\sigma}_a^2 = 0.0341$.

Using the suggested model ARMA (2,3) both LRT and ADV methods detect the outliers at t = 43, 55, and 113. The diagnosis is shown in Table 5.16

Hence, the suggested model is

$$Z_t = \theta_0 + \omega_1\xi_t^{(43)} + \omega_2\xi_t^{(55)} + \omega_3\xi_t^{(113)} + N_t$$

where

$$(1-\phi_1 B-\phi_2 B^2)\, N_t = \theta_0 + (1-\theta_1 B-\theta_2 B^2-\theta_3 B^3)\, a_t.$$

Table 5.16

Outlier Detection for Series J

Iteration	LRT[1]		ADV	
	Position	Type	Position	Type
1	43	AO	43	AO
2	55	AO	55	AO
3	113	AO	113	AO

Source: 1. Tiao, 1985.

Since both the methods detect same type of outliers at same positions, there is not much difference between the parameter estimation except for a slight variation due to different estimation procedures.

In conclusion, we illustrated how the proposed procedure ADV analyses the selected five data sets with outliers and made comparative statements of the analysis provided by the proposed procedure with the analysis available in the literature, which in all five cases was using the classical LRT procedure.

The number of detected outliers and their types as well as positions are same for two series, namely Series D and Series J. For the remaining data sets, the number of outliers identified by ADV is different and the analysis also shows difference in the type of outliers and the position of outliers as compared to the existing analysis. It is hard to confirm the actual type of outliers and their positions in these real life data sets, but ADV seems to be giving justifiable outlier types and positions in these cases. In addition, the procedure identified less number of outliers in some of the series, while keeping estimates of parameters about the same as the existing estimates available in the literature.

Conclusion

The time series data often encounters anomalous observations due to external disturbances or errors which disrupt the pattern of the time series. Such observations are called *outliers*. Apart from influencing the adjacent observations, affecting the estimates of model parameters, forecasting and so on, outliers can distort the model specification itself and the impact of outliers in time series modeling can be serious enough to affect the credibility of the model (Barnett and Lewis, 1994, Chapter 10). Thus the investigation into presence of outliers, identification of outliers, assessment of their effects on the analysis and the remedial measures to accommodate the outliers is a crucial aspect of time series analysis. The dependent structure of time series observations makes the detection of outliers difficult, since, unlike in case of general linear models, an outlier in time series need not necessarily be an extreme value (Barnett and Lewis, 1994, p. 395).

Following Fox (1972) and Abraham and Box (1979), the possible outliers in a ARMA (p,q) model are divided into two main types, Type I or *Additive outliers* (AOs) and Type II or *Innovational Outliers* (IOs). The AOs are those which do not affect adjacent observations and hence can be visualized in terms of superimposing an isolated measurement or execution error on the standard process.

Alternatively, the IOs are those which indicate inherent form of contamination influencing successive observations through the correlation structure. As a result, the realization of an outlier often gets concealed by the observations succeeding it, which are affected by the "carry-over effect".

The impact of presence of a single outlier on estimates of the time series parameters and the error variance is empirically investigated in Chapter 2. It is seen that all the estimates get affected in the presence of outlier, but the estimate of error variance shows marked difference. The error variance tends to get significantly overestimated in the presence of outlier and the estimate increases with increase in the outlier parameter ω. As a result, the estimate of error variance seems to be the right choice to build the outlier detection procedures.

In time series set up, due to the correlated nature of neighboring observations, the impact of multiple outliers depends on whether they occur isolated or in patches. It is empirically shown in Section 4.7 (Tables 4.22 to 4.25) that the error variance gets overestimated in the presence of multiple outliers of either AO or IO type. The estimate increases with the increase in the number of isolated outliers, but does not show a marked increase in comparison with the estimate in the presence of a single outlier when the multiple outliers occur in a patch, even if the patch length is increased. As a result, we conclude that the problem of detection of multiple outliers occurring in a patch cannot be satisfactorily handled using estimates of error variance.

Significant among the outlier detection procedures available in the literature are the procedures based on deletion diagnostics (Peña, 1987; Bruce and Martin, 1989; Abraham and Chuang, 1989; Ledolter 1990). Most of these procedures are adapted from deletion diagnostics procedures available for detection of outliers in regression data (Cook and Weisberg, 1982; Cook, 1986, 1987; Chatterjee and Hadi, 1988) and do not take into account of the AO or IO types of outliers separately, except for Abraham and Chuang (1989). As a result, the procedures also do not attempt to identify the outlier type after outlier detection. The procedures treat each observation as missing in turn and replace it by its estimate based on remaining observations. The estimate used to replace the missing value is the least square predictor proposed by Brubacher and Wilson (1976) which is the weighted sum of the adjacent observations.

A critical view of the available deletion diagnostic procedures is presented in Section 2.5, highlighting two issues. Firstly, since the estimate of missing value depends on the adjacent observations, the estimate can be contaminated by the succeeding observations, which contaminate themselves if the outlier is of IO type. Secondly, in the presence of an IO type of outlier, the succeeding observations also get contaminated due to correlated structure of the series. In such a case, replacing a single observation by its estimate may not be enough to remove the contamination of the series. Based on these observations, it can be claimed that most of the available outlier diagnostics procedures are suitable in the presence of AO type of outliers but not IO types. It is also shown in Section 2.3

that though presence of IO may not affect the estimates as much as that of AO, the effect is significantly high.

To fill up this gap, a new diagnostic procedure called *adjustment diagnostic* is proposed in this thesis. In this procedure, two separate models for two types of outliers are considered. Each observation is treated as a possible outlier in turn and the assumed model is appropriately adjusted to remove the effect of outlier from the subsequent affected points as well. In Section 2.6, we introduce the series adjustment in the presence of a single outlier and show that it handles the presence of both AO and IO type of outlier. It is shown that the series adjustment in case of an AO is equivalent to missing value estimation using least squares predictor. Thus in case of AO, the proposed procedure will be same as that in case of deletion diagnostic.

In Chapter 3, the effect of series adjustment on the estimate of error variance is investigated. The observed series is adjusted for each type of outlier at each adjustment position i in turn, $i \in \tau$. The adjustment may or may not be for the correct type of outlier at the correct position. It is shown in Section 3.3 that the estimate of error variance decreases when the correct type of series adjustment is made at the correct position. It is also shown in (3.11) and (3.12) that in the presence of a single outlier in time series, the biases in estimated error variance are $\frac{1}{n}\hat{\omega}^2_{A,T}\hat{\eta}^2$ and $\frac{1}{n}\hat{\omega}^2_{I,T}$ for AO and IO respectively if the presence of outlier is ignored, where $\hat{\eta}^2 = \sum_{j=0}^{n-T} \hat{\pi}_j^2$ which is greater than 1 and $\hat{\omega}_{A,T}$ and $\hat{\omega}_{I,T}$ are the estimates of outlier parameters for AO and IO types respectively. This provides a justification to the empirical evidence found in Section 2.3 that the error variance gets overestimated in the presence of an outlier (Tables 2.1 to 2.4). Also, the increase in the estimated error variance is directly proportional to the square of the estimate of outlier parameter. Thus the bias in estimate does not depend on the sign of the estimate of outlier parameter and is inversely proportional to the series length. As a result, a single outlier in a long series may not have significant effect on the estimates of various parameters.

In Chapter 4, a diagnostic measure based on likelihood displacement (Cook, 1986, 1987) is derived using series adjustment. The likelihood displacement is considered in a general set up where the model perturbation is taken to be an appropriate adjustment of the series. Based on the derived measures, a comprehensive procedure using adjustment diagnostics is proposed. The proposed procedure is called *Adjustment Diagnostic based on Variance* (ADV), which is a comprehensive procedure to detect an outlier, identify its type and position. In addition to outlier detection, is shown to perform better in identification of correct outlier type and correct outlier position in the series.

Extensive Monte Carlo simulations are carried out to compute the critical values of the proposed procedure, part of which is presented in Table 4.1 and Tables A1-A9 in Appendix A. We also suggest that the ADV diagnostic plot be used initially to guess the possible positions and types of outliers and also to get a reliable reference value. The proposed procedure is extended to iteratively identify multiple outliers in the series in Section 4.5.

An extensive simulation study is carried out to evaluate the performance of the procedure and to compare the performance with that of the existing deletion diagnostic procedure. Initially, the performance evaluation of the proposed procedure is carried out using least squares estimates of the time series parameters which is presented in Section 4.3. The evaluation shows that the proposed procedure gives a satisfactory performance irrespective of the type of outlier, unlike the deletion diagnostic procedures available in the literature. For instance, for the outlier parameter $\omega = 3$, the adjustment diagnostic identifies an IO at correct position in an AR(1) series 29.4% of times as against the deletion diagnostics for which the corresponding figure is 19.6% (Table 4.5). A brief study of the performance of the proposed procedure for various values of time series parameters is also presented here.

In Section 4.4 evaluation of the proposed procedure is carried out using robust estimates of time series parameters.

The problem of multiple outliers is addressed in Section 4.5 and its performance evaluation in the presence of two outliers of same or different types is presented in Section 4.6. The simulation based study shows satisfactory performance of the procedure in detecting the outliers when the outliers occur in isolation. When the outliers occur in a patch, the procedure detects the first outlier with high precision. If the two outliers are both IO type or IO-AO type, the performance of the procedure is satisfactory. However, if both the outliers are of AO type, or if the first outlier is an AO, the procedure detects and identifies outlier type with less accuracy. A critical evaluation of the proposed diagnostic in the presence of multiple outliers is presented in Section 4.7. It is observed that the procedure misidentifies the outlier type AO in the presence of a patch of multiple outliers and the heuristics presented to analyze the reason behind it. Based on it, possible extensions of the proposed procedures are suggested.

Chapter 5 presents data analysis using the proposed procedure where various generated data sets with different types of outliers at different positions and some real life data sets available in the literature are used. It is suggested that initially the plot of ADV be drawn, which gives a fairly good idea of type and position of outliers in the contaminated series. The use of ADV plot and residual plot is illustrated in Section 5.2 using two specific AR(1) and MA(1) series. The real life data sets considered in Section 5.4 are Truck Defects Series (Wei, 1990) and Box and Jenkins Series C, Series A, Series D and Series J (Box et al., 1994). This section presents comparison of the analysis available in the literature and analysis given by the proposed procedure for each of these series. For some data sets, the proposed procedure diagnoses outliers which are of types different from those of the existing analysis. Also in certain instances, the positions indicated are different from those suggested by existing analysis. In most of these situations, ADV seems to be giving a justifiable outlier type and outlier position. The model identification and parameter estimation using ADV is also satisfactory. As claimed earlier, the proposed adjustment diagnostic procedure is more comprehensive than the existing deletion diagnostics. Thus the proposed ADV procedure can be considered as a desirable alternative to the existing diagnostic procedures.

Based on simulation it was found that the performance of the proposed procedure is in general better in case of AR(1) series as against MA(1) series. Also, consistent with the findings in Section 2.3, the deletion of AO is handled with higher accuracy than that of IO.

It is well known that in time series analysis, the theoretical derivation and exact expression of estimators are intractable. As a result, it is impossible to theoretically compute the effect of outliers on estimates of parameters of interest for time series of finite length. Since we consider series with finite length, the emphasis here is on simulation based empirical study which is consistent with the study of outliers in time series available in the literature.

Throughout the thesis, the simulations are presented for AR(1) and MA(1) series only. An attempt was made to study the performance of the procedure in case of a contaminated ARMA(1,1) series. It showed that the detection of outliers was similar to that in case of AR(1) and MA(1) for moderate values of outlier parameter. A systematic simulation study was not carried out due to computational difficulties in handling high fluctuations in the ARMA(1,1) series.

The adjustment diagnostics is derived using the likelihood displacement criterion in the thesis. Alternatively, Akaike's information criteria (AIC) (Akaike, 1974), Bayesian information criteria (BIC) (Akaike, 1978, 1979) or similar criteria can be employed to arrive at an appropriate diagnostic. We, however, believe that there will not be much difference in the performance of procedures based on various criteria as in all cases the adjustment diagnostics will outperform the existing deletion diagnostics in certain cases and will present a more comprehensive procedure.

Several issues remain to be answered at this stage. Though the issue of multiple outliers is addressed in Chapter 4, it is well known that multiple outliers in any data set are difficult to be identified due to the masking effect. The problem is even more challenging in time series data due to the correlated structure of observations. The iterative procedure is shown to handle the multiple outliers satisfactorily provided they are well separated. In the presence of patch outliers also, the procedure detects the outliers and their positions with high accuracy. If the patch of outliers starts with an AO, then the procedure identifies it as an IO more often. At the same time, it must be pointed out that none of the existing procedures seem to satisfactorily handle the problem of masking effect in the presence of patch outliers. To overcome this drawback, it is proposed to extend the adjustment method to block adjustment which is planned to be taken up for future research work. The determination of block size, however, is a problem. Based on simulation study, it is shown in Table 4.25 that the estimate of error variance does not show significant increase with an increase in the patch length, even when the outlier parameter is of large magnitude. This indicates that handling patch outliers based on estimates of error variance will not result in a satisfactory solution. A new approach to this problem is needed and the Gibbs sampling based procedure proposed by Justel et al. (2001) suggests a new direction.

In the present study, it is assumed that the basic time series model is known, which may not always be the case. A suitable model identification procedure along with adjustment diagnostics is needed to ensure a more precise outlier analysis on the lines of Tsay (1986).

The present work focuses on two types of outliers, namely, the additive outliers (AOs) and the innovational outliers (IOs). The proposed adjustment diagnostic can be easily modified to handle other types such as level shift (LS) and temporary change (TC) etc, giving a much more comprehensive diagnostic procedure. The method can also be extended for seasonal time series models.

In conclusion, this thesis proposes a comprehensive outlier diagnostic procedure for outliers in time series which satisfactorily handles the diagnosis of outlier type and outlier position in stationary and invertible ARMA(p,q) series.

Appendix A

Critical Values for Adjustment Diagnostic

Table A1

Estimated Percentiles of Ω^*, Ω_A and Ω_I : AR(1) Series
($\sigma_a^2 = 1$; 5000 replications)

ϕ	%	n = 100 Ω^*	n = 100 Ω_A	n = 100 Ω_I	n = 125 Ω^*	n = 125 Ω_A	n = 125 Ω_I	n = 150 Ω^*	n = 150 Ω_A	n = 150 Ω_I	n = 175 Ω^*	n = 175 Ω_A	n = 175 Ω_I
0.2	10	12.28	11.68	11.85	12.83	12.14	12.23	13.05	12.27	12.29	13.09	12.36	12.33
	5	13.74	13.19	13.33	14.42	13.90	13.78	14.42	13.92	13.94	14.45	13.95	13.99
0.3	10	12.58	11.80	11.96	12.75	12.09	12.02	13.09	12.39	12.38	13.24	12.51	12.63
	5	14.22	13.40	13.64	14.35	13.60	13.67	14.65	13.80	13.86	14.78	13.94	14.05
0.4	10	12.78	11.81	11.81	12.84	11.99	11.99	13.27	12.30	12.37	13.38	12.50	12.54
	5	14.21	13.31	13.54	14.37	13.55	13.62	14.78	13.71	13.89	14.80	13.92	13.96
0.5	10	13.03	11.84	12.07	13.29	12.12	12.22	13.39	12.31	12.44	13.47	12.43	12.61
	5	14.65	13.52	13.70	14.93	13.80	13.88	15.00	13.92	14.04	15.03	13.89	14.00
0.6	10	12.86	11.61	11.90	13.13	11.84	12.18	13.16	12.11	12.35	13.57	12.40	12.47
	5	14.62	13.19	13.39	14.79	13.36	13.56	14.69	13.45	13.66	14.90	13.91	13.96
0.7	10	12.85	11.46	11.90	13.13	11.89	12.08	13.38	12.11	12.30	13.68	12.44	12.60
	5	14.49	12.90	13.29	14.88	13.51	13.88	14.89	13.56	13.90	15.12	13.98	14.22
0.8	10	12.85	11.67	11.82	13.12	11.81	12.09	13.37	12.08	12.26	13.60	12.30	12.68
	5	14.30	13.01	13.34	14.62	13.26	13.50	14.85	13.67	13.58	15.12	13.59	14.20
0.9	10	12.95	11.41	12.00	13.13	11.73	12.14	13.18	11.97	12.30	13.78	12.40	12.62
	5	14.48	12.91	13.47	14.50	13.15	13.50	14.53	13.19	13.54	15.33	13.96	14.14
-0.3	10	12.63	11.95	11.96	12.99	12.13	12.20	13.13	12.33	12.45	13.13	12.39	12.50
	5	14.11	13.48	13.44	14.59	13.74	13.90	14.63	13.77	13.88	14.73	13.91	13.95
-0.6	10	12.93	11.68	11.86	13.12	11.87	12.23	13.32	12.09	12.34	13.46	12.30	12.47
	5	14.53	13.14	13.71	14.78	13.44	13.78	14.80	13.60	13.97	15.11	13.88	14.00

ϕ	%	n = 200 Ω^*	n = 200 Ω_A	n = 200 Ω_I	n = 225 Ω^*	n = 225 Ω_A	n = 225 Ω_I	n = 250 Ω^*	n = 250 Ω_A	n = 250 Ω_I	n = 275 Ω^*	n = 275 Ω_A	n = 275 Ω_I	n = 300 Ω^*	n = 300 Ω_A	n = 300 Ω_I
0.2	10	13.12	12.54	12.66	13.51	12.99	13.03	13.64	13.13	13.21	13.66	13.15	13.31	13.80	13.28	13.34
	5	14.75	14.16	14.12	15.09	14.61	14.52	15.21	14.64	14.62	15.23	14.65	14.68	15.25	14.70	14.73
0.3	10	13.48	12.70	12.65	13.67	12.89	13.03	13.76	13.09	13.03	13.92	13.15	13.24	13.99	13.29	13.37
	5	14.84	14.05	14.18	14.99	14.22	14.34	15.23	14.49	14.59	15.50	14.64	14.69	15.60	14.82	14.73
0.4	10	13.43	12.66	12.75	13.77	12.76	12.90	13.82	12.89	12.99	13.97	13.12	13.06	14.15	13.21	13.32
	5	14.88	13.93	14.01	15.16	14.22	14.29	15.18	14.24	14.36	15.48	14.57	14.50	15.71	14.55	14.83
0.5	10	13.46	12.41	12.60	13.62	12.60	12.72	13.93	12.84	12.88	14.02	13.02	13.08	14.20	13.14	13.13
	5	15.05	13.75	14.11	15.08	13.90	14.23	15.32	14.39	14.32	15.54	14.51	14.51	15.48	14.54	14.43
0.6	10	13.72	12.59	12.77	13.90	12.76	12.86	13.96	12.93	12.98	14.16	13.01	13.19	14.36	13.17	13.39
	5	15.20	14.10	14.16	15.34	14.27	14.31	15.47	14.40	14.41	15.81	14.44	14.45	15.87	14.65	14.96
0.7	10	13.79	12.52	12.73	14.05	12.91	13.08	14.01	12.82	13.19	14.27	13.08	13.31	14.37	13.06	13.32
	5	15.14	13.91	14.23	15.53	14.36	14.45	15.53	14.25	14.59	15.66	14.47	14.72	15.72	14.49	14.64
0.8	10	13.70	12.38	12.70	14.13	12.87	12.96	14.09	12.91	12.99	14.25	12.99	13.29	14.46	13.22	13.32
	5	15.17	13.85	14.12	15.52	14.39	14.35	15.62	14.29	14.51	15.76	14.55	14.59	16.04	14.69	14.88
0.9	10	13.85	12.35	12.97	13.86	12.63	12.89	14.06	12.96	13.00	14.36	13.01	13.26	14.49	13.17	13.31
	5	15.43	13.93	14.47	15.54	13.95	14.42	15.62	14.27	14.49	15.78	14.60	14.63	15.95	14.64	14.86
-0.3	10	13.24	12.58	12.54	13.54	12.83	12.87	13.61	12.91	12.90	13.75	13.13	13.07	14.10	13.34	13.39
	5	14.60	14.01	14.02	15.10	14.35	14.34	15.11	14.29	14.40	15.05	14.42	14.47	15.48	14.81	14.77
-0.6	10	13.76	12.44	12.85	13.81	12.73	12.80	14.02	12.89	13.05	14.18	13.00	13.06	14.28	13.07	13.35
	5	15.24	14.03	14.33	15.42	14.22	14.27	15.52	14.24	14.36	15.59	14.27	14.49	15.72	14.53	14.94

Table A2

Estimated Percentiles of Ω^*, Ω_A and Ω_I : AR(1) Series
($\sigma_a^2 = 3$; 5000 replications)

ϕ	%	n = 100			n = 125			n = 150			n = 175		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	12.52	12.06	12.15	12.67	12.11	12.16	12.90	12.35	12.42	13.11	12.66	12.56
	5	14.04	13.47	13.54	14.16	13.61	13.61	14.50	13.89	13.97	14.64	14.09	14.14
0.3	10	12.55	11.68	11.95	12.88	12.11	12.16	13.13	12.37	12.45	13.15	12.39	12.44
	5	14.19	13.43	13.48	14.39	13.66	13.63	14.58	13.90	13.96	14.66	13.93	14.00
0.4	10	12.78	11.64	11.85	13.07	12.09	12.22	13.13	12.23	12.38	13.59	12.56	12.66
	5	14.53	13.29	13.58	14.58	13.56	13.59	14.61	13.70	13.76	15.02	14.15	14.26
0.5	10	12.83	11.64	11.80	13.02	11.99	12.11	13.09	12.18	12.20	13.48	12.44	12.51
	5	14.34	13.20	13.27	14.64	13.47	13.69	14.72	13.53	13.67	15.04	13.80	14.04
0.6	10	12.98	11.62	11.85	13.16	11.95	12.22	13.36	12.10	12.34	13.47	12.22	12.56
	5	14.52	13.25	13.54	14.71	13.45	13.66	15.05	13.69	13.87	15.10	13.66	14.01
0.7	10	12.96	11.79	11.85	13.10	11.79	12.18	13.23	12.04	12.35	13.49	12.21	12.57
	5	14.36	13.31	13.31	14.66	13.28	13.68	14.66	13.50	13.76	14.85	13.59	13.95
0.8	10	12.98	11.58	11.89	13.20	11.84	12.17	13.47	12.14	12.37	13.53	12.24	12.46
	5	14.55	13.20	13.48	14.69	13.44	13.52	15.03	13.67	13.76	15.15	13.65	13.86
0.9	10	13.00	11.45	12.02	13.31	11.91	12.23	13.34	12.00	12.35	13.67	12.24	12.67
	5	14.68	13.02	13.65	14.94	13.37	13.83	14.99	13.41	13.96	15.18	13.74	14.20
-0.3	10	12.53	11.88	11.95	12.90	12.14	12.16	13.17	12.27	12.48	13.14	12.39	12.55
	5	14.03	13.48	13.44	14.37	13.55	13.65	14.73	13.91	13.90	14.73	13.93	14.01
-0.6	10	12.89	11.77	11.90	13.07	11.98	12.09	13.27	12.26	12.43	13.58	12.40	12.59
	5	14.58	13.36	13.39	14.61	13.48	13.55	14.91	13.67	13.78	15.02	13.83	13.98

ϕ	%	n = 200			n = 225			n = 250			n = 275			n = 300		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	13.35	12.81	12.86	13.35	12.82	12.91	13.39	12.91	12.94	13.64	13.07	13.09	13.75	13.19	13.30
	5	14.85	14.28	14.29	14.93	14.45	14.50	14.98	14.48	14.51	15.00	14.51	14.68	15.33	14.55	14.72
0.3	10	13.35	12.69	12.78	13.53	12.75	12.84	13.88	13.09	12.99	13.84	13.12	13.20	14.03	13.28	13.30
	5	14.76	14.12	14.14	14.99	14.17	14.21	15.21	14.41	14.58	15.31	14.50	14.64	15.51	14.75	14.85
0.4	10	13.52	12.61	12.77	13.67	12.74	12.85	13.91	12.98	13.23	14.14	13.13	13.22	14.15	13.22	13.31
	5	15.21	14.20	14.40	15.25	14.23	14.44	15.33	14.33	14.64	15.50	14.42	14.65	15.62	14.61	14.65
0.5	10	13.69	12.67	12.73	13.98	12.82	13.00	14.05	12.89	13.05	14.11	13.15	13.10	14.13	13.23	13.28
	5	15.21	14.10	14.22	15.52	14.35	14.52	15.53	14.37	14.58	15.55	14.48	14.61	15.56	14.57	14.63
0.6	10	13.76	12.64	12.54	13.83	12.73	12.83	14.15	12.91	13.21	14.18	12.95	13.17	14.37	13.20	13.36
	5	15.13	14.05	14.11	15.19	14.08	14.30	15.49	14.33	14.61	15.64	14.43	14.61	15.64	14.64	14.66
0.7	10	13.77	12.40	12.65	13.92	12.68	12.89	14.04	12.74	13.07	14.11	12.98	13.10	14.19	13.06	13.09
	5	15.19	13.93	14.11	15.48	14.12	14.27	15.46	14.27	14.50	15.58	14.34	14.39	15.63	14.36	14.60
0.8	10	13.75	12.56	12.72	13.90	12.82	12.79	14.16	12.83	13.05	14.16	12.98	13.12	14.39	13.12	13.25
	5	15.25	13.86	14.05	15.46	14.20	14.24	15.69	14.36	14.59	15.78	14.38	14.64	15.85	14.54	14.75
0.9	10	13.75	12.47	12.69	14.04	12.59	12.93	14.07	12.66	12.94	14.16	12.96	12.97	14.43	12.96	13.35
	5	15.44	13.92	14.28	15.55	14.26	14.38	15.57	14.32	14.45	15.78	14.44	14.58	15.86	14.54	14.80
-0.3	10	13.29	12.72	12.73	13.64	12.76	12.86	13.61	12.95	12.88	13.96	13.19	13.28	14.11	13.21	13.48
	5	14.75	14.03	14.09	15.17	14.09	14.13	15.25	14.63	14.78	15.45	14.85	14.81	15.76	14.87	15.04
-0.6	10	13.64	12.52	12.68	13.88	12.64	12.87	14.04	12.81	12.97	14.09	13.07	13.22	14.31	13.21	13.37
	5	15.10	14.09	14.07	15.35	14.07	14.40	15.39	14.20	14.36	15.50	14.45	14.65	15.94	14.64	14.85

Table A3

Estimated Percentiles of Ω^*, Ω_A and Ω_I : AR(1) Series
($\sigma_a^2 = 5$; 5000 replications)

ϕ	%	n = 100			n = 125			n = 150			n = 175			n = 200			n = 225			n = 250			n = 275			n = 300		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	12.36	11.90	11.94	12.65	12.01	12.18	12.93	12.48	12.42	13.06	12.54	12.53	13.18	12.71	12.71	13.54	12.87	12.91	13.56	12.90	12.94	13.61	13.07	13.04	13.86	13.33	13.35
	5	14.00	13.30	13.45	14.10	13.51	13.71	14.45	13.77	14.01	14.57	14.00	14.10	14.72	14.13	14.23	14.99	14.21	14.53	15.32	14.36	14.56	15.34	14.50	14.69	15.53	14.58	14.72
0.3	10	12.57	11.89	12.00	12.89	12.07	12.27	12.92	12.20	12.31	13.20	12.35	12.59	13.37	12.58	12.75	13.50	12.76	12.85	13.87	13.00	13.11	13.80	13.16	13.15	14.06	13.29	13.32
	5	14.39	13.44	13.55	14.46	13.45	13.69	14.56	13.57	13.72	14.78	13.91	13.98	14.97	14.07	14.21	15.04	14.12	14.25	15.27	14.47	14.57	15.36	14.59	14.74	15.54	14.64	14.75
0.4	10	12.82	11.96	11.85	12.94	12.00	12.07	13.14	12.26	12.22	13.43	12.50	12.65	13.51	12.58	12.57	13.79	12.92	12.93	13.93	12.99	13.06	13.89	12.97	13.10	14.22	13.29	13.38
	5	14.23	13.26	13.39	14.70	13.41	13.63	14.75	13.57	13.73	14.86	14.09	14.15	14.92	14.12	14.20	15.31	14.41	14.27	15.35	14.43	14.56	15.42	14.53	14.60	15.56	14.61	14.65
0.5	10	12.93	11.89	11.99	13.10	11.90	12.10	13.34	12.26	12.29	13.46	12.44	12.53	13.66	12.59	12.80	13.67	12.69	12.87	13.98	12.97	13.14	14.12	13.02	13.12	14.16	13.17	13.21
	5	14.56	13.31	13.59	14.61	13.40	13.64	14.79	13.73	13.86	15.24	14.05	14.04	15.23	14.05	14.27	15.31	14.24	14.37	15.48	14.53	14.53	15.58	14.54	14.56	15.59	14.66	14.70
0.6	10	12.89	11.65	11.92	13.06	11.86	12.12	13.18	12.08	12.20	13.50	12.47	12.51	13.67	12.64	12.69	13.83	12.83	12.88	13.94	12.86	12.93	14.26	13.10	13.21	14.34	13.16	13.40
	5	14.41	13.18	13.60	14.55	13.42	13.56	14.70	13.43	13.71	14.90	13.87	13.95	15.12	14.17	14.20	15.20	14.14	14.34	15.50	14.28	14.35	15.59	14.49	14.70	15.96	14.56	14.90
0.7	10	13.06	11.66	12.07	13.14	11.82	12.10	13.30	12.13	12.18	13.48	12.16	12.48	13.87	12.52	12.69	13.88	12.71	12.70	13.92	12.80	12.92	14.12	13.16	13.07	14.48	13.18	13.39
	5	14.59	13.16	13.48	14.61	13.28	13.58	14.84	13.51	13.68	15.11	13.56	14.13	15.37	14.20	14.18	15.46	14.19	14.34	15.47	14.22	14.31	15.64	14.48	14.51	15.88	14.58	14.82
0.8	10	13.09	11.64	12.06	13.21	11.81	12.08	13.51	11.99	12.51	13.71	12.41	12.62	14.00	12.62	12.86	14.02	12.64	12.84	14.07	12.69	12.99	14.08	12.87	13.04	14.25	13.10	13.13
	5	14.70	13.19	13.78	14.75	13.31	13.74	15.19	13.58	14.06	15.32	13.93	14.21	15.34	14.20	14.33	15.44	14.23	14.36	15.49	14.24	14.42	15.54	14.38	14.45	15.66	14.51	14.67
0.9	10	12.96	11.55	11.97	13.22	11.88	12.10	13.26	12.03	12.19	13.66	12.42	12.54	13.86	12.45	12.77	13.93	12.75	12.94	14.29	12.97	13.19	14.38	12.99	13.27	14.47	13.05	13.34
	5	14.43	13.08	13.39	14.69	13.50	13.50	14.78	13.43	13.60	15.33	13.74	14.12	15.28	13.84	14.13	15.33	14.04	14.24	15.46	14.21	14.36	15.64	14.28	14.36	15.87	14.66	14.76
-0.3	10	12.62	11.92	12.00	12.90	12.07	12.11	13.03	12.28	12.38	13.30	12.56	12.52	13.50	12.71	12.82	13.51	12.84	12.88	13.73	12.92	13.08	13.77	13.07	13.14	14.04	13.35	13.30
	5	14.34	13.49	13.60	14.38	13.63	13.68	14.66	13.78	13.79	14.83	14.08	14.06	15.04	14.19	14.27	15.10	14.24	14.30	15.29	14.33	14.58	15.18	14.43	14.63	15.60	14.83	14.77
-0.6	10	12.72	11.49	11.82	13.24	12.06	12.28	13.39	12.17	12.25	13.42	12.39	12.56	13.67	12.43	12.70	13.96	12.82	12.84	14.07	12.82	13.06	14.15	12.87	13.20	14.22	13.10	13.26
	5	14.27	13.00	13.26	14.66	13.61	13.65	14.88	13.76	13.86	15.12	13.91	14.00	15.15	13.94	14.14	15.46	14.34	14.32	15.47	14.37	14.57	15.63	14.51	14.72	15.88	14.41	14.90

Table A4

Estimated Percentiles of Ω^{*}, Ω_A and Ω_I : AR(2) Series
($\sigma_a^2 = 1$; 5000 replications)

ϕ_1	ϕ_2	%	n = 100			n = 125			n = 150		
			Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I
0.3	-0.3	10	13.14	12.09	12.17	13.22	12.29	12.24	13.54	12.54	12.67
		5	14.77	13.61	13.69	14.78	13.75	13.76	15.14	14.05	14.16
0.3	-0.6	10	13.26	11.97	12.25	13.45	12.12	12.41	13.74	12.46	12.68
		5	14.96	13.35	13.98	14.96	13.45	14.07	15.36	14.00	14.07
0.3	-0.9	10	13.35	11.81	12.27	13.37	11.89	12.32	13.59	12.08	12.47
		5	14.94	13.39	13.73	14.96	13.31	13.92	14.97	13.67	13.87
0.6	-0.3	10	13.18	11.97	12.27	13.54	12.22	12.43	13.56	12.43	12.52
		5	15.13	13.28	13.91	15.13	13.72	14.06	15.29	13.84	14.11
0.6	-0.6	10	13.27	11.82	12.21	13.37	11.83	12.30	13.63	12.37	12.49
		5	15.05	13.35	13.87	15.10	13.40	13.91	15.20	14.00	13.81
0.6	-0.9	10	13.32	11.64	12.33	13.51	11.95	12.50	13.69	12.28	12.68
		5	15.14	13.25	13.88	15.02	13.48	13.90	15.28	13.68	14.10
0.9	-0.3	10	13.37	11.88	12.30	13.48	11.96	12.40	13.70	12.32	12.49
		5	15.07	13.55	13.94	15.08	13.53	14.01	15.22	13.84	14.07
0.9	-0.6	10	13.21	11.54	12.17	13.70	12.09	12.53	13.72	12.23	12.56
		5	14.85	13.21	13.76	15.41	13.62	14.21	15.43	13.68	14.22
-0.3	0.6	10	12.92	11.68	11.97	13.47	11.99	12.58	13.53	12.23	12.57
		5	14.51	13.17	13.36	15.11	13.59	13.99	15.00	13.60	14.13
-0.6	0.3	10	13.38	11.94	12.42	13.41	12.17	12.42	13.69	12.30	12.67
		5	15.05	13.43	13.78	15.10	13.58	13.87	15.37	14.03	14.18

ϕ_1	ϕ_2	%	n = 175			n = 200			n = 225		
			Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I
0.3	-0.3	10	13.53	12.68	12.65	13.85	12.86	12.85	13.92	13.03	12.98
		5	15.33	14.11	14.32	15.44	14.48	14.42	15.38	14.48	14.35
0.3	-0.6	10	13.94	12.59	12.78	14.00	12.71	12.87	14.09	12.78	12.95
		5	15.26	14.05	14.40	15.50	14.13	14.36	15.47	14.36	14.43
0.3	-0.9	10	13.80	12.38	12.75	13.98	12.60	12.83	13.94	12.65	13.13
		5	15.21	13.93	14.21	15.27	14.04	14.22	15.36	13.91	14.56
0.6	-0.3	10	13.89	12.47	12.86	13.93	12.66	12.95	14.06	12.88	13.06
		5	15.41	14.02	14.43	15.50	14.22	14.33	15.50	14.39	14.35
0.6	-0.6	10	13.87	12.41	12.83	13.94	12.71	12.90	14.09	12.71	12.99
		5	15.37	13.99	14.18	15.34	14.09	14.22	15.62	14.09	14.48
0.6	-0.9	10	13.96	12.33	12.79	14.10	12.51	12.83	14.14	12.80	13.00
		5	15.55	13.93	14.21	15.59	14.02	14.21	15.66	14.30	14.37
0.9	-0.3	10	13.95	12.55	12.84	13.96	12.54	12.87	14.09	12.59	13.07
		5	15.50	13.95	14.39	15.56	14.02	14.40	15.63	14.05	14.59
0.9	-0.6	10	13.83	12.24	12.75	14.09	12.66	12.92	14.40	12.91	13.21
		5	15.45	13.72	14.23	15.34	14.13	14.37	15.73	14.48	14.64
-0.3	0.6	10	13.76	12.48	12.81	13.87	12.65	12.89	14.07	12.81	13.03
		5	15.29	13.93	14.21	15.32	14.04	14.33	15.53	14.21	14.47
-0.6	0.3	10	13.71	12.58	12.67	14.06	12.79	12.98	13.97	12.84	12.87
		5	15.40	14.12	14.24	15.46	14.34	14.31	15.64	14.37	14.39

ϕ_1	ϕ_2	%	n = 250			n = 275			n = 300		
			Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I
0.3	-0.3	10	14.11	13.10	13.22	14.22	13.21	13.24	14.27	13.36	13.44
		5	15.76	14.63	14.66	15.80	14.66	14.72	15.81	14.72	14.92
0.3	-0.6	10	14.20	12.92	13.20	14.35	13.06	13.30	14.44	13.18	13.42
		5	15.83	14.36	14.70	15.84	14.59	14.81	15.97	14.92	14.82
0.3	-0.9	10	14.45	12.94	13.28	14.58	13.20	13.45	14.58	13.31	13.51
		5	16.02	14.63	14.67	15.96	14.62	14.96	15.97	14.64	14.86
0.6	-0.3	10	14.33	13.19	13.26	14.32	13.21	13.32	14.40	13.25	13.50
		5	15.79	14.61	14.71	15.86	14.62	14.75	15.96	14.63	14.79
0.6	-0.6	10	14.24	12.89	13.13	14.26	13.03	13.18	14.57	13.24	13.40
		5	15.52	14.30	14.52	15.81	14.43	14.54	16.13	14.69	14.93
0.6	-0.9	10	14.22	12.89	13.11	14.36	13.12	13.20	14.52	13.13	13.46
		5	15.92	14.30	14.67	15.88	14.35	14.80	16.03	14.53	14.81
0.9	-0.3	10	14.28	12.98	13.35	14.53	13.09	13.39	14.42	13.34	13.48
		5	15.99	14.36	14.70	15.89	14.59	14.81	15.98	14.72	14.85
0.9	-0.6	10	14.42	13.01	13.23	14.48	13.06	13.23	14.49	13.15	13.52
		5	15.90	14.34	14.73	15.81	14.52	14.76	15.92	14.46	14.94
-0.3	0.6	10	14.18	12.88	13.22	14.32	13.09	13.23	14.55	13.26	13.40
		5	15.67	14.21	14.65	15.76	14.65	14.66	16.11	14.73	14.92
-0.6	0.3	10	14.35	13.23	13.23	14.24	13.25	13.12	14.37	13.24	13.22
		5	15.76	14.41	14.59	15.86	14.44	14.58	15.87	14.75	14.59

Table A5

Estimated Percentiles of Ω^*, Ω_A and Ω_I : MA(1) Series
($\sigma_a^2 = 1$; 5000 replications)

θ	%	n = 100			n = 125			n = 150			n = 175		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	12.21	11.72	11.75	12.74	12.24	12.08	12.80	12.27	12.22	13.00	12.43	12.50
	5	13.76	13.22	13.23	14.37	13.82	13.87	14.37	13.83	13.68	14.51	14.01	13.95
0.3	10	12.64	11.95	11.82	12.73	11.95	11.98	12.98	12.38	12.15	13.16	12.45	12.38
	5	14.37	13.59	13.38	14.23	13.47	13.35	14.71	14.00	13.78	14.69	14.07	13.94
0.4	10	12.82	11.87	11.85	13.08	12.10	12.17	13.22	12.25	12.29	13.49	12.68	12.39
	5	14.36	13.34	13.45	14.87	13.75	13.72	14.89	13.82	13.74	14.93	14.13	13.74
0.5	10	13.22	12.15	11.89	13.24	12.03	12.15	13.42	12.42	12.12	13.79	12.64	12.51
	5	14.95	13.88	13.61	14.89	13.73	13.70	14.89	13.80	13.84	15.29	14.20	14.14
0.6	10	13.54	12.24	12.23	13.55	12.37	12.20	13.78	12.46	12.56	14.06	13.01	12.66
	5	15.37	14.14	13.94	15.46	14.07	13.83	15.60	14.13	14.29	15.83	14.69	14.32
0.7	10	13.79	12.40	12.30	13.97	12.76	12.40	13.91	12.84	12.57	14.38	13.17	12.80
	5	16.03	14.70	14.26	16.04	14.88	14.32	15.97	14.74	14.40	16.48	15.21	14.83
0.8	10	14.78	13.54	12.45	14.86	13.74	12.76	14.84	13.72	12.94	15.14	13.93	13.17
	5	17.42	16.26	14.62	17.51	16.51	14.82	17.64	16.99	15.24	18.34	17.05	15.32
0.9	10	16.53	15.84	12.31	16.83	16.12	12.81	16.95	16.30	12.90	17.35	16.44	13.00
	5	19.28	18.71	14.25	19.95	19.56	14.88	20.39	19.80	14.70	21.31	21.00	14.86
-0.3	10	12.68	11.89	11.83	12.86	12.10	12.18	12.98	12.32	12.21	13.19	12.52	12.53
	5	14.35	13.44	13.47	14.30	13.41	13.66	14.41	13.72	13.67	14.61	13.87	13.78
-0.6	10	13.45	12.31	11.94	13.51	12.29	12.24	13.47	12.50	12.25	13.84	12.58	12.53
	5	15.39	14.21	13.67	15.40	14.15	13.95	15.19	14.03	13.85	15.45	14.23	14.14

θ	%	n = 200			n = 225			n = 250			n = 275			n = 300		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	13.23	12.64	12.70	13.33	12.87	12.71	13.55	13.05	13.04	13.61	13.12	13.10	13.82	13.27	13.22
	5	14.74	14.20	14.11	14.85	14.10	14.30	14.97	14.41	14.44	14.95	14.39	14.41	15.34	14.97	14.70
0.3	10	13.57	12.71	12.80	13.60	12.90	12.83	13.54	12.93	12.78	14.04	13.28	13.21	14.13	13.39	13.35
	5	14.96	14.15	14.28	15.05	14.35	14.28	15.13	14.29	14.26	15.59	14.83	14.69	15.63	14.84	14.84
0.4	10	13.48	12.64	12.65	13.68	12.80	12.77	13.98	13.07	13.18	14.06	13.15	13.15	14.21	13.22	13.23
	5	15.08	14.15	14.07	15.11	14.33	14.04	15.58	14.51	14.61	15.51	14.51	14.63	15.57	14.65	14.61
0.5	10	13.98	12.85	12.71	14.11	13.02	12.96	14.11	13.05	12.95	14.16	13.18	13.14	14.54	13.35	13.34
	5	15.74	14.71	14.35	15.84	14.65	14.59	15.88	14.44	14.45	15.95	14.73	14.43	15.97	14.99	14.97
0.6	10	14.10	12.91	12.58	14.33	13.22	13.00	14.39	13.26	13.17	14.51	13.42	13.29	14.89	13.67	13.59
	5	15.89	14.79	14.35	15.98	14.87	14.65	15.87	14.91	14.66	16.32	15.23	14.97	16.66	15.53	15.46
0.7	10	14.42	13.16	13.03	14.59	13.37	13.20	14.69	13.45	13.36	14.98	13.66	13.60	15.15	13.72	13.77
	5	16.57	15.32	14.84	16.67	15.76	15.14	17.21	15.85	15.41	17.56	16.50	15.75	17.60	16.37	15.93
0.8	10	15.20	13.97	13.43	15.26	14.16	13.45	15.67	14.32	13.92	16.01	14.41	13.99	16.05	14.61	14.25
	5	18.41	17.31	15.89	18.47	17.57	16.00	18.59	17.80	16.40	18.92	17.86	16.56	20.06	18.67	17.72
0.9	10	17.53	16.49	13.54	17.32	16.55	13.36	17.62	16.77	13.67	17.77	16.68	13.69	18.31	17.73	14.15
	5	21.53	21.05	15.73	22.03	21.34	15.75	22.32	22.00	15.80	22.87	22.29	15.98	24.32	24.09	16.82
-0.3	10	13.43	12.71	12.69	13.74	12.92	12.88	13.70	13.00	12.94	13.96	13.20	13.27	14.16	13.38	13.38
	5	14.90	14.24	14.10	15.06	14.46	14.27	15.14	14.45	14.31	15.39	14.45	14.85	15.55	14.82	14.73
-0.6	10	14.14	12.88	12.88	14.32	13.11	13.07	14.26	13.26	13.10	14.65	13.44	13.43	14.50	13.47	13.40
	5	16.08	14.74	14.85	16.11	14.71	14.72	16.01	14.83	14.73	16.56	15.23	15.44	16.60	15.30	15.40

Table A6

Estimated Percentiles of Ω^*, Ω_A and Ω_I : MA(1) Series
($\sigma_a^2 = 3$; 5000 replications)

θ	%	n = 100			n = 125			n = 150			n = 175		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	12.34	11.92	11.72	12.52	12.00	12.00	12.86	12.38	12.33	13.02	12.48	12.50
	5	13.95	13.52	13.33	13.94	13.37	13.48	14.41	13.86	13.77	14.36	13.85	13.86
0.3	10	12.73	11.86	11.94	12.93	12.18	12.14	12.93	12.25	12.17	13.30	12.52	12.50
	5	14.31	13.57	13.67	14.39	13.73	13.53	14.55	13.74	13.55	15.04	14.11	14.00
0.4	10	12.77	11.88	11.75	13.15	12.16	12.24	13.34	12.55	12.45	13.22	12.39	12.26
	5	14.10	13.32	13.19	14.65	13.73	13.67	15.04	13.94	13.83	15.06	13.89	13.70
0.5	10	13.12	12.18	11.87	13.29	12.26	12.08	13.48	12.43	12.41	13.63	12.59	12.47
	5	14.76	13.62	13.51	14.93	13.94	13.65	15.13	13.94	14.02	15.03	14.09	13.89
0.6	10	13.15	12.03	11.73	13.41	12.24	12.24	13.69	12.46	12.37	13.85	12.69	12.65
	5	14.74	13.73	13.22	14.97	13.61	13.72	15.39	14.16	13.86	15.29	14.15	14.29
0.7	10	13.39	12.18	11.76	13.72	12.26	12.25	13.60	12.35	12.29	13.91	12.43	12.44
	5	15.32	13.88	13.30	15.21	13.97	13.86	15.20	13.82	13.94	15.49	14.36	13.94
0.8	10	14.10	12.78	11.96	14.12	12.79	12.29	14.19	12.97	12.43	14.16	12.92	12.32
	5	16.24	15.05	13.59	16.02	14.99	13.77	15.99	14.78	13.90	15.90	14.88	13.83
0.9	10	15.76	14.70	11.96	15.80	14.79	12.34	16.08	15.01	12.43	16.01	15.16	12.56
	5	18.73	18.27	13.71	18.63	18.39	13.98	19.08	18.50	14.31	19.23	18.79	14.21
-0.3	10	12.62	11.87	11.82	13.09	12.23	12.23	13.13	12.38	12.30	13.26	12.57	12.40
	5	14.18	13.34	13.42	14.75	13.91	13.96	14.75	13.86	13.79	14.85	13.99	14.09
-0.6	10	13.19	12.11	11.83	13.38	12.07	12.10	13.62	12.46	12.41	13.82	12.69	12.53
	5	14.98	13.76	13.25	14.98	13.69	13.59	14.95	13.96	13.86	15.33	14.22	14.07

θ	%	n = 200			n = 225			n = 250			n = 275			n = 300		
		Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.2	10	13.19	12.78	12.60	13.41	12.99	12.84	13.47	12.91	12.96	13.64	13.06	13.09	13.82	13.32	13.26
	5	14.71	14.14	14.06	14.78	14.36	14.32	14.94	14.34	14.41	15.21	14.64	14.65	15.23	14.63	14.63
0.3	10	13.48	12.73	12.85	13.56	12.96	12.76	13.84	13.08	13.08	13.91	13.16	13.24	13.99	13.39	13.29
	5	14.99	14.32	14.24	15.08	14.31	14.39	15.33	14.59	14.62	15.49	14.67	14.72	15.46	14.74	14.63
0.4	10	13.67	12.62	12.67	13.92	12.87	13.00	14.01	13.03	13.18	13.98	13.04	13.05	14.13	13.23	13.22
	5	15.10	14.21	14.20	15.35	14.37	14.40	15.47	14.37	14.44	15.53	14.50	14.64	15.79	14.80	14.61
0.5	10	13.86	12.68	12.70	13.99	12.91	12.91	14.21	13.22	13.12	14.29	13.25	13.21	14.37	13.34	13.23
	5	15.33	14.12	14.27	15.56	14.38	14.39	15.64	14.41	14.50	15.69	14.52	14.60	15.88	14.74	14.78
0.6	10	14.02	12.89	12.67	14.10	12.94	12.89	14.22	13.05	13.00	14.30	13.15	13.05	14.54	13.41	13.31
	5	15.37	14.28	14.11	15.55	14.35	14.40	15.72	14.49	14.53	15.67	14.67	14.42	15.95	14.86	14.72
0.7	10	14.06	12.72	12.76	14.14	12.81	12.97	14.41	13.07	13.16	14.37	13.11	13.12	14.48	13.38	13.27
	5	15.81	14.29	14.24	15.69	14.25	14.39	16.03	14.68	14.66	16.11	14.54	14.52	16.12	14.66	14.79
0.8	10	14.48	13.09	12.74	14.66	13.23	12.84	14.64	13.26	13.03	14.93	13.53	13.37	14.81	13.58	13.29
	5	16.33	15.16	14.36	16.34	15.38	14.37	16.39	15.22	14.47	16.53	15.44	14.79	16.42	15.33	14.68
0.9	10	16.19	15.23	12.94	16.21	15.37	13.08	16.34	15.60	13.20	16.37	15.64	13.30	16.44	15.67	13.46
	5	19.36	18.85	14.57	19.47	18.89	14.46	19.45	18.86	14.74	19.44	19.13	14.76	19.43	19.28	14.80
-0.3	10	13.49	12.67	12.80	13.51	12.81	12.78	13.72	12.99	12.96	13.77	13.10	13.00	13.98	13.25	13.23
	5	15.04	14.25	14.27	14.93	14.17	14.19	15.14	14.30	14.25	15.10	14.45	14.33	15.29	14.59	14.50
-0.6	10	14.02	12.77	12.77	13.86	12.68	12.68	14.10	12.99	12.91	14.15	12.94	13.00	14.33	13.18	13.11
	5	15.53	14.34	14.24	15.54	14.10	14.22	15.39	14.31	14.33	15.64	14.40	14.48	15.60	14.59	14.49

Table A7

Estimated Percentiles of Ω^{*}, Ω_{A} and Ω_{1} : MA(1) Series
($\sigma_a^2 = 5$; 5000 replications)

θ	%	n = 100			n = 125			n = 150			n = 175			n = 200			n = 225			n = 250			n = 275			n = 300		
		Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1	Ω^*	Ω_A	Ω_1
0.2	10	12.41	11.85	11.89	12.65	12.16	12.08	12.65	12.08	12.07	13.03	12.48	12.49	13.06	12.62	12.56	13.41	12.94	12.76	13.54	13.07	13.07	13.50	12.95	13.03	13.83	13.23	13.25
	5	13.89	13.27	13.41	14.26	13.69	13.77	14.09	13.35	13.57	14.53	14.05	13.92	14.68	14.03	14.15	14.91	14.38	14.35	14.93	14.53	14.39	15.08	14.46	14.43	15.32	14.79	14.74
0.3	10	12.64	11.88	11.85	12.87	12.09	12.01	13.28	12.43	12.39	13.27	12.53	12.51	13.55	12.72	12.72	13.55	12.81	12.86	13.70	12.79	12.93	13.92	13.17	13.20	13.89	13.20	13.06
	5	14.33	13.54	13.47	14.44	13.57	13.64	14.86	14.01	14.09	14.89	14.03	14.03	15.12	14.21	14.26	15.07	14.26	14.25	15.09	14.26	14.38	15.36	14.59	14.43	15.37	14.63	14.47
0.4	10	12.88	11.98	11.91	13.08	12.13	12.00	13.31	12.44	12.37	13.46	12.51	12.53	13.65	12.70	12.68	13.80	12.94	12.86	13.95	12.96	13.00	14.09	13.10	13.26	14.21	13.29	13.29
	5	14.44	13.48	13.47	14.64	13.60	13.54	14.86	13.90	13.83	15.00	14.06	14.09	15.33	14.30	14.35	15.31	14.33	14.30	15.41	14.43	14.42	15.65	14.47	14.76	15.44	14.61	14.61
0.5	10	12.93	11.84	11.80	13.34	12.28	12.24	13.37	12.42	12.26	13.46	12.41	12.22	13.74	12.71	12.67	13.75	12.63	12.78	14.09	13.11	12.94	14.30	13.18	13.14	14.23	13.11	13.31
	5	14.45	13.30	13.35	14.93	13.89	13.61	14.99	13.76	13.81	14.83	13.94	13.67	15.24	14.13	14.05	15.26	14.10	14.30	15.66	14.54	14.39	15.98	14.70	14.75	15.72	14.45	14.67
0.6	10	13.33	12.19	11.94	13.45	12.49	12.11	13.54	12.32	12.34	13.83	12.63	12.50	13.90	12.73	12.78	14.17	12.93	12.90	14.31	13.10	13.10	14.44	13.18	13.19	14.50	13.37	13.37
	5	15.13	13.88	13.61	14.93	13.99	13.67	15.20	14.01	13.78	15.26	14.21	14.02	15.56	14.26	14.10	15.78	14.43	14.30	15.79	14.81	14.49	15.98	14.57	14.69	15.97	14.64	14.83
0.7	10	13.51	12.43	11.99	13.68	12.29	12.18	13.63	12.60	12.34	13.97	12.64	12.60	14.11	12.75	12.69	14.13	12.86	12.72	14.39	13.13	13.06	14.43	13.09	13.28	14.63	13.37	13.41
	5	15.31	14.24	13.38	15.27	14.02	13.76	15.18	14.03	13.65	15.74	14.37	14.11	15.59	14.21	14.35	15.79	14.34	14.24	16.03	14.53	14.61	15.96	14.53	14.77	16.08	14.91	14.75
0.8	10	13.88	12.81	12.07	14.19	12.83	12.36	14.32	12.94	12.46	14.29	12.96	12.58	14.36	13.10	12.58	14.78	13.19	13.03	14.62	13.42	13.03	14.74	13.44	13.28	14.76	13.47	13.37
	5	15.93	14.86	13.51	16.14	15.06	13.89	16.20	14.97	13.99	16.22	15.03	14.13	16.27	15.08	13.94	16.24	15.18	14.54	16.22	15.20	14.44	16.31	15.31	14.71	16.49	15.37	14.83
0.9	10	15.70	14.95	12.32	16.02	15.25	12.47	16.29	15.32	12.51	16.51	15.33	12.53	16.13	15.36	12.75	16.25	15.37	13.05	16.49	15.45	13.26	16.55	15.50	13.31	16.61	15.66	13.33
	5	18.36	17.81	13.80	19.14	18.71	14.00	19.32	18.90	14.14	19.27	18.99	14.10	19.30	18.97	14.22	19.48	19.05	14.43	19.54	19.14	14.65	19.66	19.17	14.78	19.61	19.32	14.86
-0.3	10	12.64	12.01	11.73	12.86	12.09	12.15	13.15	12.34	12.33	13.36	12.65	12.49	13.48	12.73	12.74	13.55	12.70	12.80	13.79	13.07	13.07	13.87	13.04	13.12	14.05	13.35	13.29
	5	14.24	13.58	13.37	14.36	13.61	13.47	14.91	13.91	13.93	14.81	14.13	14.27	15.07	14.16	14.32	14.78	14.12	14.14	15.27	14.49	14.58	15.47	14.62	14.57	15.51	14.69	14.79
-0.6	10	13.04	11.93	11.60	13.30	12.26	11.98	13.83	12.59	12.37	13.72	12.60	12.35	14.07	12.88	12.79	13.96	12.70	12.80	14.08	13.04	12.86	14.21	13.11	13.02	14.49	13.34	13.23
	5	14.82	13.51	13.15	14.82	13.74	13.39	15.21	14.25	13.93	15.17	14.14	13.75	15.79	14.28	14.36	15.54	14.31	14.22	15.52	14.35	14.26	15.46	14.54	14.37	16.00	14.70	14.64

Appendix C

List of Computer Programs

CP-1: Series Generation of ARMA(p,q) Model

CP-2: Average of Estimated Parameters for AR(1)/MA(1) with One Outlier

CP-3: Bias Calculation of Incorrect Type Adjustment for AR(1)/MA(1)

CP-4: Estimated Percentiles of ADV for ARMA(p,q)

CP-5: Significance Level of ADV for Given C Value

CP-6: Comparison of Adjustment and Deletion Diagnostics

CP-7: Outlier Detection and Correct Type Identification (One Outlier)

CP-8: Outlier Detection and Correct Type Identification (Two Outliers)

CP-9: Average of Estimated Parameters for AR(1)/MA(1) with Isolated Outliers

CP-10: Average of Estimated Parameters for AR(1)/MA(1) with Patch Outliers

Note: 1. Please see the attached CD for the programs.
2. All programs are written in C under Sun OS with IMSL Fortran Libraries.

Appendix B

Contents of STDS Manual

Table A9

Estimated Percentiles of Ω^{*}, Ω_A and Ω_I : ARMA(1,1) Series
($\sigma_a^2 = 1$; 5000 replications)

ϕ	θ	%	n = 100			n = 125			n = 150			n = 175			n = 200			n = 225			n = 250			n = 275			n = 300		
			Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I	Ω^*	Ω_A	Ω_I
0.3	-0.3	10	13.05	11.95	12.03	13.59	12.40	12.44	13.67	12.56	12.63	13.79	12.66	12.68	13.88	12.73	12.79	14.09	12.96	13.03	14.29	13.12	13.12	14.28	13.14	13.11	14.59	13.41	13.50
		5	14.68	13.32	13.56	15.05	13.94	13.97	15.04	14.08	14.07	15.29	14.16	14.27	15.54	14.18	14.45	15.59	14.42	14.56	15.67	14.56	14.56	15.75	14.59	14.63	16.14	14.83	14.99
0.3	-0.6	10	13.67	12.18	12.21	13.65	12.27	12.36	13.76	12.38	12.60	13.92	12.57	12.70	14.09	12.82	12.88	14.18	12.83	12.99	14.32	12.88	13.09	14.37	12.89	13.25	14.71	13.28	13.48
		5	15.28	13.63	13.75	15.38	13.69	13.70	15.45	13.89	13.95	15.68	14.12	14.10	15.64	14.32	14.28	15.87	14.24	14.52	15.80	14.27	14.72	15.92	14.36	14.79	16.28	14.72	15.14
0.3	-0.8	10	14.10	12.11	12.56	13.96	12.14	12.62	13.97	12.17	12.64	14.23	12.32	13.09	14.36	12.36	13.24	14.33	12.43	13.29	14.51	12.64	13.38	14.68	12.80	13.48	14.63	12.78	13.60
		5	15.65	14.20	14.10	15.75	14.08	14.15	15.89	13.82	14.14	16.09	13.96	14.73	16.02	14.00	14.82	15.78	13.98	14.71	16.17	14.38	14.99	16.43	14.45	15.13	16.29	14.40	15.02
0.6	-0.3	10	13.35	12.04	12.23	13.53	12.06	12.42	13.74	12.33	12.60	13.72	12.39	12.66	13.92	12.49	12.90	14.29	12.93	13.01	14.35	13.08	13.06	14.36	13.00	13.12	14.47	13.22	13.39
		5	14.86	13.50	13.76	14.99	13.66	13.97	15.37	14.07	14.00	15.29	13.80	14.06	15.68	14.03	14.49	15.77	14.42	14.72	15.85	14.56	14.55	16.12	14.47	14.79	15.96	14.66	14.85
0.6	-0.6	10	13.74	11.97	12.45	13.66	12.04	12.31	13.77	12.23	12.54	13.95	12.41	12.70	14.05	12.49	12.89	14.16	12.68	12.96	14.19	12.55	13.20	14.45	12.92	13.28	14.58	13.18	13.42
		5	15.38	13.61	14.23	15.35	13.84	13.89	15.32	13.71	14.10	15.60	14.01	14.28	15.58	14.05	14.35	15.57	14.11	14.43	15.68	13.99	14.78	15.94	14.45	14.70	16.19	14.57	14.90
0.6	-0.8	10	14.00	12.12	12.52	14.01	12.10	12.76	14.03	11.86	12.83	14.10	11.86	13.07	14.19	12.08	13.34	14.53	12.41	13.50	14.54	12.33	13.56	14.66	12.59	13.74	14.79	12.78	13.79
		5	15.70	13.59	14.32	15.82	13.64	14.38	15.84	13.69	14.58	15.66	13.78	14.65	15.94	13.89	14.94	16.08	13.91	14.97	16.29	13.93	15.21	16.08	14.05	15.23	16.41	14.19	15.56
0.9	-0.3	10	13.40	11.98	12.23	13.49	12.11	12.28	13.58	12.34	12.42	13.83	12.36	12.72	14.01	12.57	12.94	14.06	12.83	12.94	14.37	12.92	13.12	14.33	13.07	13.14	14.42	13.25	13.26
		5	15.07	13.56	13.82	15.00	13.67	13.79	15.14	13.79	13.90	15.28	13.91	14.19	15.54	14.19	14.21	15.70	14.20	14.41	15.83	14.51	14.54	16.00	14.62	14.61	16.00	14.59	14.79
0.9	-0.6	10	13.33	11.64	12.13	13.39	12.07	12.22	13.62	12.09	12.29	13.82	12.35	12.64	13.97	12.38	12.85	14.05	12.56	12.92	14.18	12.70	13.22	14.35	12.77	13.25	14.47	13.00	13.45
		5	14.92	13.28	13.77	15.06	13.60	13.62	15.13	13.81	13.78	15.36	13.87	14.10	15.50	14.03	14.33	15.69	14.08	14.41	15.47	14.05	14.65	15.75	14.32	14.78	15.96	14.48	14.84
-0.3	0.6	10	13.59	12.12	12.16	13.58	12.22	12.26	13.77	12.35	12.48	14.08	12.66	12.76	13.96	12.60	12.75	14.34	12.73	13.10	14.26	12.88	13.04	14.37	13.22	13.27	14.61	13.20	13.42
		5	15.25	13.81	13.71	15.14	13.95	13.77	15.26	13.85	13.95	15.52	14.37	14.16	15.39	14.28	13.99	15.72	14.31	14.68	15.87	14.44	14.39	15.74	14.53	14.57	15.91	14.75	14.78
-0.6	0.3	10	13.41	11.94	12.10	13.63	12.28	12.43	13.72	12.38	12.54	13.79	12.50	12.53	13.86	12.44	12.75	14.21	12.94	12.93	14.17	12.98	12.96	14.37	13.16	13.32	14.61	13.28	13.38
		5	15.17	13.50	13.84	15.20	13.82	14.09	15.20	13.95	13.92	15.31	13.98	14.07	15.32	13.79	14.21	15.72	14.37	14.42	15.63	14.34	14.47	15.89	14.38	14.85	16.03	14.85	14.88

Table A8

Estimated Percentiles of Ω^{*}, Ω_{A} and Ω_{I} : MA(2) Series
($\sigma_a^2 = 1$; 5000 replications)

θ_1	θ_2	%	n = 100			n = 125			n = 150			n = 175			n = 200			n = 225			n = 250			n = 275			n = 300		
			Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I	Ω^{*}	Ω_A	Ω_I
0.3	-0.3	10	13.20	12.19	12.25	13.34	12.50	12.45	13.28	12.52	12.35	13.60	12.67	12.74	13.96	12.86	13.05	13.91	13.01	12.96	13.95	13.03	13.20	14.15	13.33	13.32	14.31	13.38	13.31
		5	14.95	13.84	13.70	14.86	13.82	13.80	14.90	13.98	13.88	15.01	14.12	14.03	15.59	14.58	14.46	15.40	14.55	14.54	15.40	14.47	14.60	15.71	14.76	14.77	15.67	14.91	14.72
0.3	-0.6	10	13.85	12.64	12.28	13.91	12.78	12.40	14.06	12.68	12.60	13.91	12.84	12.61	14.27	13.07	12.92	14.26	13.18	13.01	14.48	13.44	13.27	14.66	13.40	13.32	14.82	13.58	13.51
		5	15.62	14.36	13.95	15.74	14.54	14.00	15.76	14.46	14.13	15.62	14.33	14.00	15.69	14.65	14.49	15.70	14.54	14.46	15.90	14.90	14.63	16.19	15.08	14.73	16.36	15.15	15.12
0.3	-0.9	10	18.12	17.59	12.76	18.45	18.11	12.93	18.23	17.94	13.10	18.81	18.55	13.07	18.90	18.65	13.39	18.93	18.64	13.21	19.17	18.84	13.61	19.14	18.82	13.66	19.63	19.16	13.70
		5	20.94	20.71	14.48	21.55	21.23	14.63	21.59	21.26	14.85	22.12	22.02	14.98	22.23	22.08	14.99	22.47	22.34	15.08	23.08	22.90	15.30	22.92	22.66	15.23	23.46	23.35	15.31
0.6	-0.3	10	13.40	12.24	12.11	13.43	12.36	12.29	13.55	12.52	12.42	13.71	12.66	12.55	13.97	12.80	12.88	14.21	13.05	13.05	14.31	13.15	13.26	14.36	13.19	13.17	14.46	13.28	13.45
		5	15.23	13.96	13.79	15.10	13.88	13.62	15.07	13.96	13.86	15.22	14.15	14.10	15.74	14.34	14.45	15.72	14.56	14.57	15.74	14.62	14.74	15.83	14.64	14.68	16.11	14.89	14.83
0.6	-0.6	10	14.12	12.78	12.54	14.01	12.63	12.43	14.05	12.76	12.70	14.25	13.00	12.87	14.28	12.90	12.96	14.59	13.30	13.14	14.33	13.17	13.18	14.70	13.47	13.37	14.83	13.67	13.43
		5	15.46	14.28	14.01	15.52	14.30	14.05	15.58	14.43	14.23	16.01	14.52	14.45	16.03	14.55	14.50	16.15	14.88	14.75	16.09	14.66	14.71	16.12	14.81	14.94	16.40	15.34	14.89
0.6	-0.9	10	18.51	18.23	12.64	18.60	18.27	12.78	18.91	18.64	13.14	19.76	19.55	13.20	19.69	19.45	13.44	20.03	19.70	13.57	20.46	20.38	13.67	20.62	20.45	13.72	20.60	20.43	13.66
		5	21.78	21.38	14.54	21.85	21.40	14.56	22.43	22.23	14.80	23.41	23.32	14.90	23.73	23.63	15.06	23.78	23.55	15.18	24.56	24.55	15.21	25.22	25.07	15.19	25.73	25.72	15.33
0.9	-0.3	10	13.70	12.39	12.25	14.13	12.70	12.50	14.03	12.94	12.60	14.17	12.72	12.89	14.35	12.99	12.92	14.30	13.09	13.12	14.41	13.22	13.18	14.61	13.41	13.29	15.00	13.79	13.49
		5	15.46	14.02	13.97	15.91	14.72	14.24	15.68	14.49	14.08	15.86	14.48	14.49	16.05	14.66	14.55	16.05	14.66	14.41	16.04	14.83	14.54	16.21	14.96	14.96	16.79	15.61	14.93
0.9	-0.6	10	14.20	12.75	12.58	14.22	12.94	12.49	14.21	13.10	12.76	14.42	13.25	13.00	14.58	13.34	13.00	14.72	13.54	13.23	14.80	13.73	13.29	14.86	13.57	13.44	15.04	13.81	13.52
		5	16.01	14.77	14.50	16.10	14.66	14.53	16.16	14.78	14.61	16.29	15.16	14.50	16.25	15.05	14.46	16.39	15.31	14.77	16.57	15.36	14.84	16.27	15.22	14.93	16.62	15.56	15.02
-0.3	0.6	10	15.29	14.51	12.43	15.55	14.86	12.40	16.12	15.37	12.75	16.13	15.30	12.77	16.10	15.31	12.95	16.48	15.46	13.20	16.33	15.51	13.10	16.43	15.61	13.19	16.81	15.93	13.55
		5	17.54	16.85	13.86	17.75	16.88	14.06	18.37	17.98	14.37	18.33	17.78	14.38	18.27	17.61	14.46	18.98	18.44	14.86	18.88	18.47	14.66	18.95	18.43	14.68	19.07	18.49	15.04
-0.6	0.3	10	15.58	14.85	12.17	15.78	14.99	12.62	16.10	15.35	12.58	15.94	15.17	12.82	15.77	15.00	12.90	16.20	15.31	13.04	16.38	15.58	13.09	16.38	15.48	13.22	16.43	15.50	13.37
		5	17.90	17.19	13.84	18.08	17.44	14.01	18.19	17.48	14.21	18.24	17.53	14.38	18.26	17.51	14.45	18.28	17.71	14.47	18.34	17.74	14.58	18.74	18.35	14.64	18.88	18.41	14.92

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