Arc-Disjoint Paths and Internally Disjoint Paths in a Digraph

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Abstract

In this paper, some basic definitions and notations on digraphs are introduced. Some basic concepts on strong connectivity are discussed. Finally, the fundamental theorem concerning with arc-disjoint paths and internally-disjoint paths is studied with illustrated examples.

Keywords: digraph, path, arc-disjoint and internally disjoint.

1. Introduction

Digraph is introduced the theorems on digraph can be used to determine the number of minimum paths, maximum paths, different paths, shortest path, longest path. Some problems are also illustrated.

2. Basic Definitions and Notations

There are non-empty finite vertex set V(G) and non-empty finite arc set A(G) in a digraph G and is written G(V, A). V(G) has elements or vertices and A(G) has order pairs of distinct vertices or arcs. The numbers of vertices are denoted by the order of G and the numbers of arcs in G are denoted by the size of G.

The vertex **a** is **tail** as well as the vertex **b** is **head** for an arc (a, b). The arc (a, b) is from leaves **a** to enters **b**. The head and tail of an arc are end-vertices that they are **adjacent**. So, not only a is adjacent to b but also b is adjacent to a. If u is the head or tail of a , a vertex u is **incident** to an arc a. Moreover, arc (x, y) will be denoted by xy.

We define $(X,Z)_G = \{xz \in A(G) : x \in X, z \in Z\}$, for a pair X, Z of vertex sets of a digraph G where $(X,Z)_G$ is the set of arcs with tail in X and head in Z.

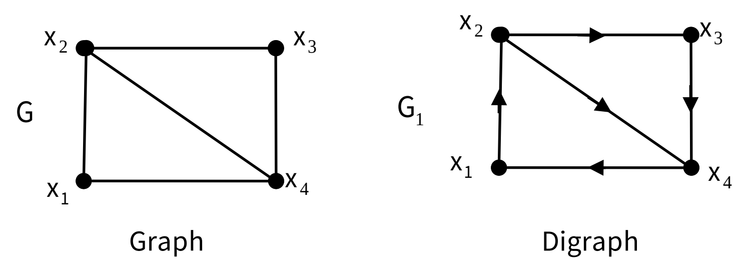

Figure 1. Graph G and digraph G.

2.1 Example

The digraph G in Figure 2 is of order and size 6; $V = \{\text{vertices}\}, A = \{\text{arcs}\}$, d^+ =out degree and d^-= in degree.

$V(G) = \{x, y, z, a, b, c\}$, $A(G) = \{(x,z),(y,z),(z,a),(a,b),(a,c),(c,a)\}$, $d^+(a)$=3, $d^-(a)$=1.

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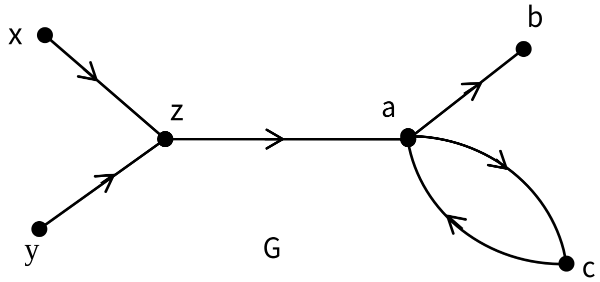


Figure 2. A digraph G

2.2 Example

$$\left(\{x_1,x_2\},\{x_3,x_4\}\right)_G = \{x_1x_3\} \text{ where } X = \{x_1,x_2\}, Y = \{x_3,x_4\},$$
$$\left(\{x_3,x_4\},\{x_1,x_2\}\right)_G = \{x_3x_2\} \text{ where } X = \{x_3,x_4\}, Y = \{x_1,x_2\},$$
$$\left(\{x_1,x_2\},\{x_1,x_2\}\right)_G = \{x_1x_2, x_2x_1\} \text{ where } X = \{x_1,x_2\}, Y = \{x_1,x_2\}.$$

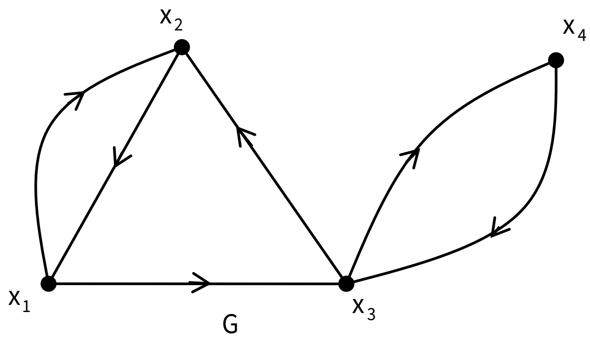


Figure 3. A digraph G

A digraph is called **directed pseudograph** when it contains parallel and loop. There are no loops in directed pseudographs that it is called **directed multigraphs**. By adding a loop x_4x_4 and two parallel arcs from x_1 to x_3 the directed pseudograph G' is obtained from G.

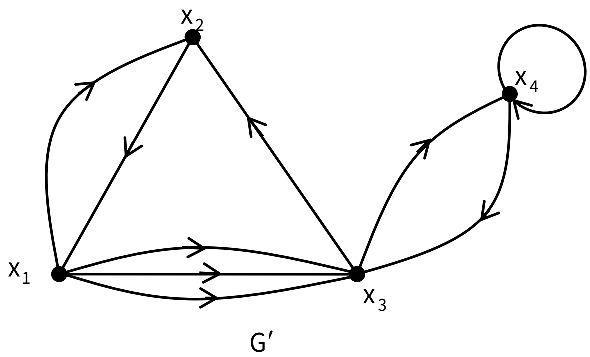


Figure 4. A directed pseudograph G'

3. Walk, Trail and Path

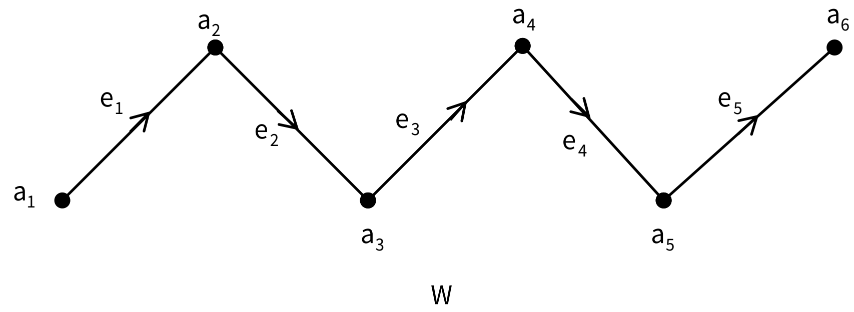


Figure 5. A simple directed graph G

In Figure 4, W has 6 vertices and 5 edges. Vertex a_1 is origin vertex and vertex a_6 is terminus vertex. Vertices $a_2a_3a_4a_5$ are called internal vertices.

A finite sequence $W = a_1e_1a_2e_2a_3e_3a_4e_4a_5e_5a_6$ is a **walk**. W is a walk from a_1 to a_6 or (a_1, a_6)-walk.

If the edges $e_1e_2e_3e_4e_5$ of a walk of W are not same edges, W is a **trail**.

If $a_1a_2a_3a_4a_5a_6$ and $e_1e_2e_3e_4e_5$ are distinct, W is a **path** or (a_1, a_6)-path. Number of edges of a path is called the length of the path.

W is a path and $[e_1e_2\ldots e_{k-1}] \in W$ are distinct. If $k \geq 3$ and $e_1 = e_k$, W is a **cycle**.

Two vertices a_1 and a_6 of W are said to be **connected** if there is a (a_1, a_6)-path in W.

3.1 Example

For example, a directed graph G in Figure 6 contains walk, trail, path and cycle.

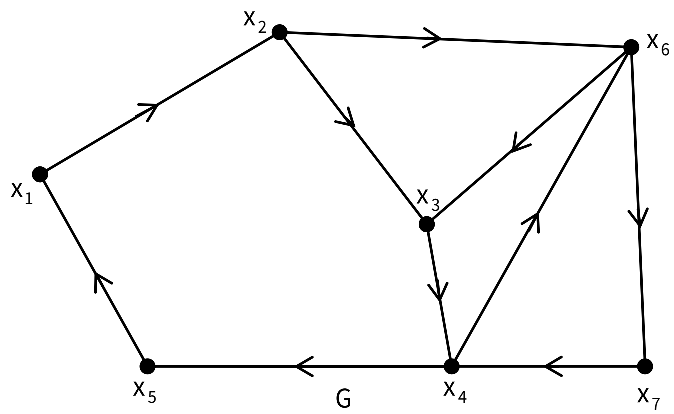


Figure 6. A directed graph G

$x_1\, x_2\, x_6\, x_3\, x_4\, x_6\, x_7\, x_4\, x_5$ is a walk.

$x_5\, x_1\, x_2\, x_3\, x_4\, x_6\, x_7$ is a path.

$x_1\, x_2\, x_6\, x_7\, x_4 x_5$ is an (x_1, x_5)-path.

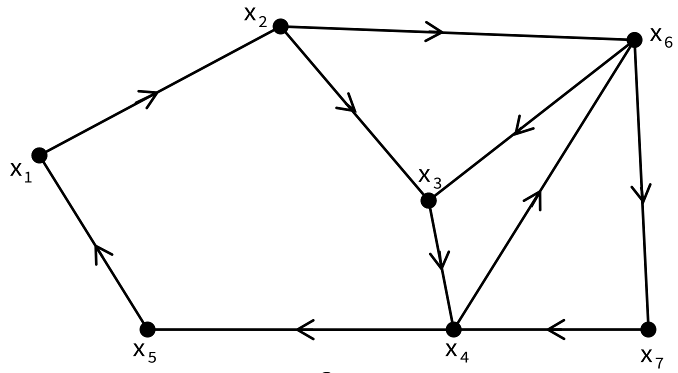
$x_2\, x_3\, x_4\, x_6\, x_3$ is an (x_2, x_3)-trail.

$x_1\, x_2\, x_3\, x_4\, x_5\, x_1$ is a cycle.

4. Vertex-disjoint and Arc-disjoint Paths

In a digraph G, it is supposed that $X = x_1 x_2 . . . x_k$ and $Y = y_1 y_2 . . . y_t$ are a pair of walks. The walks X and Y are **vertex-disjoint** if $V(X) \cap V(Y) = \varnothing$ and **arc-disjoint** if $A(W) \cap A(Y) = \varnothing$. X and Y are internally disjoint when $\{x_2, x_3, ..., x_{k-1}\} \cap V(Y) = \varnothing$ and $V(X) \cap \{y_2, y_3, ..., y_{t-1}\} = \varnothing$.

4.1 Example



G

Figure 7. A directed graph G

$x_1x_2x_3$ and $x_6x_7x_4x_5$ are vertex-disjoint paths.

$x_1x_2x_3x_4x_5$ and $x_2x_6x_7x_4$ are arc-disjoint paths.

$x_2x_3x_4$ and $x_2x_6x_7x_4$ are internally disjoint paths.

4.2 Proposition

G is a digraph and x, y are a pair of distinct vertices in G. We can prove that G contains an (x, y)-path P such that $A(P) \in A(W)$ when G has an (x, y)-walk W. G has a cycle C through x such that $A(C) \subset A(W)$ when G has a closed (x, x)-walk W.

Proof:

Let a walk P be minimum length from x to y in (x, y)-walks.

Consider $P = x_1 x_2 \dots x_k$ where $x = x_1$ and $y = x_k$.

The walk $P[x_1, x_i]P[x_{j+1}, x_k]$ is shorter than P if $x_i = x_j$, (where $1 \le i < j \le k$).

So, it contradics.

Since P is a path, all vertices of P are distinct.

Therefore, P is a path with $A(P) \subseteq A(W)$.

Let $W = z_1 z_2 \dots z_k$ be a walk from $x = z_1$ to $x = z_k$.

Since G has no loop, $z_{k-1} \neq z_k$.

Let $y_1y_2 \cdots y_t$ be a shortest walk form $y_1 = z_1$ to $y_t = z_{k-1}$.

We have proved above that $y_1y_2 \cdots y_t$ is a path. Thus $y_1y_2 \cdots y_ty_1$ is a cycle with $y_1 = x$. □

5. Strong Connectivity

For every distinct vertex in G, there are (u, v)-walk and (v, u)-walk vice versa that G is called strongly connected. For example, the digraph G in Figure 6 is **strongly connected**. For a strong digraph $G=(V,A)$, a set $S \subset V$ is a separator if $G-S$ is not strong. For example, the digraph G in Figure 6, $\{x_2, x_3\}$ is a **separator**. For a pair x, y of distinct vertices of a digraph G, a set $S \subseteq V(G)-\{x,y\}$ is an **(x, y)-separator** if G – S has no (x, y)-paths. For a digraph G in Figure 6, $\{x_2\}$ is an (x, y)-separator.

A digraph G is **p-strongly** connected if $|V| \geq p+1$ and G has no separator with less than p vertices.

For a strong digraph G = (V, A), a set of arc $W \subseteq A$ is a **cut** or **cutset** if G – W is not strong. In Figure 6, $\{x_1x_2, x_4x_5\}$ is a cutset. A digraph G is **p-arc-strong** if G has no cut with less than p arcs. (x, y)-cut is a set of arcs $(W,\overline{W})$, where $\overline{W}=A-W$ and $x \in W, y \in \overline{W}$.

5.1 Menger's Theorem

Let G= (V, A) be a multigraph and $x, y \in V(G)$ be distinct vertices.

(i) The minimum number of (x, y)- cut equals the maximum number of arc-disjoint (x, y)-path.

(ii) If $xy \notin A$, then the minimum number of (x, y)-separator equals the maximum number of internally disjoint (x, y)-path.

5.2 Example

This example illustrates Theorem 5.1

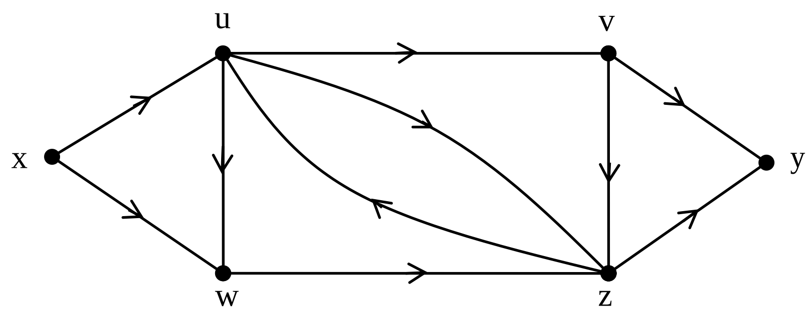


Figure 8. A directed graph H

(i) xuzy and xwzuvy are two arc-disjoint (x, y)- paths.

Let $X=\{x\}$.

Then $\overline{X}=\{u,v,w,z,y\}$.

(x,y)-cut $=\{xu, xw\}$.

The maximum number of arc-disjoint (x, y)- paths equals the minimum number of (x, y)-cut.

(ii) xuvy and xwzy are two internally disjoint (x, y)-paths.

(x,y)-separator $=\{u,w\}$ or $\{v,z\}$ or $\{u,z\}$.

The maximum number of internally disjoint paths equals the minimum number of (x,y)-separator.

5.3 Corollary

If $G=(V,A)$ is a multigraph, then (i) and (ii) can be proved.

(i) G contains p arc-disjoint (x, y)-paths for every distinct vertices $x,y\in V$ iff it is p-arc-strong and

(ii) G is p-strong iff $|V(D)|\geq p+1$ and D contains p internally disjoint (x, y)-paths for all distinct vertices $x,y\in V$.

5.4 Example

This example illustrates Corollary 5.3.

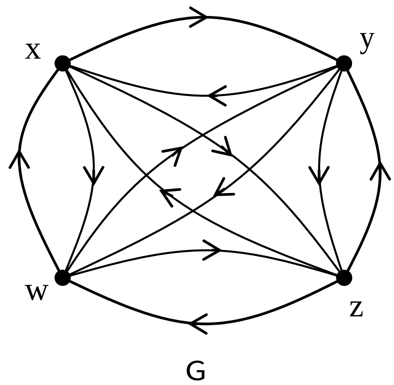


Figure 9. A strong digraph G

(i) For all distinct vertices $x,y\in V$, 3 arc-disjoint (x, y)-paths.

Therefore, G is 3-arc-strong.

(ii) $|V(D)|=4$ and G contains 3 internally disjoint (x, y)-paths for all distinct vertices $x,y\in V$. Therefore, G is 3-strong.

6. Result and Discussion

A digraph $G_2=(V,A)$ can be constructed from a digraph $G_1=(V,A)$. For every vertex $v\in G_1$, there exist two new vertices $v_s,v_t\in G_2$ as follow.

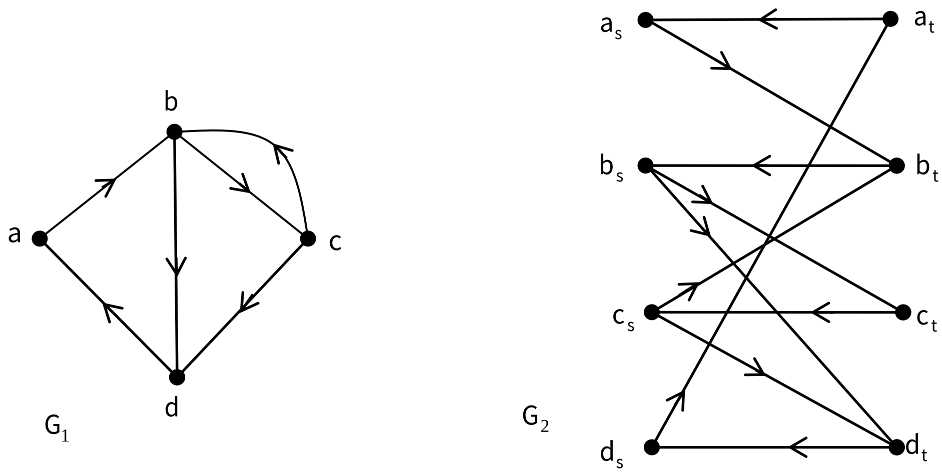


Figure 10. The vertex splitting procedure

By using Menger's Theorem, a new directed graph with a arc-disjoint paths and internally disjoint paths are constructed. It can be solved to a network flow problems and links problems. Moreover a disjoint path, shortest path and longest path in a graph can be found in a directed graph.

Conclusion

The graph which contains the direction is called digraph. Two distinct vertices are connected if there is a path. The digraph is Arc-disjoint and internally disjoint respectively if there are no common edge and no common vertex. When initial vertex and end vertex are the same of a path in digraph, it has a cycle. Every pair of vertices has to connect for strong connected digraph. The purpose of the paper is to construct network flows and connect links.

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