# Basic Concepts of the Sets and Functions

**Than Thar Yee**[1]

**Abstract**

This paper expresses the basic concepts of sets and single-valued functions. Moreover, inverses and multi-valued functions are introduced. Then transforming equations are discussed.

## Introduction

In section 1, the basic concepts of sets are expressed. And, single-valued functions with operations, properties are introduced. Moreover, $\phi$, Z, Q, R, C, $Z^+$, $Q^+$, $R^+$, $Z^{0+,}$ $Q^{0+}$ and $R^{0+}$ etc… with examples are discussed. And then, exponential functions with examples are also discussed.

In section 2, some functions such as inverse, onto, one-to-one are studied.

In section 3, the transforming equations are discussed.

1 Tutor, Department of Mathematics, Co- operative University, Sagaing

# 1 Sets and Single-Valued Functions

## 1.1 Sets

A set is a collection of objects. We call the objects, elements. A set is denoted by listing the elements between braces. For example: $\{e, i, \pi, 1\}$ is the set of the integer 1, the pure imaginary number $i = \sqrt{-1}$ and the transcendental numbers $e = 2.7182818 \ldots$ and

$\pi = 3.1415926 \ldots$ . For elements of a set, we do not count multiplicities. We regard the set $\{1,2,2,3,3,3\}$ as identical to the set $\{1,2,3\}$. Order is not significant in sets. The set $\{1,2,3\}$ is equivalent to $\{3,2,1\}$.

In enumerating the elements of a set, we use ellipses to indicate patterns. We denote the set of positive integers as $\{1,2,3, \ldots\}$. We also denote sets with the notation $\{x| \, conditions \; on \; x\}$ for sets that are more easily described than enumerated. This is read as "the set of elements x such that…".x $\epsilon$ $S$ is the notation for "x is an element of the set S". To express the opposite we have x $\notin S$ for "x is not an element of the set S".

## 1.2 Examples

We have notations for denoting some of the commonly encountered sets.

- $\emptyset = \{\ \}$ is the empty set, the set containing no elements.
- $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers.
- $Q = \{p\,/\,q|\, p, q \in Z, q \neq 0\}$ is the set of rational numbers.
- $R = \{x \mid x = a_1 a_2 \ldots a_n \,.\, b_1 b_2 \ldots\}$ is the set of real numbers,
  i.e. the set of numbers with decimal expansions.
- C= $\{a + \, ib|a, b \in R, i^2 = -1\}$ is the set of complex numbers.
  $i$ is the square root of $-1$.
- $Z^+$, $Q^+$and $R^+$are the sets of positive integers, rationals and reals, respectively.

  For example, $Z^+ = \{1,2,3, \ldots\}$.We use a – superscript to denote the sets of negative numbers.
- $Z^{0+}$, $Q^{0+}$ and $R^{0+}$ are the sets of non-negative integers, rationals and reals, respectively.
  For example , $Z^{0+} = \{0,1,2, \ldots\}$.
- (a…b) denotes an open interval on the real axis.
  (a…b) ≡ $\{x \mid x \in R, a < x < b\}$
- We use brackets to denote the closed interval.
  [a…b] ≡ $\{x \mid x \in R, a \leq x \leq b\}$

The cardinality or order of a set S is denoted |S|. For finite sets, the cardinality is the number of elements in the set. The Cartesian product of two sets is the set of ordered pairs.

$$X \times Y \equiv \{(x, y) \mid x \in X, y \in Y\}.$$

The Cartesian product of n sets is the set of ordered n-tuples.

$$X_1 \times X_2 \times \ldots \times X_n \equiv \{(x_1, x_2, \ldots, x_n) \mid x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}.$$

## 1.3 Equality and Inequality

Equality. Two sets S and T are equal if each element of S is an element of T and vice versa. This is denoted, $S = T$. Inequality is $S \neq T$, of course. S is a subset of T, $S \subseteq T$, if every element of S is an element of T. S is a proper subset of T, $S \subset T$, if $S \subseteq T$ and $S \neq T$. For example: The empty set is a subset of every set, $\emptyset \subseteq S$. The rational numbers are a proper subset of the real numbers, $Q \subset R$.

## 1.4 Operations

The union of two sets, $S \cup T$, is the set whose elements are in either of the two sets. The union of n sets,

$$\cup_{j=1}^{n} S_j \equiv S_1 \cup S_2 \cup \dots \cup S_n$$

is the set whose elements are in any of the sets $S_j$.

The intersection of two sets, $S \cap T$, is the set whose elements are in both of the two sets. In other words, the intersection of two sets is the set of elements that the two sets have in common. The intersection of n sets,

$$\cap_{j=1}^{n} S_j \equiv S_1 \cap S_2 \cap \dots \cap S_n$$

is the set whose elements are in all of the sets $S_j$. If two sets have no elements in common, $S \cap T = \emptyset$, then the sets are disjoint.

If $T \subseteq S$, then the difference between S and T, $S \setminus T$, is the set of elements in S which are not in T.

$$S \setminus T \equiv \{x \mid x \in S, x \notin T\}$$

The difference of sets is also denoted $S - T$.

## 1.5 Properties

The following properties are verified from the above definitions.

- $S \cup \emptyset = S$,
- $S \cap \emptyset = \emptyset$.
- $S \setminus \emptyset = S$,
- $S \setminus S = \emptyset$.
- Commutative - $S \cup T = T \cup S$,
  $S \cap T = T \cap S$.
- Associative - $(S \cup T) \cup U = S \cup (T \cup U) = S \cup T \cup U$,
  $(S \cap T) \cap U = S \cap (T \cap U) = S \cap T \cap U$.
- Distributive - $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$,
  $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$.

## 1.6 Single-Valued Functions

A single-valued function or single-valued mapping is a mapping of the elements $x \in X$ into elements $y \in Y$. This is expressed as $f: X \to Y$ or $X \xrightarrow{f} Y$. If such a function is well-defined, then for each $x \epsilon X$ there exists a unique element of y such that $f(x) = y$. The set X is the domain of the function, Y is the codomain. To denote the value of a function on a particular element we can use any of the notations: $f(x) = y$, $f : x \mapsto y$ or simply $x \mapsto y$. f is the identity map on X if $f(x) = x$ for all $x \epsilon X$.

Let $f : X \rightarrow Y$. The range or image of f is

$$f(X) = \{y \mid y = f(x) \text{ for some } x \epsilon X \}.$$

The range is a subset of the codomain. For each $Z \subseteq Y$, the inverse image of Z is defined.

$$f^{-1}(Z) = \{x \epsilon X \mid f(x) = z \text{ for some } z \epsilon Z\}.$$

## 1.7 Examples

- Finite polynomials, $f(x) = \sum_{k=0}^{n} a_k x^k, a_k \in R$, and the exponential function, $f(x) = e^x$, are examples of single-valued functions which map real numbers to real numbers.
- The greatest integer function, $f(x) = \lfloor x \rfloor$, is a mapping from R to Z. $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x. Likewise, the least integer function, $f(x) = \lceil x \rceil$, is the least integer greater than or equal to x.

## 1.8 The Injective, Surjective and Bijective Functions

A function is injective if for each $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. In other words, distinct elements are mapped to distinct elements. f is surjective if for each y in the codomain, there is an x such that $y = f(x)$. If a function is both injective and surjective, then it is bijective. A bijective function is also called a one-to-one mapping.

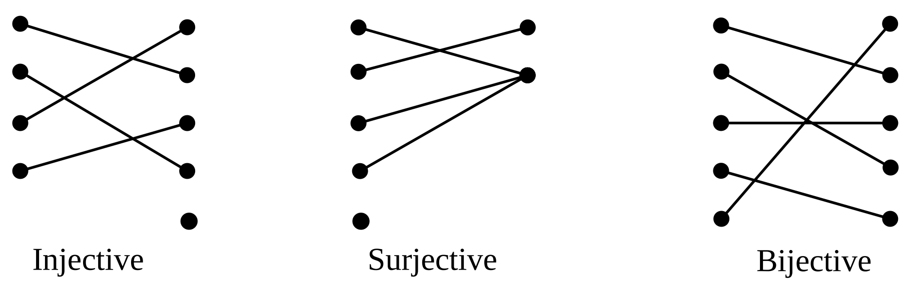


*Figure 1.1: Depictions of Injective, Surjective and Bijective Functions*

## 1.9 Examples

- The exponential function $f(x) = e^x$, considered as a mapping from R to $R^+$, is bijective, (a one-to-one mapping).
- $f(x) = x^2$ is a bijection from $R^+$ to $R^+$. f is not injective from R to $R^+$. For each positive y in the range, there are two values of x such that $y = x^2$.
- $f(x) = \sin x$ is not injective from R to $[-1,1]$. For each $y \in [-1,1]$ there exists an infinite number of values of x such that $y = \sin x$.

# 2 Inverses and Multi-Valued Functions

If y = f(x), then we can write x = $f^{-1}$(y) where $f^{-1}$ is the inverse of f. If y = f(x) is a one-to-one function, then $f^{-1}$(y) is also a one-to-one function. In this case,

x = $f^{-1}$(f(x)) = f ($f^{-1}$(x)) for values of x where both f(x) and $f^{-1}$(x) are defined. For example

ln x, which maps $R^+$ to R is the inverse of $e^x$. x = $e^{lnx}$ = ln($e^x$) for all x$\epsilon$ $R^+$.

If y = f(x) is a many-to-one function, then x = $f^{-1}$(y) is a one-to-many function. $f^{-1}$(y) is a multi-valued function. We have x = f($f^{-1}$(x)) for values of x where $f^{-1}$(x) is defined, however x ≠ $f^{-1}$(f(x)). There are diagrams showing one-to-one, many-to-one and one-to-many functions in figure (2.1).

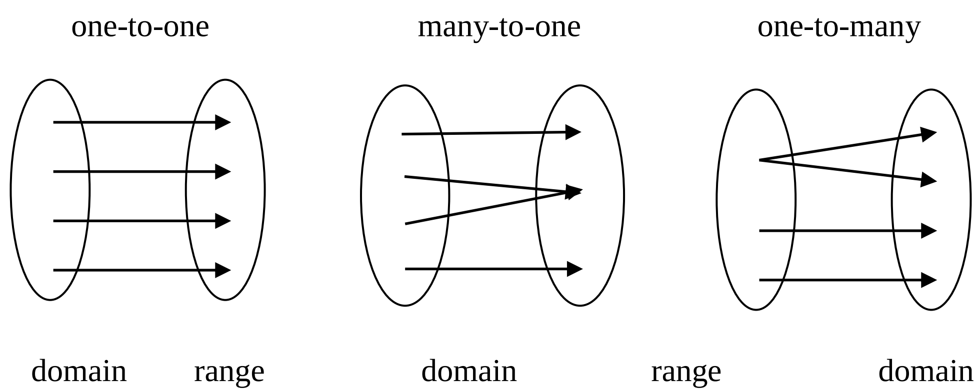


*Figure 2.1: Diagrams of one-to-one, many-to-one and one-to-many functions.*

## 2.1 Example

y = $x^2$, a many-to-one function has the inverse x = $y^{1/2}$. For each positive y, there are two values of x such that x = $y^{1/2}$. y = $x^2$ and y = $x^{1/2}$ are graphed in figure (2.2).

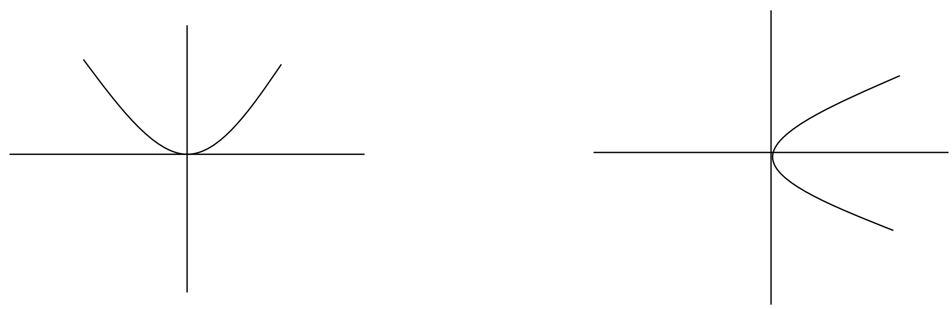

*Figure 2.2: y = $x^2$ and y = $x^{1/2}$*

We say that there are two branches of $y = x^{1/2}$: the positive and the negative branch. We denote the positive branch as $y = \sqrt{x}$; the negative branch is $y = -\sqrt{x}$. We call $\sqrt{x}$ the principal branch of $x^{1/2}$. Note that $\sqrt{x}$ is a one-to-one function. Finally, $x = (x^{1/2})^2$ since $(\pm\sqrt{x})^2 = x$, but $x \neq (x^2)^{1/2}$ since $(x^2)^{1/2} = \pm x$. $y = \sqrt{x}$ is graphed in figure (2.3).

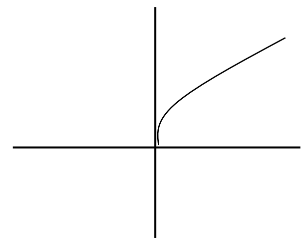

*Figure 2.3: y = $\sqrt{x}$*

Now consider the many-to-one function $y = \sin x$. The inverse is $x = \arcsin y$. For each $y \in [-1,1\,]$ there are an infinite number of values of x such that $x = \arcsin y$. In figure (2.4) is a graph of $y = \sin x$ and a graph of a few branches of $y = \arcsin x$.

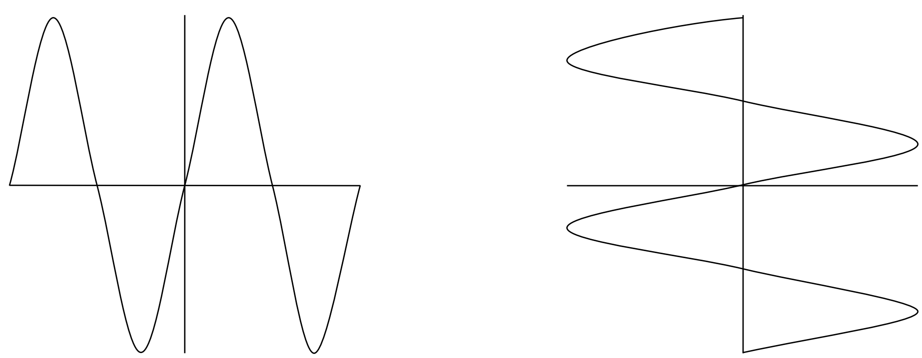

*Figure 2.4: y = sin x and y = arcsin x*

**2.2 Example**

arcsin x has an infinite number of branches. We will denote the principal branch by arcsin x which maps $[-1,1\,]$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Note that $x = \sin(\arcsin x)$, but $x \neq \arcsin(\sin x)$.

$y = \arcsin x$ in figure (2.5).

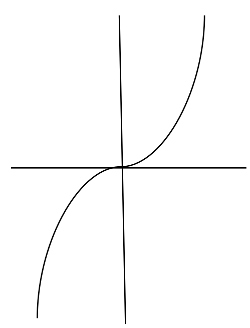

*Figure 2.5: y = arcsin x*

**2.3 Example**

Consider $1^{1/3}$. Since $x^3$ is a one-to-one function, $x^{1/3}$ is a single-valued function in figure(2.6) $1^{1/3} = 1$.

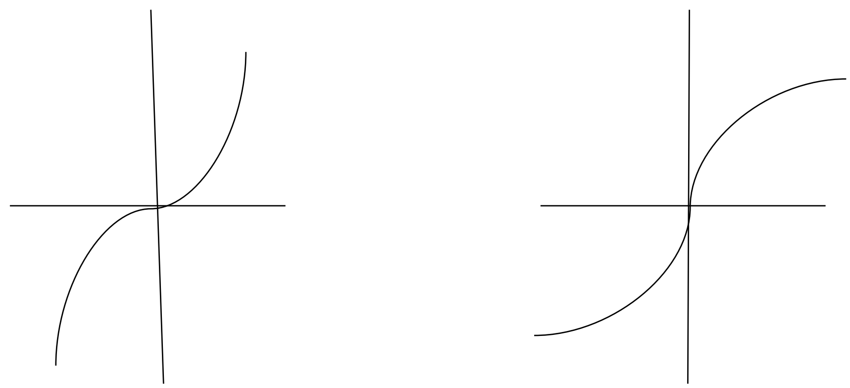

*Figure 2.6 : y = $x^3$ and y = $x^{1/3}$*

**2.4 Example**

Consider arccos $(^1/_2)$. cos x and a portion of arccos x are graphed in figure(2.7). The equation cos x $= {^1/_2}$ has the two solutions x $= \pm {^\pi/_3}$ in the range x $\in (-\pi, \pi]$. We use the periodicity of the cosine.

cos (x+$2\pi$) = cos x, to find the remaining solutions.

arccos $(^1/_2) = \{\pm {^\pi/_3} + 2n\pi\}$, n∈Z.

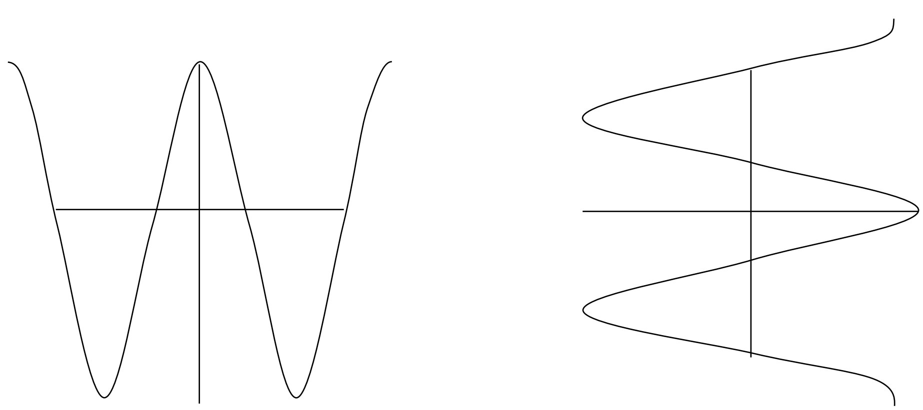

*Figure 2.7: y = cosx and y = arccos x*

# 3 Transforming Equations

Consider the equation g(x) = h(x) and the single-valued function f(x). A particular value of x is a solution of the equation if substituting that value into the equation results in an identity. In determining the solutions of an equation, we often apply functions to each side of the equation in order to simplify its form. We apply the function to obtain a second equation, f(g(x)) = f(h(x)). If $x = \xi$ is a solution of the former equation, (Let $\psi = g(\xi) = h(\xi)$), then it is necessarily a solution of latter. This is because $f(g(\xi)) = f(h(\xi))$ reduces to the identity $f(\psi) = f(\psi)$. If f(x) is bijective, then the converse is true: any solution of the latter equation is a solution of the former equation. Suppose that $x = \xi$ is a solution of the latter, $f(g(\xi)) = f(h(\xi))$. That f(x) is a one-to-one mapping implies that $g(\xi) = h(\xi)$. Thus $x = \xi$ is a solution of the former equation.

It is always safe to apply a one-to-one, (bijective), function to an equation. For example, we can apply $f(x) = x^3$ or $f(x) = e^x$, considered as mappings on R, to the equation $x = 1$. The equation $x^3 = 1$ and $e^x = e$ each have the unique solution $x = 1$ for $x \in R$.

In general, we must take care in applying functions to equations. If we apply a many-to-one function, we may introduce spurious solutions. Applying $f(x) = x^2$ to the equation $x = \pi/2$ results in $x^2 = \pi^2/4$, which has the two solutions, $x = \{\pm \pi/2\}$.

Applying f(x) = sin x results in $x^2 = \pi^2/4$, which has an infinite number of solutions,

$x = \{\pi/2 + 2n\pi \mid n \in Z\}$.

We do not generally apply a one-to-many, (multi-valued), function to both sides of an equation. Consider the equation,

$$\sin^2 x = 1$$

Applying the function $f(x) = x^{1/2}$ to the equation would not get us anywhere.

$$(sin^2 x)^{1/2} = 1^{1/2}$$

Since $(sin^2 x)^{1/2} \neq \sin x$,

we cannot simplify the left side of the equation. Instead we could use the definition of

$f(x) = x^{1/2}$ as the inverse of the $x^2$ function to obtain

$$\sin x = 1^{1/2} = \pm 1$$

Now note that we should not just apply arcsin to both sides of the equation as arcsin (sin x) $\neq x$. Instead we use the definition of arcsin as the inverse of sin.

$$x = \arcsin(\pm 1)$$

$$x = \arcsin(1)$$

has the solutions x = ${}^{\pi}/_{2} + 2n\pi$ and

$$x = \arcsin(-1)$$

has the solutions x = $-{}^{\pi}/_{2} + 2n\pi$.

We enumerate the solutions.

$$x = \{\pm {}^{\pi}/_{2} + n\pi \mid n \in Z\}$$

### 3.1 Example

Consider the equation $\frac{x+1}{y-2} = \frac{x^2-1}{y^2-4}$ .

If we multiply the equation by $y^2 - 4$ and divide by x +1, we obtain the equation of a line.

$$y + 2 = x - 1$$

We factor the quadratics on the right side of the equation.

$$\frac{x+1}{y-2} = \frac{(x+1)(x-1)}{(y-2)(y+2)}.$$

We note that one or both sides of the equation are undefined at y $= \pm 2$ because of division by zero. There are no solutions for these two values of y and we assume from this point that y $\neq \pm 2$.

We multiply by $(y - 2)(y + 2)$.

$$(x+1)(y+2) = (x+1)(x-1).$$

For x $= -1$, the equation becomes the identity $0 = 0$.

Now we consider x $\neq -1$.We divide by x +1 to obtain the equation of a line.

$$y + 2 = x - 1,$$

$$y = x - 3.$$

Now we collect the solutions we have found.

$$\{(-1,\ y\ ): y \neq \pm 2\} \cup \{(\ x, x-3): x \neq 1{,}5\}$$

The solutions are depicted in figure (3.1)

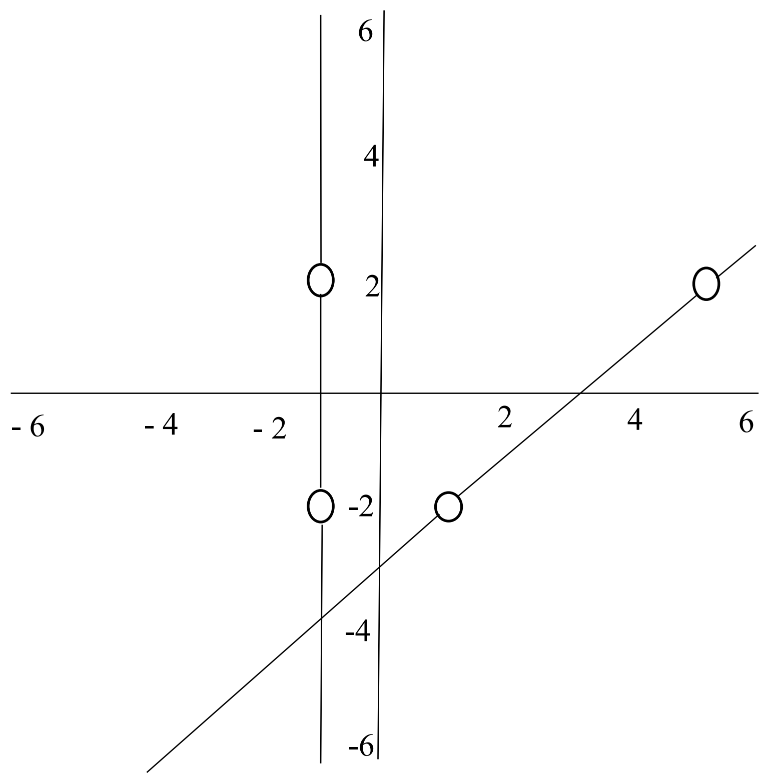


*Figure 3.1: The solutions of* $\frac{x+1}{y-2} = \frac{x^2-1}{y^2-4}$

**3.2 Example**

Consider p(x) and q(x) as general quadratic polynomials,

$$f(x) = \frac{p(x)}{q(x)}$$

$$= \frac{ax^2+bx+c}{\alpha x^2+\beta x+\chi}.$$

We will use the properties of function to solve for the unknown parameters.

p(x) and q(x) appear as a ratio, they are determined only up to a multiplicative constant. Let the leading coefficient of q(x) to be unity.

$$f(x) = \frac{p(x)}{q(x)}$$

$$= \frac{ax^2+bx+c}{x^2+\beta x+\chi}$$

f(x) has a second order zero at x = 0.

This means that p(x) has a second order zero there and that $\chi \neq 0$.

$$\mathrm{f(x)} = \frac{ax^2}{x^2+\beta x+\chi}$$

We note that f(x)→ 2 as x → ∞. This determines the parameter a.

$$\lim_{x\to\infty} \mathrm{f}(x) = \lim_{x\to\infty} \frac{ax^2}{x^2 + \beta x + \chi}$$

$$= \lim_{x\to\infty} \frac{2ax}{2x+\beta}$$

$$= \lim_{x\to\infty} \frac{2a}{2}$$

$$= \quad a$$

$$\mathrm{f}(x) = \frac{2x^2}{x^2+\beta x+\chi}$$

Now we use the fact that f(x) is even to conclude that q(x) is even and thus $\beta = 0$.

$$\mathrm{f}(x) = \frac{2x^2}{x^2+\chi}$$

Finally, we use that f(1) = 1 to determine $\chi$.

$$\mathrm{f}(x) = \frac{2x^2}{x^2+1}$$

## Conclusion

In this research paper, fundamental mathematics are known because of basic concepts of sets and functions. Sets are important everywhere in mathematics because every field of mathematics uses or refers to sets in some way. They are important for building more complex mathematical structures. Functions are widely used in science and most field of mathematics. It is very important for students and researchers by understanding basic concepts of sets and functions.

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