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All Authors	Zaw Myint
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Characterization of Bipartite Graphs with given γ_2(G)

Zaw Myint

Abstract

In this paper, we first give a characterization of the bipartite graphs G satisfying $\gamma_2(G) = 3\beta(G)/2$. Moreover, we compare the value of 2-domination number and independence number in trees and give bounds on these two parameters in terms of the order and the number of pendant vertices of the tree. More precisely we show that for a nontrivial tree T, $\beta(T) \le \gamma_2(T) \le 3\beta(T)/2$, $(n(T) + \ell(T) + 2)/3 \le \gamma_2(T) \le (n(T) + \ell(T))/2$ and $\beta(T) \le (n(T) + \ell(T) - 1)/2$. Finally, we characterize the trees achieving equality in each bound.

Introduction

In this paper, all graphs will be finite, undirected and simple. Unless defined or mentioned otherwise, we refer the reader to F.Harary for standard terminology and notation in graph theory.

A **dominating set** of a graph G = (V(G), E(G)) is a subset D of V(G) such that every vertex of V(G)–D is adjacent to at least one vertex in D. That is, for every vertex v in V(G)–D, $|N(v) \cap D| \ge 1$. The minimum cardinality of the dominating sets of G is called the **domination number** of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as **minimum dominating set** and is denoted by $\gamma(G)$-set. A **minimal dominating set** in a graph G is a dominating set that contains no dominating set as a proper subset. $\lceil x \rceil$ is the smallest integer greater than or equal to a real number x. $\lfloor x \rfloor$ is the greatest integer smaller than or equal to a real number x. A vertex v in V(G) – D is **k-dominated** if it is dominated by at least k vertices in D, that is $|N(v) \cap D| \ge k$. If every vertex in V(G) – D is k-dominated, then D is called a **k-dominating set**. The minimum cardinality of a k-dominating set is called the **k-domination number** and is denoted by $\gamma_k(G)$. A dominating set of cardinality $\gamma_k(G)$ is called **minimum k-dominating set** and is denoted by $\gamma_k(G)$-set. Two vertices that are not adjacent in a graph G are said to be **independent**. A set S is an **independent set** if every two vertices of S are independent. The **vertex independence number** or simply the **independence number** $\beta(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G. An independent

Assistant Lecturer, Department of Mathematics, Myingyan Degree College

set of cardinality $\beta(G)$ is denoted by $\beta(G)$-set. An independent set S with the property that any vertex set properly containing S is not independent is called a **maximal independent set**. For $k \geq 2$, a subset $S \subseteq V(G)$ is called a **k-dependent set** if and only if the maximum degree of a vertex in the subgraph induced by S is less than k, that is , $\Delta(G[S]) < k$. The maximum cardinality of a k-dependent set in G is the **k-dependence number** $\beta_k(G)$. For k = 1, $\beta_1(G) = \beta(G)$ is the independence number of G.

Main Results

Proposition 1. If a connected graph G is the corona of a corona graph or the corona of the cycle C_4, then $\gamma_2(G) = \frac{3}{2}\beta(G) = \frac{3}{4}n(G)$.

Proof. Let G be a corona of a corona graph of H. In this case, clearly $\gamma_2(G) = 3n(H)$ and $\beta(G) = 2n(H)$. Therefore, $\gamma_2(G) = \frac{3}{2}\beta(G)$ in Figure 1.

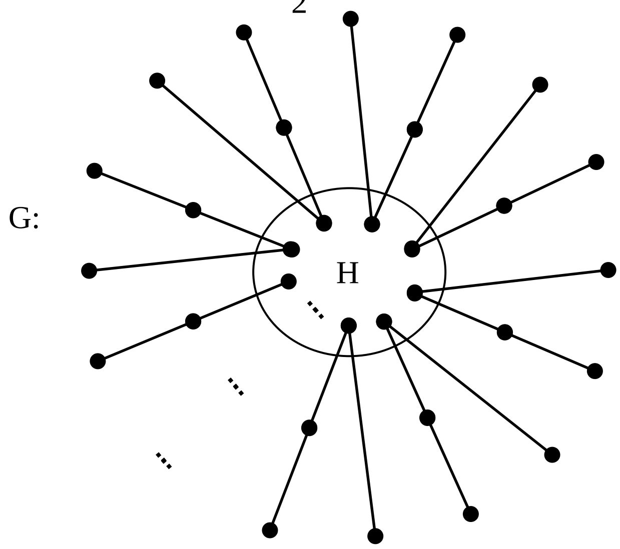

Figure 1. A corona of a corona graph of H.

Again, let G be a corona of cycle C_4. Then $\gamma_2(G) = 6$ and $\beta(G) = 4$. Thus, $\gamma_2(G) = \frac{3}{2}\beta(G)$ in the following Figure 2.

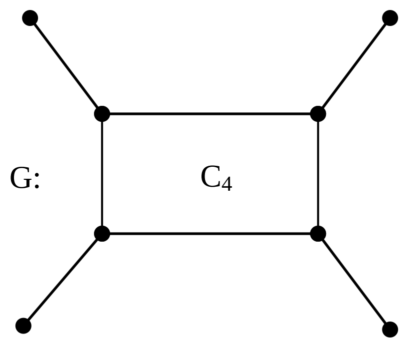

Figure 2. A corona of C_4.

In each case, clearly $\gamma_2(G) = \frac{3}{4}n(G)$. □

Theorem(Ore's inequality) 2. If G is a graph without isolated vertices, then $\gamma(G) \le n(G)/2$.

Proof. See [Chartrand, G. and Lesniak, L., ***Graphs and Digraphs***, 4th ed. CRC Press Company, London, 2004]. □

Theorem 3. Let G be a connected graph. Then $\gamma(G) = n(G)/2$ if and only if G is the K_1-corona graph of any connected graph J or G is isomorphic to the cycle C_4.

Proof. See [Chartrand, G. and Lesniak, L., ***Graphs and Digraphs***, 4th ed. CRC Press Company, London, 2004]. □

Proposition 4. If G is the corona graph of a connected graph H of order at least two, then $\gamma_2(G) \le \frac{3}{4} n(G)$ with equality if and only if H is either the corona graph of a connected graph or H is isomorphic to the cycle C_4.

Proof. Let L be the set of pendant vertices of G and let D be a minimum dominating set of H = V(G) – L. Then, since G is a corona graph, $D \cup L$ is a minimum 2-dominating set of G and hence we obtain with Ore's inequality

$$\gamma_2(G) = \gamma(H) + |L| \le \frac{|V(G) - L|}{2} + |L|.$$

Since G is the corona graph of H, n(G) = 2 n(H) and |L| = n(H) = $\frac{n(G)}{2}$.

Therefore, $\gamma_2(G) \le \frac{n(G)}{4} + \frac{n(G)}{2} = \frac{3}{4} n(G)$. By Theorem 3, equality holds if and only if H is the corona of a connected graph or if $H \cong C_4$. □

Theorem 5. If G is a connected bipartite graph of order at least 3, then $\gamma_2(G) \le \frac{3}{2}\beta(G)$ and equality holds if and only if G is the corona of the corona of a connected bipartite graph or G is the corona of the cycle C_4.

Proof. Let L be the set of pendant vertices in G, and let I be a maximum independent set of G. We can assume, without loss of generality, that $L \subseteq I$ and thus it follows that $\ell(G) \le \beta(G)$. Since G is bipartite, evidently $2\beta(G) \ge n(G)$.

Let A and B be the partition sets of G. Define $A_1 = A - L$ and $B_1 = B - L$ and assume, without loss of generality, that $|A_1| \le |B_1|$. Then $|A_1| \le \frac{n(G)-\ell(G)}{2}$. Since every vertex in B_1 has at least two neighbors in $A_1 \cup L$, we see that the latter is a 2-dominating set of G and hence

$$\gamma_2(G) \le |A_1 \cup L| \le \frac{n(G)-\ell(G)}{2} + \ell(G) = \frac{n(G)+\ell(G)}{2}.$$

Combining this inequality with $\ell(G) \le \beta(G)$ and $n(G) \le 2\beta(G)$, we obtain the desired bound

$$\gamma_2(G) \le \frac{n(G)+\ell(G)}{2} \le \frac{2\beta(G)+\beta(G)}{2} = \frac{3}{2}\beta(G).$$

Thus G is a bipartite graph with $\gamma_2(G) = \frac{3}{2}\beta(G)$ if and only if $n(G) = 2\beta(G)$,

$$\ell(G) = \beta(G) \text{ and } \gamma_2(G) = \frac{n(G)+\ell(G)}{2}.$$

The facts that $\ell(G) = \beta(G)$ and $n(G) = 2\beta(G) = 2\ell(G)$ show that G is a corona graph. Furthermore, the identity $\gamma_2(G) = \frac{n(G)+\ell(G)}{2}$ leads to $\gamma_2(G) = \frac{3}{4}n(G)$ and, by Proposition 4, G is either the corona of the corona of a connected bipartite graph or G is the corona of the cycle C_4.

Conversely, if G is either the corona of the corona of a bipartite graph or G is the corona of the cycle C_4, then Proposition 2.1 implies that $\gamma_2(G) = \frac{3}{2}\beta(G)$. □

Corollary 6. If T is a tree of order at least 3, then $\gamma_2(G) \le \frac{3}{2}\beta(G)$ with equality if and only if T is the corona of the corona of a tree.

Proof. Since every tree is a bipartite graph, this result immediately follows from Theorem 5. □

Theorem 7. Every tree T with n vertices and ℓ pendant vertices satisfies $\beta(T) \ge (n(T)+\ell(T))/3$ with equality if and only if the tree is well covered.

Proof. See [Favaron, O., ***A Bound on the Independent Domination Number of a Tree***, Vishwa internat. J. Graph Theory 1(1)(1992), 19-27]. □

Lemma 8. Let G be a graph different from a star. Let u be a support vertex of G having only one nonpendant neighbor and let $G' = G - (L_u \cup \{u\})$. Then $\beta(G') = \beta(G) - |L_u|$, there exists a $\gamma_2(G)$-set not containing u and $\gamma_2(G') \leq \gamma_2(G) - |L_u|$. If moreover u is a strong support vertex, then $\gamma_2(G') = \gamma_2(G) - |L_u|$.

Proof. Let v be the unique nonpendant neighbor of u. If I is a $\beta(G)$-set, then $I - L_u$ is a $\beta(G')$-set and conversely if I' is a $\beta(G')$-set, then $I' \cup L_u$ is a $\beta(G)$-set. Hence $\beta(G') = \beta(G) - |L_u|$.

Let S be a $\gamma_2(G)$-set. Clearly, $L_u \subset S$. If $u \in S$, then $(S - \{u\}) \cup \{v\}$ is another $\gamma_2(G)$-set not containing u. Let D be a $\gamma_2(G)$-set not containing u. Then $D - L_u$ is a 2-dominating set of G' and thus $\gamma_2(G') \leq \gamma_2(G) - |L_u|$. If moreover $|L_u| \geq 2$, let conversely D' be a $\gamma_2(G')$-set. Then $D' \cup L_u$ is a 2-dominating set of G and thus

$$\gamma_2(G) \leq \gamma_2(G') + |L_u|,\ \gamma_2(G) - |L_u| \leq \gamma_2(G').$$

Therefore, $\gamma_2(G') = \gamma_2(G) - |L_u|$. □

Next, we show that for every tree T, the 2-domination number is bounded below by the independence number.

Theorem 9. If T is a tree of order n, then $\gamma_2(T) \geq \beta(T)$.

Proof. We proceed by induction on the order of T. Clearly, the result holds for n =1,2 establishing the basic cases. Let $n \geq 3$ and assume that for every tree T' of order $n' < n$ we have $\gamma_2(T') \geq \beta(T')$. If T is a star, then $\gamma_2(T) = \beta(T) = n - 1$ and hence the result is valid. So assume that T is not a star and let u be a support vertex of T for which the subgraph induced by $V(T) - (L_u \cup \{u\})$ is a tree. For instance, u is the support vertex of a pendant vertex of maximum eccentricity. Let $T' = T - (L_u \cup \{u\})$. Since T is not a star, T' has order at least two and u has a unique neighbour in T'.

By Lemma 8, $\gamma_2(T') \leq \gamma_2(T) - |L_u|$ and $\beta(T') = \beta(T) - |L_u|$. By the induction hypothesis applied to T', we have $\gamma_2(T') \geq \beta(T')$.

Then $\gamma_2(T) - |L_u| \geq \beta(T) - |L_u|$. Therefore, $\gamma_2(T) \geq \beta(T)$. □

In order to characterize the trees with equal 2-domination and independence numbers we define the **family** $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where T_1 is a star $K_{1,t}$ $(t \geq 2)$ of center vertex w, $T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the three operations defined below. Put $A(T_1) = L_w$.

- Operation O_1: Add a star $K_{1,p}$, $p \geq 1$, centered at a vertex x and join x by an edge to a pendant vertex y of T_i. Put $A(T_{i+1}) = A(T_i) \cup L_x$.
- Operation O_2: Suppose that at least one operation O_1 has been performed before O_2. Add a star $K_{1,p}$, $p \geq 1$, centered at a vertex x and join x by an edge to a nonpendant vertex y of $A(T_i)$. Put $A(T_{i+1}) = A(T_i) \cup L_x$.
- Operation O_3: Add a star $K_{1,p}$, $p \geq 2$, centered at a vertex x and join x by an edge to a vertex y of $V(T_i) - A(T_i)$. Put $A(T_{i+1}) = A(T_i) \cup L_x$.

We also define $\mathcal{F}_1$ as the subfamily of $\mathcal{F}$ consisting of trees constructed from T_1 by recursively applying operation O_1. For instance an odd path P_{2q+1} belongs to $\mathcal{F}_1$ since it is obtained from $K_{1,2}$ by applying q – 1 times O_1 with each p which is equal to 1.

We present in the following Lemma 10 and Lemma 11 some properties of the trees in $\mathcal{F}$.

Lemma 10. If T is a tree of $\mathcal{F}$, then A(T) is both the unique γ_2(T)-set and the unique β(T)-set.

Proof. Let T be a tree of $\mathcal{F}$. From the way in which T is constructed, A(T) is both a 2-dominating set and an independent set. Thus $\beta(T) \geq |A(T)| \geq \gamma_2(T)$. The equality follows from Theorem 9.

To show that A(T) is both the unique γ_2(T)-set and the unique β(T)-set, we proceed by induction on k where k – 1 is the number of operations performed to construct T from T_1. If k = 1, then $T = K_{1,t}$ with $t \geq 2$ and so A(T) is the unique γ_2(T)-set and the unique β(T)-set. This establishes the basic case.

Assume now that $k \geq 2$ and the result holds for all trees of $\mathcal{F}$ that can be constructed from a sequence of at most k – 2 operations. Let T be a tree of $\mathcal{F}$ constructed by k – 1operations, $T' = T_{k-1}$, x the center of the star $K_{1,p}$ added to T′to get T, and y the neighbour of x in T′. Then

$$\beta(T) = |A(T)| = \beta(T') + |L_x| \text{ and}$$

$$\gamma_2(T) = |A(T)| = \gamma_2(T') + |L_x|.$$

By induction hypothesis applied to T', we know that $A(T')$ is the unique $\gamma_2(T')$-set and the unique $\beta(T')$-set.

Suppose that $A(T) = A(T') \cup L_x$ is not the unique $\gamma_2(T)$-set and let D be a second $\gamma_2(T)$-set . Then D must contain x for otherwise $A(T')$ and $D - L_x$ are two different $\gamma_2(T')$ - sets, a contradiction. Also $y \notin D$ and y is dominated by x and by exactly one vertex of $V(T') \cap D$ for otherwise $D - (L_x \cup \{x\})$ would be a 2-dominating set of T' of order less than $\gamma_2(T')$, a contradiction. Then $(D \cap V(T')) \cup \{y\}$is a 2-dominating set of $T'$of order $|D| - (|L_x| + 1) + 1 = |A(T)| - |L_x| = |A(T')|$. By the unicity of the $\gamma_2(T')$-set, $(D \cap V(T')) \cup \{y\}$ is the independent set $A(T')$. This contradicts the fact that y has a neighbour in $V(T') \cap D$. Consequently, $A(T)$ is a unique $\gamma_2(T)$-set.

Let I be a $\beta(T)$-set. If L_x is not contained in I, then clearly $|L_x| = 1$. Thus T is constructed from T' by operation O_1 or O_2, and $y \in A(T')$. If u is the pendant vertex of T adjacent to x, $u \notin I$, $x \in I$ and $y \notin I$. The set $I_1 = (I - \{x\}) \cup \{u\}$ is a $\beta(T)$-set such that $I_1 \cap V(T')$ is a $\beta(T')$-set and $I_1 \cap V(T') = I \cap V(T')$. By the unicity of the $\beta(T')$-set, $I \cap V(T') = A(T')$ contradicting $y \in A(T')$ but $y \notin I$. Therefore $L_x \subset I$ and $I - L_x = I \cap V(T')$ is a $\beta(T')$-set. By the unicity of the $\beta(T')$-set, $I - L_x = A(T')$ and thus $I = A(T)$. □

Lemma 11. If T is obtained from T_1 by applying $r \geq 0$ times operations O_2 or O_3 and any number of times operation O_1, then $|A(T)| = (n(T) + \ell(T) - r - 1)/2$. If moreover $r \geq 1$(that is if $T \in \mathcal{F} - \mathcal{F}_1$), then $n(T) + \ell(T) \geq 3r+7$ with equality if and only if T is a subdivided star SS_q with $q \geq 3$ or the double star $S_{2,2}$.

Proof. If a tree T is obtained from a tree $T'$of $\mathcal{F}$ by an operation O_1, then

$$n(T) = n(T') + p + 1,$$

$$\ell(T) = \ell(T') + p - 1,$$

$$\gamma_2(T) = \gamma_2(T') + p.$$

Therefore, $\gamma_2(T) - (n(T) + \ell(T))/2 = \gamma_2(T') - (n(T') + \ell(T'))/2$.

If T is obtained from T' of $\mathcal{F}$ by an operation O_2 or O_3, then

$$n(T) = n(T') + p + 1, \ \ell(T) = \ell(T') + p, \gamma_2(T) = \gamma_2(T') + p.$$

Therefore, $\gamma_2(T) - (n(T) + \ell(T))/2 = \gamma_2(T') - (n(T') + \ell(T') + 1)/2$.

Hence the value of $\gamma_2(T) - (n(T) + \ell(T))/2$ is unchanged in each application of O_1 and decreases by 1/2 in each of the $r \geq 0$ applications of O_2 or O_3. Since for $T_1 = K_{1,t}$ with $t \geq 2$, $\gamma_2(T_1) = (n(T_1) + \ell(T_1) - 1)/2$, we get $|A(T)| = (n(T) + \ell(T) - r - 1)/2$.

We note in particular that if T is in $\mathcal{F}_1$, then r = 0 and

$$\beta(T) = \gamma_2(T) = (n(T) + \ell(T) - 1)/2.$$

Each operation O_3 increases $n(T_i)$ by $p + 1 \geq 3$ and $\ell(T_i)$ by $p \geq 2$. Hence if the tree T of $\mathcal{F}$ is constructed from $T_1 = K_{1,t}$ by $r \geq 1$ operations O_3 and any number of operations O_1,

$$n(T) \geq n(T_1) + 3r = 3r + t + 1, \ \ell(T) \geq \ell(T_1) + 2r = 2r + t \text{ and}$$

$$n(T) + \ell(T) \geq 5r + 2t + 1 \geq 3r + 7, \text{ since } t \geq 2 \text{ and } r \geq 1.$$

Moreover $n(T) + \ell(T) = 3r + 7$ if and only if t = 2 and in the construction of T, no operation O_1 and exactly one operation O_3 with p = 2 has been used, that is if T is the double star $S_{2,2}$.

If the construction of T uses at least one operation O_2 joining the center x of a star $K_{1,p}$ to a vertex y of T′, then y is a nonpendant vertex of A(T′). Hence n(T) is at least equal to n (T_1) plus two vertices added in O_1 plus 2r vertices added in the r operations O_2 and O_3. Therefore, exactly 2r if and only if these r operations are of type O_2 with p = 1. Similarly, $\ell(T)$ is at least equal to $\ell(T_1)$ plus r pendant vertices added in the r operations O_2 and O_3. Therefore, exactly r if and only if these r operations are of type O_2 with p = 1. In this case,

$$n(T) \geq t + 3 + 2r, \ \ell(T) \geq t + r \text{ and}$$

$$n(T) + \ell(T) \geq 3r + 2t + 3 \geq 3r + 7, \text{ since } t \geq 2.$$

Moreover, $n(T) + \ell(T) = 3r + 7$ if and only if T is constructed from $K_{1,2}$ by using one operation O_1 and $r \geq 1$ operations O_2, each of them with p = 1. This tree T is a subdivided star SS_q with $q = r + 2 \geq 3$. □

We are now ready to characterize the trees achieving equality in Theorem 9.

Theorem 12. Let T be a tree of order n. Then the following statements are equivalent:

(a) $\gamma_2(T) = \beta(T)$,

(b) T = K_1 or T $\in \mathcal{F}$,

(c) T has a unique $\gamma_2(T)$-set that also is a unique $\beta(T)$-set.

Proof. Let T be a tree of order n.

(b) implies (c). The result is obvious if T = K_1 and it follows from Lemma 10 if T $\in \mathcal{F}$.

(c) implies (a). Obvious.

(a) implies (b). We proceed by induction on the order of T. If n = 1, then T = K_1. The only tree T of order n = 2 does not satisfy $\gamma_2(T) = \beta(G)$. If n = 3, then T = $K_{1,2}$ is a tree T_1 of $\mathcal{F}$. Let $n \geq 4$ and assume that any tree T′of order n′< n with $\gamma_2(T') = \beta(T')$ is in $\mathcal{F}$. Let T be a tree of order n such that $\gamma_2(T) = \beta(T)$. If T is star, then T $\in \mathcal{F}$. So we assume that T is not a star. Let x be a support vertex of T such that T – ($L_x \cup \{x\}$) is a tree denoted by T′ . Then there exist the unique neighbour y of x in T′. If T′ has order two, then $\beta(T) = |L_x|+1$ and $\gamma_2(T) = |L_x| + 2$, a contradiction. Hence T′ has order at least three. Then by Lemma 8,

$$\beta(T') = \beta(T) - |L_x| \text{ and } \gamma_2(T') \leq \gamma_2(T) - |L_x|.$$

Therefore, $\gamma_2(T') \leq \beta(T')$ and by Theorem 9, $\gamma_2(T') = \beta(T')$. By induction hypothesis, T′ is in $\mathcal{F}$. The set A(T′) $\cup$ L_x is then both a $\beta(T)$-set and a $\gamma_2(T)$-set. The addition of the star $\{x\} \cup L_x = K_{1,p}$ to T′in an operation O_1 if y is a pendant vertex of T′, O_2 if y is a nonpendant vertex of A(T′), O_3 if $y \notin$ A(T′) where $|L_x| \geq 2$. Therefore, T $\in \mathcal{F}$. □

We present the upper bounds on the 2-domination number $\gamma_2(G)$ and the independence number $\beta(G)$.

Theorem 13. If T is a nontrivial tree of order n, then $\gamma_2(T) \leq (n(T) + \ell(T))/2$.

Proof. Let T be a nontrivial tree. If T is a star then the result holds. So we assume that T is not a star. The tree T′ obtained from T by removing all its pendant vertices is not trivial and admits a bipartition A, B. Every vertex of degree one in T′ is a support vertex in T that is adjacent to at least one vertex of L(T). Every vertex of degree at least two of A(resp. B) is dominated twice by B(resp. by A). Thus $L(T) \cup A$ and $L(T) \cup B$ are two 2-dominating sets of T.

So $\gamma_2(T) \leq \min\{|L(T) \cup A|, |L(T) \cup B|\}$

$\leq \ell(T) + (n(T) - \ell(T))/2 = (n(T) + \ell(T))/2.$ □

In order to characterize the nontrivial trees attaining the upper bound in Theorem 13, we define the collection $\mathcal{G}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k (k \geq 1)$ of trees, where T_1 is the path P_2, $T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the two operations listed below.

- Operation $\mathcal{T}_1$: Add a star $K_{1,p}$, $p \geq 2$, centered at a vertex x and join x by an edge to a pendant vertex y of T_i.
- Operation $\mathcal{T}_2$: Add a path P_2 = x z, join x by an edge to a pendant vertex y of T_i, and add $r \geq 0$ new vertices adjacent to y.

For instance an even path P_{2q} is constructed from P_2 by performing q – 1 times $\mathcal{T}_2$ with r = 0. Hence $P_{2q} \in \mathcal{G}$.

Theorem 14. Let T be a nontrivial tree. Then $\gamma_2(T) = (n(T) + \ell(T))/2$ if and only if $T \in \mathcal{G}$.

Proof. We first prove the part "if" by induction on the number k – 1 of operations performed to construct T from $T_1 = P_2$. If k = 1, then $T = P_2$, and so

$$\gamma_2(T) = (n(T) + \ell(T))/2 = 2.$$

This establishes the basic case.

Assume now that $k \geq 2$ and that the result holds for all trees in $\mathcal{G}$ that can be constructed from less than k – 1 operations. Let $T = T_k \in \mathcal{G}$, and let $T' = T_{k-1}$. By induction hypothesis applied to T′ we have $\gamma_2(T') = (n(T') + \ell(T'))/2$. We consider two cases depending on whether T is obtained from T′ by using operation $\mathcal{T}_1$ or $\mathcal{T}_2$.

Case 1. T is obtained from T′ by using operation $\mathcal{T}_1$.

Let $|L_x| = p \geq 2$. Then $n(T) = n(T') + p + 1$ and $\ell(T) = \ell(T') + p - 1$.

By Lemma 2.8, $\gamma_2(T) = \gamma_2(T') + |L_x| = (n(T') + \ell(T'))/2 + p = (n(T) + \ell(T))/2$.

Case 2. T is obtained from T′ by using operation $\mathcal{T}_2$.

If Y denotes the set (possibly empty) of vertices attached at y by this operation, $n(T) = n(T') + |Y| + 2$ and $\ell(T) = \ell(T') + |Y|$. Let S be a 2-dominating set of T not containing x (but necessarily containing $Y \cup \{z\}$). To 2-dominate x, S also contains y which implies that $S - (Y \cup \{z\})$ is a 2-dominating set of T'. Hence $\gamma_2(T') \le \gamma_2(T) - |Y| - 1$. On the other hand if S' is a $\gamma_2(T')$-set, then it contains the pendant vertex y of T' and thus $S' \cup (Y \cup \{z\})$ is a 2-dominating set of T. Hence $\gamma_2(T) \le \gamma_2(T') + |Y| + 1$. Therefore,

$$\gamma_2(T) = \gamma_2(T') + |Y| + 1 = (n(T') + \ell(T'))/2 + |Y| + 1$$
$$= (n(T) + \ell(T))/2.$$

We prove the part "only if" by induction on the order of T. If n = 2, then $T = P_2$ which belongs to $\mathcal{G}$. The only tree of order three does not satisfy

$$\gamma_2(T) = (n(T) + \ell(T))/2.$$

One of the two trees of order four, namely P_4, satisfies $\gamma_2(T) = (n(T) + \ell(T))/2$ and it belongs to $\mathcal{G}$. Clearly, the other tree of order four does not satisfy

$$\gamma_2(T) = (n(T) + \ell(T))/2.$$

For $n \ge 5$, suppose that every tree of order less than n and satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$ is in $\mathcal{G}$ and let T be a tree of order n satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$. Among the stars and double stars, clearly the only trees of diameter 2 or 3 satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$ are the double stars $S_{1,q}$ with $q \ge 1$. These trees are obtained from P_2 by using one operation $\mathcal{T}_2$ with r = q – 1 and thus belong to $\mathcal{G}$. So we suppose $\text{diam}(T) \ge 4$ and consider a $\gamma_2(T)$-set S of T. We consider a vertex v_0 of maximum eccentricity as a root of T. Let v be a support vertex at maximum distance from v_0, u the parent of v and $T' = T - (L_v \cup \{v\})$. Since $\text{diam}(T) \ge 4$, T' is not trivial and has order $n(T') = n(T) - |L_v| - 1$. We consider two cases.

Case A. Suppose that v is a strong support vertex and thus $\gamma_2(T) = \gamma_2(T') + |L_v|$ by Lemma 8. If $\deg_T(u) \ge 3$, then $\ell(T') = \ell(T) - |L_v|$. By Theorem 13, we have

$$\gamma_2(T) = |L_v| + \gamma_2(T') \le |L_v| + (n(T') + \ell(T'))/2$$
$$= |L_v| + \frac{n(T) - |L_v| - 1 + \ell(T) - |L_v|}{2}$$
$$= (n(T) + \ell(T))/2 - \frac{1}{2} < (n(T) + \ell(T))/2, \text{ a contradiction.}$$

So u is a pendant vertex of T' and $\ell(T') = \ell(T) - |L_v| + 1$. Hence

$$\gamma_2(T') = \gamma_2(T) - |L_v| = (n(T) + \ell(T))/2 - |L_v|$$
$$= \frac{n(T') + |L_v| + 1 + \ell(T') + |L_v| - 1}{2} - |L_v|$$
$$= (n(T') + \ell(T'))/2.$$

By induction hypothesis applied to T', we have $T' \in \mathcal{G}$. Since T is obtained from T' by performing $\mathcal{T}_1$, $T \in \mathcal{G}$.

Case B. From now on we may assume that no child of u is a strong support vertex. Let v' be the unique pendant vertex adjacent to v. We claim that u has no child besides v as a support vertex. Suppose to the contrary that a child w of u is a support vertex with $L_w = \{w'\}$ and let T' be the nontrivial tree $T - (L_v \cup \{v\})$. Let S' be a $\gamma_2(T')$-set not containing w. Then S' contains w' and u. Hence $\{v'\} \cup S'$ is a 2-dominating set of T and so by Theorem 13,

$$\gamma_2(T) \le 1 + |S'| \le 1 + (n(T') + \ell(T'))/2.$$

Since $n(T') = n(T) - 2$ and $\ell(T') = \ell(T) - 1$, we get $\gamma_2(T) < (n(T) + \ell(T))/2$, a contradiction. Thus every child (if any) of u besides v is a pendant vertex.

Now let $T'' = T - (L_u \cup L_v \cup \{v\})$. Then T'' is not trivial. If S is a $\gamma_2(T)$-set not containing v, then S contains u to 2-dominate v, $S - (\{v'\} \cup L_u)$ is a 2-dominating set of T'' and thus $\gamma_2(T'') \le \gamma_2(T) - 1 - |L_u|$. On the other hand every $\gamma_2(T'')$-set S'' contains the pendant vertex u of T'' and thus $S'' \cup (L_u \cup \{v'\})$ is a 2-dominating set of T, implying $\gamma_2(T) \le \gamma_2(T'') + 1 + |L_u|$. Since

$$n(T) = n(T'') + 2 + |L_u| \text{ and } \ell(T) = \ell(T'') + |L_u|,$$

we get
$$\gamma_2(T'') = \gamma_2(T) - 1 - |L_u| = (n(T) + \ell(T))/2 - 1 - |L_u|$$
$$= (n(T'') + \ell(T''))/2.$$

By induction hypothesis applied to T'', we have $T'' \in \mathcal{G}$. Since T is obtained from T'' by using $\mathcal{T}_2$, $T \in \mathcal{G}$. This completes the proof of the theorem. □

By Theorem 9 and Theorem 13, every nontrivial tree satisfies $\beta(T) \le (n(T) + \ell(T))/2$. Therefore, the following theorem slightly improves this bound.

Theorem 15. If T is a nontrivial tree, then $\beta(T) \leq (n(T)+\ell(T)-1)/2$ with equality if and only if T $\in \mathcal{F}_1$.

Proof. By Theorem 9 and Theorem 13, we get the inequalities chain $\beta(T) \leq \gamma_2(T) \leq (n(T)+\ell(T))/2$. But $\beta(T)$ and $\gamma_2(T)$ are integers. Hence each equality $\beta(T) = (n(T)+\ell(T))/2$ or $\beta(T) = (n(T)+\ell(T)-1)/2$ implies $\beta(T) = \gamma_2(T)$. Thus, by Theorem 12, T $\in \mathcal{F}$. By Lemma 10 and Lemma 11,

$$\beta(T) = \gamma_2(T) = (n(T)+\ell(T)-r-1)/2 \text{ for some } r \geq 0.$$

Therefore, $\beta(T) = (n(T)+\ell(T))/2$ is impossible and $\beta(T) = (n(T)+\ell(T)-1)/2$ implies T $\in \mathcal{F}$, with r = 0, that is T $\in \mathcal{F}_1$.

Conversely we already observed that each tree T in $\mathcal{F}_1$ satisfies $\beta(T) = (n(T)+\ell(T)-1)/2$. □

From Theorem 7 and Theorem 9 we deduce that every nontrivial tree satisfies $\gamma_2(T) \geq (n(T)+\ell(T))/3$. Now, we slightly improve this bound.

Theorem 16. If T is a nontrivial tree, then $\gamma_2(T) \geq (n(T)+\ell(T)+2)/3$ with equality if and only if T is a subdivided star SS_q with $q \geq 3$ or the double star $S_{2,2}$.

Proof. By Theorem 7 and Theorem 9, we get the inequalities chain $\gamma_2(T) \geq \beta(T) \geq (n(T)+\ell(T))/3$. But $\beta(T)$ and $\gamma_2(T)$ are integers. Hence each of the equalities $\gamma_2(T) = (n(T)+\ell(T))/3$, $\gamma_2(T) = (n(T)+\ell(T))+1)/3$,

$$\gamma_2(T) = (n(T)+\ell(T))+2)/3 \text{ implies } \gamma_2(T) = \beta(T) .$$

Thus, by Theorem 12 and Lemma 11, $\gamma_2(T) = \beta(T) = (n(T)+\ell(T)-r-1)/2$ and

$$n(T) + \ell(T) \geq 3r + 7, \text{ for some } r \geq 0.$$

But if $\gamma_2(T) \leq (n(T)+\ell(T))+1)/3$, then

$$(n(T)+\ell(T)-r-1)/2 \leq (n(T)+\ell(T))+1)/3,$$

thus implying $n(T) + \ell(T) \leq 3r + 5$ which is impossible. Therefore,

$$\gamma_2(T) \geq (n(T)+\ell(T))+2)/3$$

with equality if and only if $n(T) + \ell(T) = 3r + 7$. The result follows from Lemma 11. □

Conclusion

The problem of finding minimum k-dominating set is interesting. In this paper, we present different bounds on the 2-domination number. Next, we have mentioned that for a non-trivial tree T, $\gamma_2(T) \le (n(T) + \ell(T))/2$ and $\gamma_2(T) \ge (n(T) + \ell(T))/3$. Moreover, we slightly improve these bounds and characterize the trees achieving equality in each bound. We think that finding the minimum k-dominating set problem is very useful in real world problems, still interesting and remaining to analyze in this field of research.

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