Force Exerted by the Flowing Incompressible Fluid on a Pipe Bend

Zin Thwe Khine Phay*, San Win Oo**

Abstract

This paper is concerned with the hydrodynamics of the fluid flow. The equations of motion such as Reynold's equation of motion, Navier-Stokes equation and Euler's equation of motion are firstly studied. For an inviscid fluid, Euler's equation of motion and Bernoulli's equation are derived. And then, the impulse-momentum equation for an inviscid fluid is expressed. It is based on the law of conservation of momentum, which states that the net force acting on the fluid mass is equal to the change in momentum of flow per unit time in that direction. Finally, the force exerted by the flowing incompressible fluid on a pipe bend is mainly discussed with examples.

Keywords: Hydrodynamics, Inviscid fluid, Euler's equation of motion, Bernoulli's equation, Momentum

1. Introduction

This paper includes the study of the forces causing the fluid flow. Thus the dynamics of the fluid flow is the study of the fluid motion with the forces causing flow. The dynamics behaviour of the fluid flow is analyzed by the Newton's second law of motion, which relates the acceleration with the forces. The fluid is assumed to be incompressible and non-viscous.

Consider the fluid flowing through a pipe in which

A = the cross - sectional area of the pipe ,

v = the velocity of the fluid across the section ,

d = the diameter of the pipe ,

ρ = the density of the fluid and

Q = the rate of flow or discharge .

2. Equation of Motion

According to Newton's second law of motion, the net force F_x acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a_x in the x-direction. Thus mathematically,

$$F_x = m a_x . \tag{1}$$

In the fluid flow, the following forces are present:

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* Associate Professor, Dr, Department of Mathematics, Yadanabon University

** Associate Professor, Dr, Department of Mathematics, Yadanabon University

(i) F_g = the gravity force.

(ii) F_p = the pressure force.

(iii) F_v = the force due to viscosity.

(iv) F_t = the force due to turbulence.

(v) F_c = the force due to compressibility.

Thus, the net force is $F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x$.

Neglecting the compressibility force F_c, the resulting net force is

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and the equation of motion is called **Reynold's equation of motion**.

Neglecting F_c and F_t, the resulting equation of motion is known as **Navier-Stokes Equation**. If the flow is assumed to be ideal, the viscous force F_v is zero, and the equation of motion is known as **Euler's equation of motion**.

3. Euler's Equation of Motion

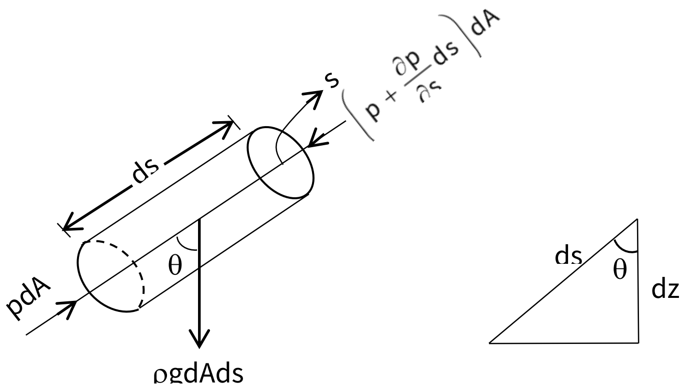
This is the equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a streamline. Consider a streamline in which the flow is taking place in s-direction as shown in Figure 1.

Consider a cylindrical element of cross-section dA and lengths ds. The forces acting on the cylindrical element are

(i) the pressure force pdA in the direction of flow,

(ii) the pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow,

(iii) the weight of element $\rho g dA ds$.



(a) (b)

Figure 1. Forces on a fluid element

Let θ be the angle between the direction of flow and the line of action of the weight of element.

The resultant force F_s on the fluid element in the direction of s must be equal to the product of the mass $\rho dAds$ of the fluid element and the acceleration a_s in the direction s.

That is, $F_s = \rho dAds\, a_s$

$$pdA - \left(p + \frac{\partial p}{\partial s}ds\right)dA - \rho gdAds\cos\theta = \rho dAds\, a_s. \quad (2)$$

Now $a_s = \dfrac{dv}{dt}$, where v is a function of s and t.

Therefore,

$$a_s = \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = v\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}, \text{ since } v = \frac{ds}{dt}.$$

If the flow is steady, then $\dfrac{\partial v}{\partial t} = 0.$

So, $a_s = v\dfrac{\partial v}{\partial s}.$

Substituting the value of a_s in (2) and simplifying the equation,

$$-\frac{\partial p}{\partial s}dAds - \rho gdAds\cos\theta = \rho dAds\, v\frac{\partial v}{\partial s}.$$

Dividing by ρdAds,

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0. \quad (3)$$

But from Fig. 1(b),

$$\cos\theta = \frac{dz}{ds}.$$

Therefore, (3) becomes

$$\frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + v\frac{dv}{ds} = 0$$

$$\frac{dp}{\rho} + gdz + vdv = 0. \quad (4)$$

Equation (4) is known as **Euler's equation of motion**.

4. Bernoulli's Equation from Euler's Equation

Bernoulli's equation is obtained by integrating the Euler's equation of motion (4) as

$$\int\frac{dp}{\rho} + \int gdz + \int vdv = \text{constant}.$$

If the flow is incompressible, ρ is constant and

$$\frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}. \quad (5)$$

Equation (5) is a **Bernoulli's equation** in which

$\frac{p}{\rho g}$ = the pressure energy per unit weight of the fluid or pressure head,

$\frac{v^2}{2g}$ = the kinetic energy per unit weight or kinetic head,

z = the potential energy per unit weight or potential head.

4.1 Assumptions

The following are the assumptions made in the derivation of Bernoulli's equation:

(i) The fluid is ideal, i.e., viscosity is zero.

(ii) The flow is steady.

(iii) The flow is incompressible.

(iv) The flow is irrotational.

4.2 Example

The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 litres/s. The section 1 is 6m above the datum and the section 2 is 4m above the datum. If the pressure at section 1 is $39.24 N/cm^2$, then the intensity of pressure at section 2 can be found.

At section 1,

$$d_1 = 20cm = 0.2m,$$

$$A_1 = \frac{\pi}{4}(0.2)^2 = 0.0314m^2,$$

$$p_1 = 39.24N/cm^2 = 39.24(10)^4 N/m^2,$$

$$z_1 = 6m.$$

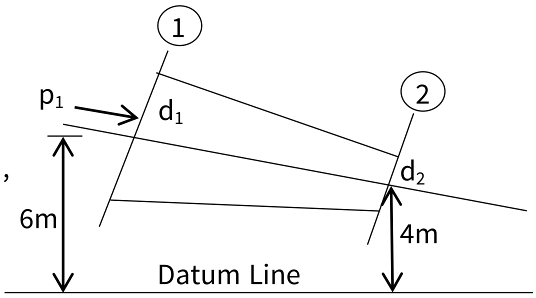


Figure 2.

At section 2, $d_2 = 0.1m$,

$$A_2 = \frac{\pi}{4}(0.1)^2 = 0.00785m^2,$$

$$z_2 = 4m.$$

Then the intensity of pressure at section 2 can be found.

The rate of flow is

$$Q = 35 lit/s = \frac{35}{1000} m^3/s = 0.035m^3/s.$$

Applying the continuity equation at sections 1 and 2,

$$Q = A_1 v_1 = A_2 v_2.$$

Therefore, $v_1 = \frac{Q}{A_1} = \frac{0.035}{0.0314} = 1.114m/s,$

and $v_2 = \frac{Q}{A_2} = \frac{0.035}{0.00785} = 4.456m/s.$

Applying Bernoulli's equation at sections 1 and 2,

$$\frac{p_1}{rg}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{rg}+\frac{v_2^2}{2g}+z_2$$

$$\frac{39.24(10^4)}{1000(9.81)}+\frac{(1.114)^2}{2(9.81)}+6=\frac{p_2}{1000(9.81)}+\frac{(4.456)^2}{2(9.81)}+4$$

$$46.063=\frac{p_2}{9810}+5.012.$$

So, $p_2=41.051(9810)N/m^2=40.27N/cm^2.=40.27N/cm^2.$

5. The Impulse-Momentum Equation

It is based on the law of conservation of momentum or on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in that direction.

The force acting on a fluid mass 'm' is given by the Newton's second law of motion, $F=ma$, where a is the acceleration acting in the same direction as force F.

But $a=\frac{dv}{dt}$, then $F=m\frac{dv}{dt}$.

Since m is constant and can be taken inside the differential,

$$F=\frac{d(mv)}{dt}. \qquad (6)$$

Equation (6) is known as the **momentum principle**. Equation (6) can be written as

$$Fdt=d(mv) \qquad (7)$$

which is known as the **impulse-momentum equation** and sates that the impulse of a force F acting on a fluid of mass m in a short interval of time dt is equal to the change of momentum d(mv) in the direction of force.

6. Result and Discussion

6. 1 Force exerted by the flowing fluid on a pipe bend

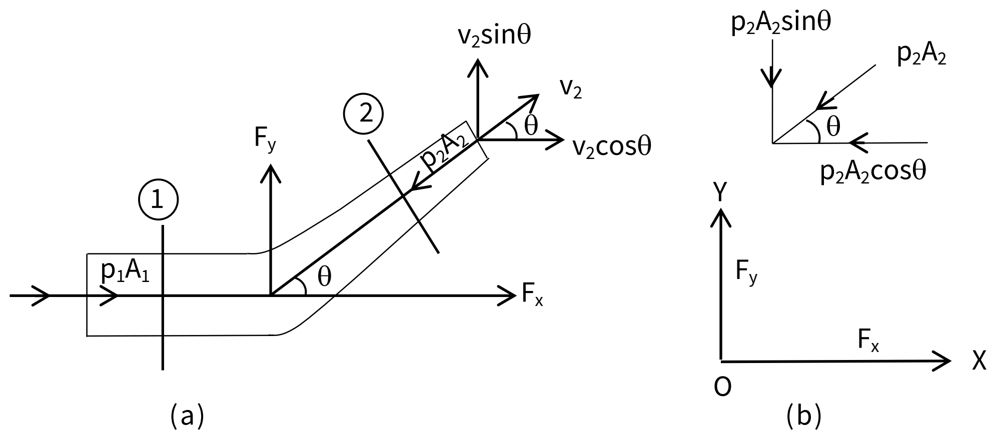


Figure 3. Forces on bend

The impulse-momentum equation (7) is used to determine the resultant force exerted by the flowing fluid on a pipe bend.

Consider two sections 1 and 2, as shown in Figure 3.

Let v_1 = the velocity of flow at section 1,

p_1 = the pressure intensity at section 1,

A_1 = the area of cross-section of pipe at section 1, and

v_2, p_2, A_2 = the corresponding values of velocity, pressure and area at section 2.

Let F_x and F_y be the components of the forces exerted by the flowing fluid on the pipe bend in X and Y directions respectively. Then the force exerted by the pipe bend on the fluid in the directions of X and Y will be equal to F_x and F_y but in the opposite directions.

Hence the component of the force exerted by the pipe bend on the fluid in the direction of x is $-F_x$ and in the direction of y is $-F_y$.

The other external forces acting on the fluid are p_1A_1 and p_2A_2 on the section 1 and 2 respectively. Then the momentum equation in x-direction is given by

$$p_1A_1 - p_2A_2\cos\theta - F_x = (\text{Mass per sec.})(\text{change of velocity})$$
$$= \rho Q(v_2\cos\theta - v_1).$$

Then, the force exerted by the flowing fluid on a pipe bend in the direction of x is

$$F_x = \rho Q(v_1 - v_2\cos\theta) + p_1A_1 - p_2A_2\cos\theta. \qquad (8)$$

Similarly, the momentum equation y-direction gives

$$0 - p_2A_2\sin\theta - F_y = \rho Q(v_2\sin\theta - 0).$$

Then, the force exerted by the flowing fluid on the pipe bend in the direction of y is

$$F_y = -(\rho Q v_2\sin\theta + p_2A_2\sin\theta). \qquad (9)$$

Now, the resultant force acting on the pipe bend is

$$F_R = \sqrt{F_x^2 + F_y^2}.$$

And the angle α made by the resultant force with horizontal direction is given by

$$\tan\alpha = \frac{F_y}{F_x}.$$

6.2 Example

250 litres/s of water is flowing in a pipe having a diameter of 300 mm. If the pipe is bent by 135° (that is, the change from the initial to the final direction is 135°), then the magnitude and the direction of the resultant force on the pipe bend can be calculated. The pressure of water flowing is $39.24 N/cm^2$.

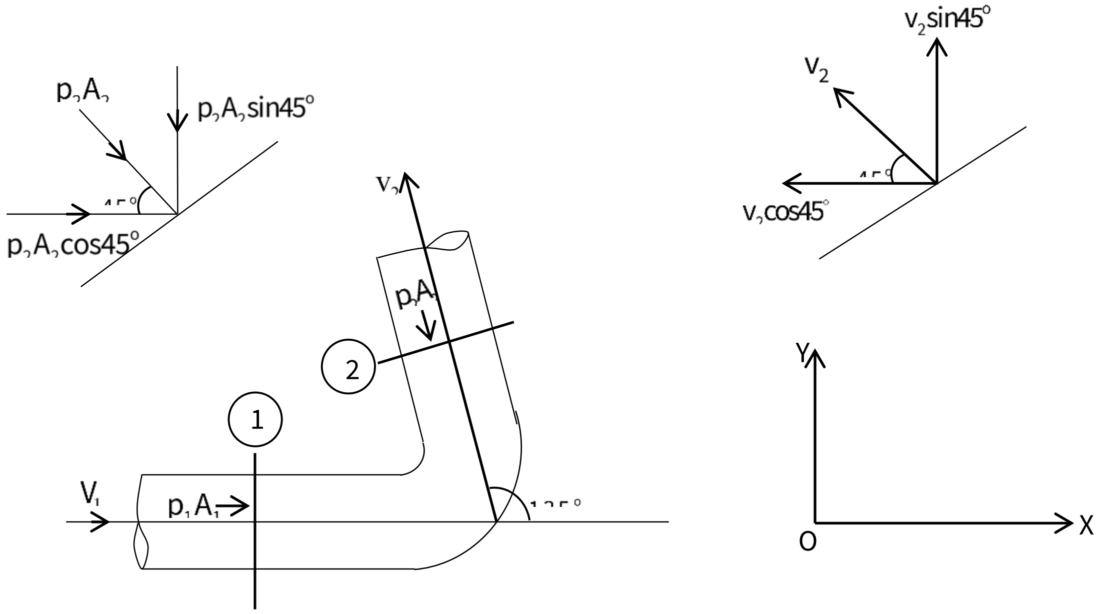


Figure 4.

The discharge is $Q = 250\,\text{litres}/\text{s} = 0.25\text{m}^3/\text{s}$.

The diameter of bend at inlet and outlet is

$$d = d_1 = d_2 = 300\text{mm} = 0.3\text{m}.$$

The area of the cross-section of pipe at sections 1 and 2 is

$$A = A_1 = A_2 = \frac{p}{4}d_1^2 = \frac{p}{4}(0.3)^2 = 0.07069\text{m}^2.$$

The velocity of water at sections 1 and 2 is

$$v = v_1 = v_2$$

$$= \frac{Q}{\text{Area}} = \frac{0.25}{0.07069} = 3.537\text{m}/\text{s}.$$

Since the pressure of water flowing is $39.24\text{N}/\text{cm}^2$,

$$p = p_1 = p_2$$

$$= 39.24\text{N}/\text{cm}^2 = 39.24(10)^4\text{N}/\text{m}^2.$$

Along x-axis, (8) becomes

$$F_x = rQ(v_1 - v_2\cos 135^\circ) + p_1A_1 - p_2A_2\cos 135^\circ$$

$$= rQ\left(v_1 - v_2\cos(180^\circ - 45^\circ)\right) + p_1A_1 - p_2A_2\cos(180^\circ - 45^\circ)$$

$$= rQ(v_1 + v_2\cos 45^\circ) + p_1A_1 + p_2A_2\cos 45^\circ$$

$$= 1000(0.25)\left[3.537 + (3.537)(0.7071)\right] + (39.24)(10)^4(0.07069)(1 + 0.7071)$$

$$= 1509.5 + 47352.83$$

$$= 48862.33\text{N}.$$

Along y-axis, (9) becomes

$$F_y = -\left(rQ\,v_2\,\sin 135^\circ + p_2A_2\,\sin 135^\circ\right)$$

$$= -\left(rQ\,v_2\,\sin(180^\circ - 45^\circ) + p_2A_2\,\sin(180^\circ - 45^\circ)\right)$$

$$= -\left(rQ\, v_2 \sin 45^o + p_2 A_2 \sin 45^o\right)$$

$$= -\left(1000(0.25)(3.537)(0.7071) + 39.24(10)^4(0.07069)(0.7071)\right).$$

Therefore, $F_y = -(625.25 + 19614.074) = -20239.324\text{N}$,

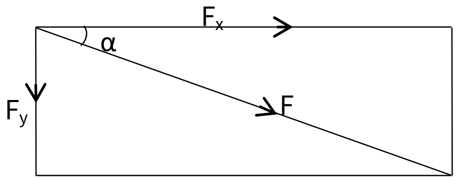
negative sign means F_y is acting in the downward direction.

Then, the resultant force acting on the bend is

$$F_R = \sqrt{F_x^2 + F_y^2}$$

$$= \sqrt{(48862.33)^2 + (20239.32)^2}$$

$$= 52888.16\text{N}.$$



Figure

The direction of the resultant force F_R, with the x-axis is given as

$$\tan a = \frac{F_y}{F_x} = \frac{20239.32}{48862.33} = 0.4142,$$

$$\alpha = 22^{\circ}30'.$$

Conclusion

This paper focuses on the various basic concepts such as Euler's equation of motion, Bernoulli's equation from Euler's equation with the expression of velocity of flow at any point in the pipe. In order to secure the expression of the resultant force exerted by the flowing incompressible fluid on a pipe bend, the basic concept of impulse momentum equation is used. From the above discussions, the magnitude and the direction of the resultant force acting on a pipe bend can also be found by using momentum equation.

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