Comparative Study of Root Finding Methods with Microsoft Excel

May Myint Thwe*

Abstract

Microsoft Excel is a very simple system to use and it is a widely used spreadsheet application. A spreadsheet can be used to resolve problems that are iterative, recursive, or tabular in conceptual format in the problem-solving procedure. It can also be used to make complex analysis on data without having to study programming. This paper describes a method to execute spreadsheet computations for root finding methods. The common root-finding methods are Newton-Raphson, Secant and Bisection methods, etc., and root finding problem is a problem of finding a root of the equation f(x)=0, where f(x) is a function of x. Additionally, this work shows how Microsoft Excel can be used to make a comparative study between Newton-Raphson method, Secant method and Bisection method with the least number of iterations when applied to resolve a single variable nonlinear equation.

Key words: Microsoft Excel, Numerical methods, Convergence, Roots, Algorithm.

1. Introduction

Microsoft Excel is the influential software to use and it is the most commonly used spreadsheet. Its usage seems to be at the increase by the accessibility of the program in almost all personal computers. Moreover, Microsoft Excel can be used effectively by all branches of science and engineering. The program has been used in solving several mathematical problems in many different ways.[8] One of the most related computational problems is the root finding problem which arises in a wide variety of practical applications in Physics, Chemistry, Engineering, etc.[4] By the nature of the program, numerical methods for root finding problems can be made much easier to be implemented.[8] Root finding problem is a problem of finding a root of the equation f(x)=0 and different methods such as Newton-Raphson, Secant, Bisection methods etc., converge to the root at different rates.[4] This paper aims at comparing the performance in relation to the rate of convergence of three root finding methods: Newton-Raphson method, Secant method, and the Bisection method. Moreover, this study is to compare the number of iterations needed by a given numerical method to reach a solution and the rate of convergence of the methods.

In this work, manual computational algorithms for each of the methods are used to find the root of a function in order to reach the aim of this study. Furthermore, this paper illustrates how excel can be used to make comparative study of three well-known root finding methods for

* Professor, Dr, Head of the Department of Computer Studies, Yadanabon University

solving non-linear equations.[1] Therefore, this system is at comparing the rate of performance with the least number of iterations of Newton-Raphson, Secant and Bisection as methods of root-finding with MS excel.

2. Numerical Methods of Root Finding Problems

Numerical methods play a significant role to engineers in many applications. In mathematical analysis, these methods are the study of set of procedure that uses numerical approximation to solve engineering problems.[3]

2.1. Newton-Raphson Method

Newton-Raphson method can be derived by using the concept of finding the slope of a function and using the Taylor series. In all of numerical analysis, Newton-Raphson method for solving nonlinear equations is one of the most powerful procedures. It converges when the initial approximation is sufficiently close to the root.[1] Derivation of Newton-Raphson method begins with a tangent to a curve that cuts the x-axis as illustrated in Figure 1. The derivative f'(x) of the nonlinear function f(x) must be evaluated.[7] Newton-Raphson method starts with an initial guess that is close to the root. If the guess is close to the root, Newton-Raphson method is efficient. The advantages of Newton-Raphson method are that it converges fast, if it converges, and it consumes less time and less iterations to find the root.[6]

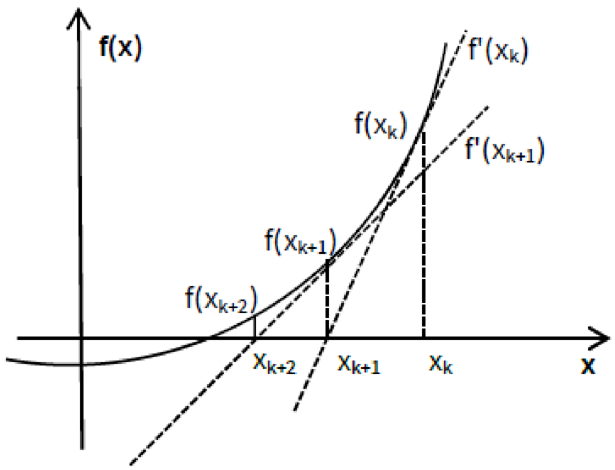


Figure 1. Newton-Raphson Method

In the above Figure 1, the slope of $(x_k, f(x_k))$ is given by

$$f'(x_k) = \frac{f(x_k)-0}{x_k-x_{k+1}} \qquad (1)$$

x_{k+1} in equation (1) represents Newton-Raphson equation.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad (2)$$

From Taylor's series of the function f(x) about a value $x=x_0$,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + f^{IV}(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$

The first approximation of the root of the given equation f(x)=0,

$$f(x) = 0 \approx f(x_0) + f'(x_0)(x - x_0)$$

it denotes that $$f(x_0) + f'(x_0)(x - x_0) = 0$$

for $x=x_1$, $$f(x_0) + x_1 f'(x_0) - x_0 f'(x_0) = 0$$

$$x_1 f'(x_0) = f'(x_0)x_0 - f(x_0)$$

The above equation is divided by $f'(x_0)$, then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

It is the first approximation. The second and third approximation to the root will be as follows.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general, this iterative procedure can be written as the following;

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

It is the Newton iterative method for non-linear equation. "k" denotes the number of iteration.

2.2. Algorithm for Newton Raphson Method

x_0 : initial guess

epsilon : error/tolerance

N : number of steps/calculations

For k=1 to N

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, f'(x_k) \neq 0$$

If $|x_{k+1} - x_k| \leq \text{epsilon}$, break.

2.3. The Secant Method

A different method to Newton's method is the secant method. The secant method is illustrated in Figure 2. A secant is the straight line which passes through two points on the curve. So, two initial approximations are required to initiate the secant method. It is required when the derivative function, f'(x), is unavailable or time consuming to evaluate.[7] To approximate the derivative by knowing the values of the function at point and the previous approximation is the only way to avoid such problems.[1] In this method, the procedure is applied repetitively to convergence.[7] The slope of the secant passes through two points, x_{k-1} and x_k. Therefore, $f(x_{k-1})$ and $f(x_k)$, the derivative of $f(x_k)$ which is $f'(x_k)$ can be approximated as:[1]

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

The general Newton-Raphson equation will be [1]

$$x_{k+1} \approx x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

The Secant method can be seen as:[1]

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

The Secant formula for solving non-linear equations can be expressed as; [6]

$$x_{k+1} = x_k - f(x_k)\frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

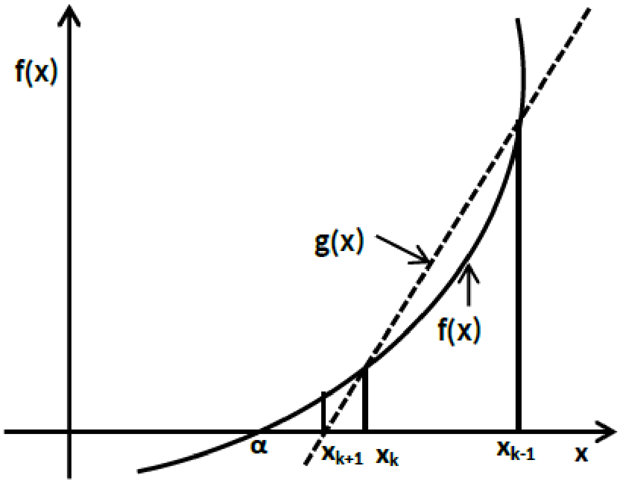


Figure 2. Secant Method

2.4. Algorithm for Secant Method

x_0 , x_1 : initial guesses

epsilon : error/tolerance

N : number of steps/calculations

For k=1 to N

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \left(f(x_k) - f(x_{k-1})\right) \neq 0$$

If $|x_{k+1} - x_k| \leq \text{epsilon}, \text{break}.$

2.5. Bisection Method

Bisection method is one of the simplest methods for finding a root of a nonlinear equation.[7] The concept of this method is based on continuity. The root of a function, f(x) = 0, lies in $[x_1, x_2]$. There is at least one root between x_1 and x_2, if f(x) is real and continuous and $f(x_1).\ f(x_2) < 0$. Since two initial guesses for the root are required, this method is classified under bracketing methods.[1] In bisection method, two estimates of the root, $x = x_1$ and $x = x_2$, which bracket the root, must first be obtained, as illustrated in Figure 3. Set $x_m = (x_1 + x_2)/2$. So, there are two intervals (x_1, x_m) and (x_m, x_2). If $f(x_1)f(x_m) < 0$, the root is in the interval (x_1, x_m). Thus, set $x_2 = x_m$ and continue. If $f(x_1)f(x_m) > 0$, the root is in the interval (x_m, x_2). Thus, set $x_1 = x_m$ and continue. If $f(x_1)f(x_m) = 0$, it is the root and terminate the iteration.[7]

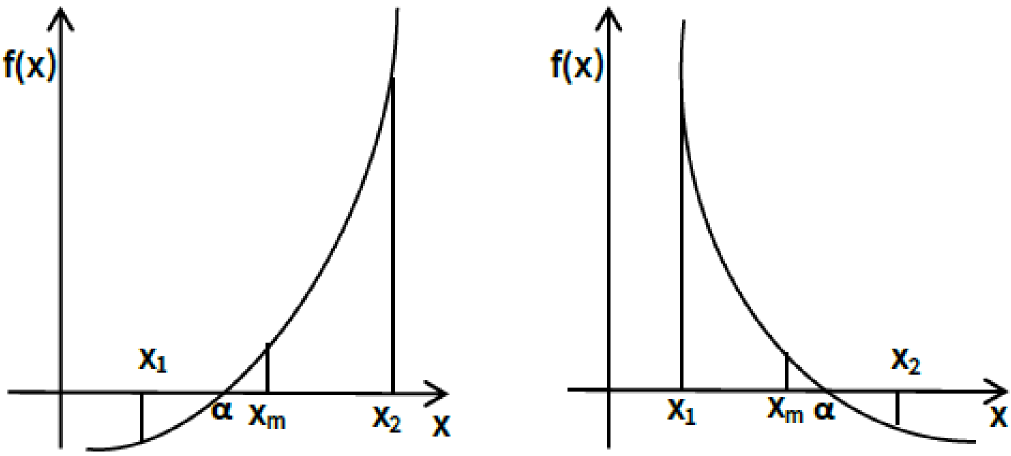


Figure 3. Bisection Method

2.6. Algorithm for Bisection Method

Step 1: Take the initial guesses x_1 and x_2 such that $f(x_1)f(x_2) < 0$ where $y_1=f(x_1)$, $y_2=f(x_2)$.

Step 2: Calculate the midpoint $x_m = (x_1+x_2)/2$, and then $y_m=f(x_m)$.

Step 3: If $y_1*y_m < 0$, then $x_1 = x_1$; $x_2 = x_m$.

If $y_1*y_m > 0$, then $x_1 = x_m$; $x_2 = x_2$.

If $y_1*y_m = 0$ then the root of the function = x_m .

Step 4: Find $x_m=(x_1+x_2)/2$.

2.7. Tolerance criteria

To measure tolerance limits, measuring when a given iterate is "close enough" to the true root is important in its own right. It powers significantly the number of times a given method must be executed. There are a quantity of altered ways to measure how close an iterate x_k is to a root r for a given function f(x). To measure how close $f(x_k)$ is to the true root f(r).[5] Note f(r) = 0,

$$|f(x_k)-f(r)| = |f(x_k)| \leq TOL, \quad \text{a given predefined tolerance.}$$

This typical way can be used to measure absolute error.[5] Similarly, another form of absolute error is,

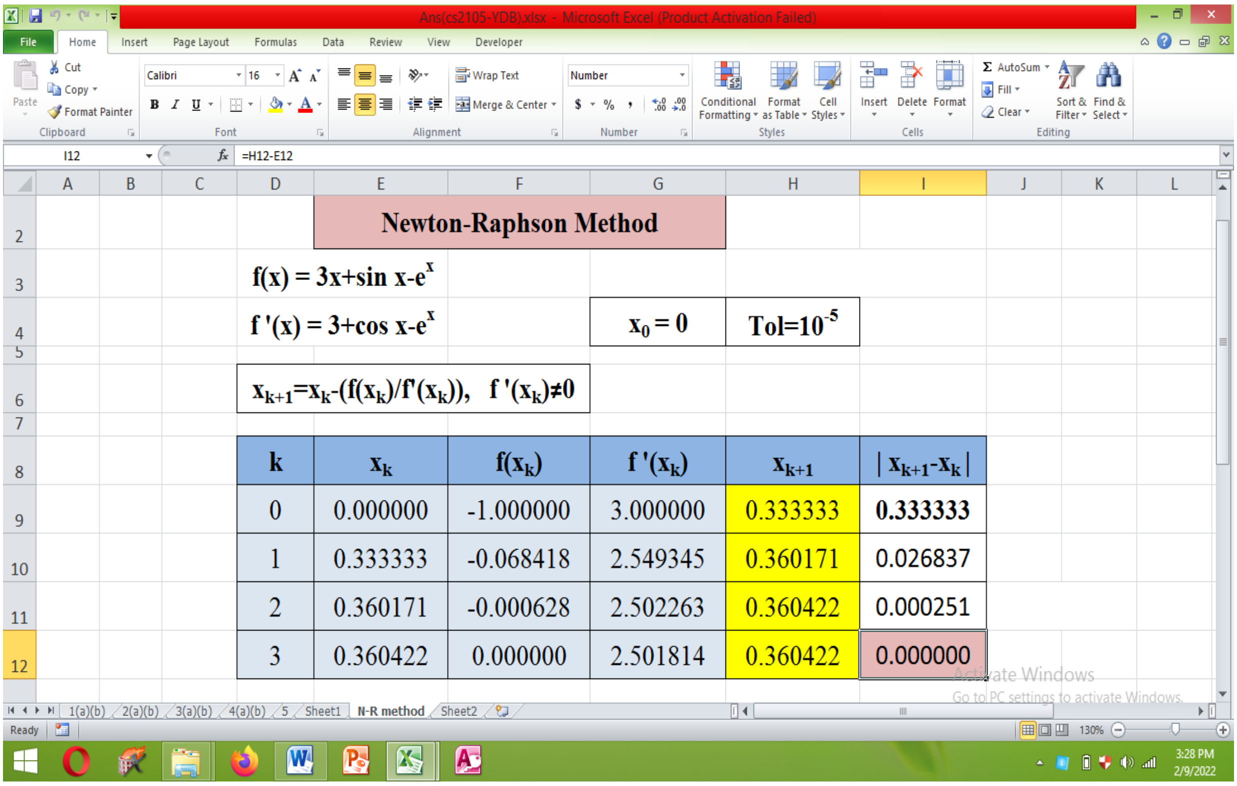
$$|x_k - x_{k-1}| \leq TOL$$

3. Results and Discussion

This study is comparing the rate of performance, rate of convergence of root findings of Newton-Raphson, Secant and Bisection methods. It also shows an approach of calculation using nonlinear equation and this is used for the iteration process with Microsoft Excel. To compare the results of the three methods under investigation, the rates of convergence of the methods are in the following. The solution of $f(x) = 3x + \sin x - e^x$ using Newton Raphson method, Secant method and Bisection method is computed in the form $f(x) = 0$ with a tolerance value of 10^{-5}. Newton-Raphson method starts with an initial guess that is close to the root. It requires the evaluation of both the function f(x) and its derivative at every iterations. This implies that $f'(x)= 3 + \cos x - e^x$.[2]

So, Newton iteration takes about two functions evaluation per iteration. For Secant method, it only requires the evaluation of f(x). For Bisection method, there is a continuous function f(x)=0 on a closed interval [x_1, x_2], i.e., $f(x_1)f(x_2) < 0$, then, the function f(x)=0 has at least a root or zero in the interval [x_1, x_2].

Table 1. Root of f(x) = 3x + sin x - e^x Using Newton-Raphson Method with MS Excel.



Newton-Raphson Method

$f(x) = 3x+\sin x-e^x$

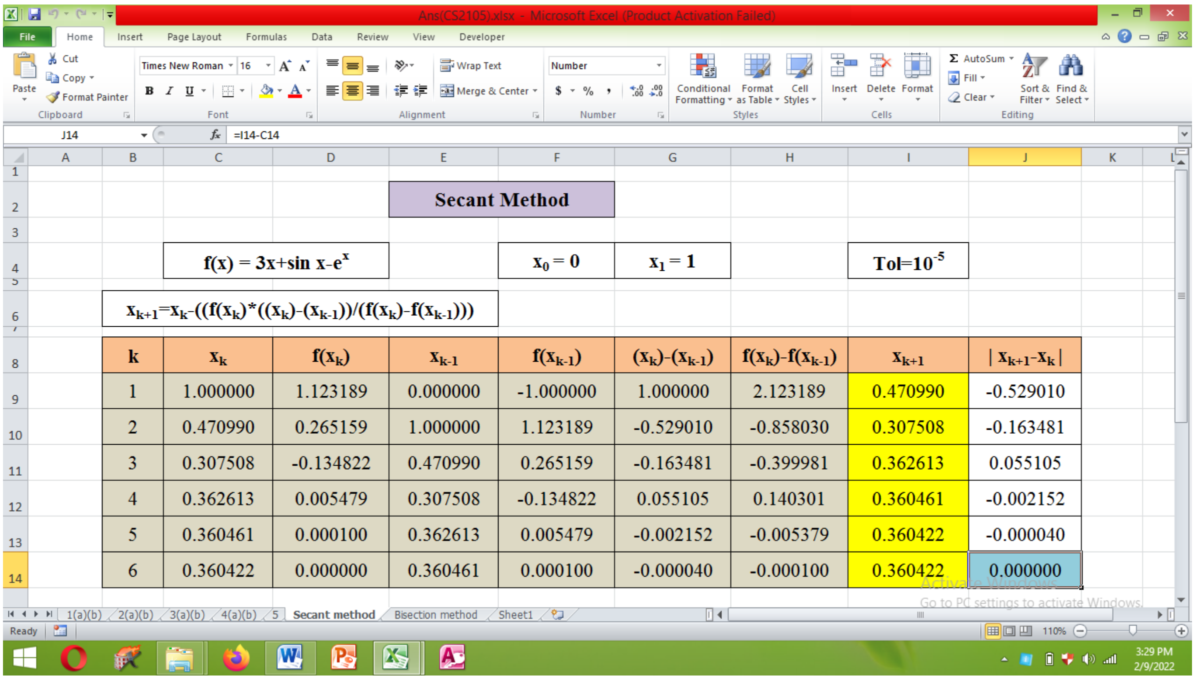
$f'(x) = 3+\cos x-e^x$ $x_0 = 0$ $Tol=10^{-5}$

$x_{k+1}=x_k-(f(x_k)/f'(x_k)),\quad f'(x_k)\neq 0$

k	x_k	$f(x_k)$	$f'(x_k)$	x_{k+1}	$\lvert x_{k+1}-x_k \rvert$
0	0.000000	-1.000000	3.000000	0.333333	0.333333
1	0.333333	-0.068418	2.549345	0.360171	0.026837
2	0.360171	-0.000628	2.502263	0.360422	0.000251
3	0.360422	0.000000	2.501814	0.360422	0.000000

Table 1 shows the iteration data obtained for Newton-Raphson method with the aid of Microsoft Excel. It was observed in Table 1 that using the Newton-Raphson method, the function, f(x)= 3x + sin x-e^x = 0 converges to 0.360422 after the 3rd iteration with error 0.000000.

Table 2. Root of f(x) = 3x + sin x - e^x Using Secant Method with MS Excel.



Secant Method

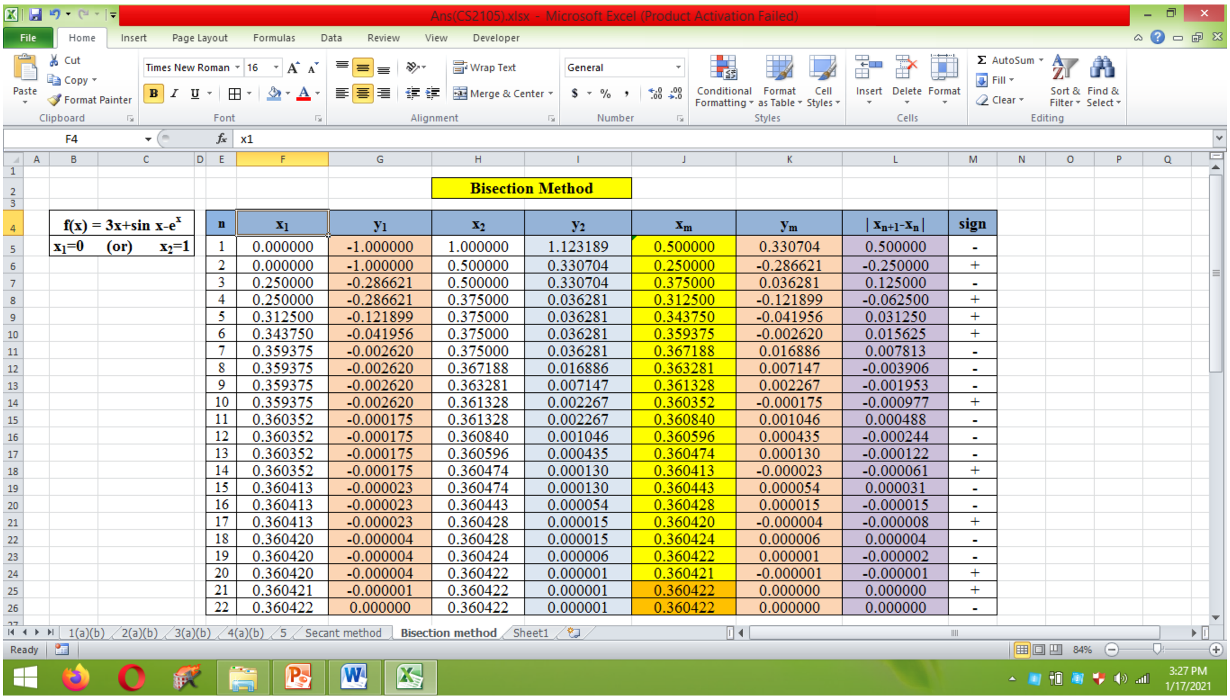
$f(x) = 3x+\sin x-e^x$ $x_0 = 0$ $x_1 = 1$ $Tol=10^{-5}$

$x_{k+1}=x_k-((f(x_k)*((x_k)-(x_{k-1}))/(f(x_k)-f(x_{k-1})))$

k	x_k	$f(x_k)$	x_{k-1}	$f(x_{k-1})$	$(x_k)-(x_{k-1})$	$f(x_k)-f(x_{k-1})$	x_{k+1}	$\lvert x_{k+1}-x_k \rvert$
1	1.000000	1.123189	0.000000	-1.000000	1.000000	2.123189	0.470990	-0.529010
2	0.470990	0.265159	1.000000	1.123189	-0.529010	-0.858030	0.307508	-0.163481
3	0.307508	-0.134822	0.470990	0.265159	-0.163481	-0.399981	0.362613	0.055105
4	0.362613	0.005479	0.307508	-0.134822	0.055105	0.140301	0.360461	-0.002152
5	0.360461	0.000100	0.362613	0.005479	-0.002152	-0.005379	0.360422	-0.000040
6	0.360422	0.000000	0.360461	0.000100	-0.000040	-0.000100	0.360422	0.000000

Table 2 shows the iteration data obtained for Secant method with the aid of MS Excel. It was observed in Table 2 that using the Secant method, $f(x)= 3x + \sin x-e^x = 0$ converges to 0.360422 after the 5th iteration with error 0.000000.

Table 3. Root of $f(x) = 3x + \sin x - e^x$ Using Bisection Method with MS Excel.



Bisection Method

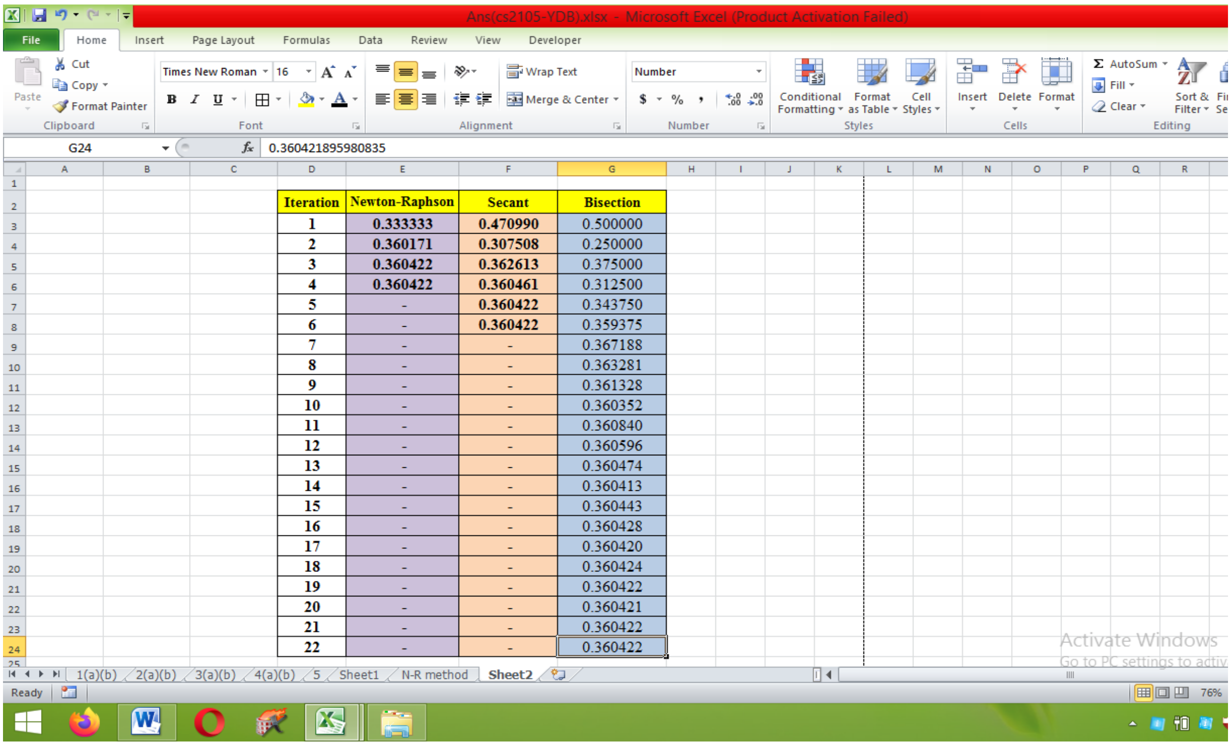
$f(x) = 3x+\sin x-e^x$

$x_1=0$ (or) $x_2=1$

n	x_1	y_1	x_2	y_2	x_m	y_m	$\lvert x_{n+1}-x_n \rvert$	sign
1	0.000000	-1.000000	1.000000	1.123189	0.500000	0.330704	0.500000	-
2	0.000000	-1.000000	0.500000	0.330704	0.250000	-0.286621	-0.250000	+
3	0.250000	-0.286621	0.500000	0.330704	0.375000	0.036281	0.125000	-
4	0.250000	-0.286621	0.375000	0.036281	0.312500	-0.121899	-0.062500	+
5	0.312500	-0.121899	0.375000	0.036281	0.343750	-0.041956	0.031250	+
6	0.343750	-0.041956	0.375000	0.036281	0.359375	-0.002620	0.015625	+
7	0.359375	-0.002620	0.375000	0.036281	0.367188	0.016886	0.007813	-
8	0.359375	-0.002620	0.367188	0.016886	0.363281	0.007147	-0.003906	-
9	0.359375	-0.002620	0.363281	0.007147	0.361328	0.002267	-0.001953	-
10	0.359375	-0.002620	0.361328	0.002267	0.360352	-0.000175	-0.000977	+
11	0.360352	-0.000175	0.361328	0.002267	0.360840	0.001046	0.000488	-
12	0.360352	-0.000175	0.360840	0.001046	0.360596	0.000435	-0.000244	-
13	0.360352	-0.000175	0.360596	0.000435	0.360474	0.000130	-0.000122	-
14	0.360352	-0.000175	0.360474	0.000130	0.360413	-0.000023	-0.000061	+
15	0.360413	-0.000023	0.360474	0.000130	0.360443	0.000054	0.000031	-
16	0.360413	-0.000023	0.360443	0.000054	0.360428	0.000015	-0.000015	-
17	0.360413	-0.000023	0.360428	0.000015	0.360420	-0.000004	-0.000008	+
18	0.360420	-0.000004	0.360428	0.000015	0.360424	0.000006	0.000004	-
19	0.360420	-0.000004	0.360424	0.000006	0.360422	0.000001	-0.000002	-
20	0.360420	-0.000004	0.360422	0.000001	0.360421	-0.000001	-0.000001	+
21	0.360421	-0.000001	0.360422	0.000001	0.360422	0.000000	0.000000	+
22	0.360422	0.000000	0.360422	0.000001	0.360422	0.000000	0.000000	-

Table 3 shows the iteration data obtained for Bisection method with the aid of MS Excel. It was observed in Table 3 that using the Bisection method, the function, $f(x)= 3x+\sin x-e^x = 0$ converges to 0.360422 after the 21st iteration with error 0.000000.

Table 4. Comparing the results of the three methods under investigation.



Iteration	Newton-Raphson	Secant	Bisection
1	0.333333	0.470990	0.500000
2	0.360171	0.307508	0.250000
3	0.360422	0.362613	0.375000
4	0.360422	0.360461	0.312500
5	-	0.360422	0.343750
6	-	0.360422	0.359375
7	-	-	0.367188
8	-	-	0.363281
9	-	-	0.361328
10	-	-	0.360352
11	-	-	0.360840
12	-	-	0.360596
13	-	-	0.360474
14	-	-	0.360413
15	-	-	0.360443
16	-	-	0.360428
17	-	-	0.360420
18	-	-	0.360424
19	-	-	0.360422
20	-	-	0.360421
21	-	-	0.360422
22	-	-	0.360422

As shown in Table 4, comparing the Newton-Raphson method and the Secant method, Newton's method converges faster than Secant method. It can also be seen that the Bisection method always converges on the root, but the rate of convergence is very slow. Again, comparing the Secant method and the Bisection method, Secant method converges faster than Bisection method.

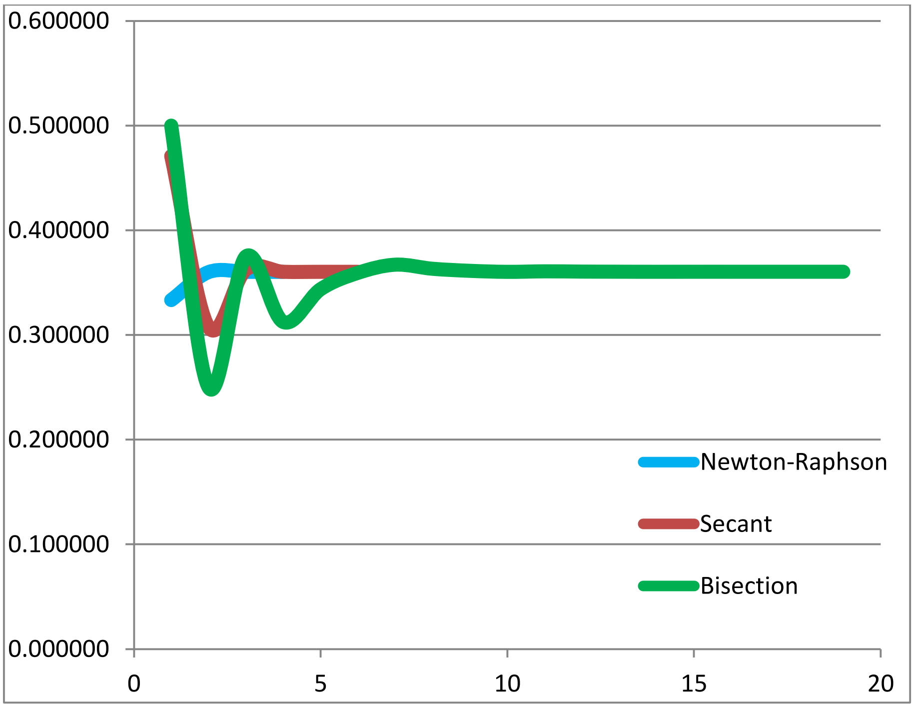


Figure 4. Comparison of the results of the three numerical methods

Figure 4 is shown comparing the results of the three methods under investigation. The rates of convergence of the methods can be seen in the order: Bisection method > Secant method > Newton-Raphson method in term of convergence. Therefore, the Newton-Raphson method is the fastest method for converging on a single root of a function and it is the most widely used method.

4. Conclusion

This paper intends to compare the performance in relation to the rate of convergence of three numerical methods namely, the Bisection method, Newton Raphson method, and Secant method. Furthermore, the basic objective of this system is to get the faster calculation time and better performance of the MS Excel application for root finding methods. Manual calculation is time consuming and prone to error, though it is important in understanding the procedure. The advantage of using MS Excel is that it consumes less time to find the root and the familiar GUI of Excel and the easy-to-use interface of Excel can provide the calculations as a very powerful and useful tool. From the results obtained in the computations with the help of Microsoft excel, Newton-Raphson method converges faster to the solution of the function compared to the other methods. Moreover, this can be seen in the number of iterations taken by each of the methods to converge to the solution. Therefore, it is clear that Newton-Raphson method is the best of the three methods employed in solving nonlinear equations and this method converges faster to the exact solution than other methods at each step of the iterations.

Acknowledgments

First of all, I wish to express my sincere thanks to Dr Tint Moe Thuzar (Rector), Dr U Khin Myo (Pro-Rector), Dr Myint Myint Oo (Pro-Rector), and Dr Khin Maw Maw Soe (Pro-Rector), Yadanabon University, for their kind permissions for the submission of this research paper. Finally, I would like to express my sincere gratitude to Dr Pho Kaung, Rector(Retired), University of Yangon, for his enthusiastic guidance and various suggestions.

References

1. Azure I., Aloliga G., Doabil L., ISSN: 2575-503X, Ghana, (2019), *Comparative Study of Numerical Methods for Solving Non-linear Equations Using Manual Computation.*
2. Curtis F. Gerald, Patrick O. Wheatley, California Polytechnic State University, (2004), *Applied Numerical Analysis*.
3. Djamila H., Faculty of Engineering, Universiti Malaysia Sabah, ICoAIMS, (2017), *Excel spreadsheet in teaching numerical methods*.
4. Ehiwario, J.C., Aghamie, S.O., Department of Mathematics, College of Education, Agbor, Delta State., ISSN (e): 2250-3021, ISSN (p): 2278-8719 Vol. 04, Issue 04, (2014), *Comparative Study of Bisection, Newton-Raphson and Secant Methods of Root- Finding Problems*.

5. Fuhrer C., Sopasakis A., FMN050, (2014), *Newton-Raphson Method, Regula Falsi Method, Secant Method, Fixed Point Method.*
6. Jinadu, Adeleke O., Federal University of Agriculture, Abeokuta, (2014), *Comparison of Newton, Secant and Fixed-Point Iteration Methods in the Solution of Non-linear Equations.*
7. Joe D. Hoffman, Department of Mechanical Engineering, Purdue University, West Lafayette, Indiana, (2001), *Numerical Methods for Engineer and Scientists*.

M. El-Gebeily, B. Yushau, ISSN 1551-3440, Vol. 4, no.1, (2007), *Numerical ethods with MS Excel*.