COMBINE TECHNIQUE FOR EVALUATION OF RATIONAL AND POLYNOMIAL FUNCTIONS

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Abstract

In this paper the evaluation of the polynomial and rational functions in digital computer system is considered and general evaluation method (E-method) is expressed. We observed combine technique, the correspondence rule in E-method and Microsoft Excel program are combined as a new technique for the evaluation of the polynomial and rational functions.

Keywords:
E-method, polynomial and rational functions

1. Introduction of the E-Method

The E-method evaluates a polynomial $P_\mu(x)$ or a rational function $R_{\mu,\nu}(x)$ by mapping it into a linear system. The system is solved using a left-to-right digit-by-digit approach, in a radix r representation system, on a regular hardware. For a result of m digits, in the range (−1, 1), the computation takes m iterations. The first component of the solution vector corresponds to the value of $P_\mu(x)$ or $R_{\mu,\nu}(x)$. Let

$$R_{\mu,\nu}(x) = \frac{P_\mu(x)}{Q_\nu(x)} = \frac{p_\mu x^\mu + p_{\mu-1} x^{\mu-1} + \dots + p_0}{q_\nu x^\nu + q_{\nu-1} x^{\nu-1} + \dots + q_1 x + 1}$$

where the p_i's and q_i's are real numbers. Let $n = \max\{\mu, \nu\}$, $p_j = 0$ for $\mu + 1 \leq j \leq n$, and $q_j = 0$ for $\nu + 1 \leq j \leq n$. According to the E-method $R_{\mu,\nu}(x)$ is mapped to a linear system L : **A y = b**:

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$$
\mathbf{L:}\quad
\begin{bmatrix}
1 & -x & 0 & & \dots & 0 \\
q_1 & 1 & -x & 0 & \dots & 0 \\
q_2 & 0 & 1 & -x & \dots & 0 \\
 & & . & . & . & . \\
\vdots & \vdots & \vdots & & \vdots & . \\
. & & & & \vdots & \\
\vdots & & & & . & 0 \\
q_{n-1} & & & & 1 & -x \\
q_n & & & \dots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_0 \\ y_1 \\ y_2 \\ \vdots \\ . \\ . \\ \vdots \\ y_{n-1} \\ y_n
\end{bmatrix}
=
\begin{bmatrix}
p_0 \\ p_1 \\ p_2 \\ \vdots \\ . \\ . \\ \vdots \\ p_{n-1} \\ p_n
\end{bmatrix}
$$

so that $y_0 = R_{\mu,\nu}(x)$. Likewise, y_0 = (x) when all $q_i = 0$.

2. Formulation of the Evaluation Method

2. 1 Introduction

In this section a general evaluation technique, named E-method, is introduced. The E-method, in general terms, can be described as

(i) A correspondence rule, C_f, which associates independent variables $\mathbf{x_f}$, dependent variable$\mathbf{y_f}$, and parameters $\mathbf{p_f}$ of a given computational problem $f(\mathbf{x_f}, \mathbf{p_f})$ with a system L of simultaneous linear equations $\mathbf{A_f\, y = b_f}$ in such a way that there is a one-one correspondence between dependent variables $\mathbf{y_f}$, i.e., the results of f, and the solution **y** of the system L. The elements of the matrix $\mathbf{A_f}$ and vector $\mathbf{b_f}$ must satisfy certain conditions, as specified later. Symbolically,

$$(C_f, \Longrightarrow \mathbf{A_f}, \mathbf{b_f}) \Rightarrow (\mathbf{y_f} \Leftrightarrow \mathbf{y} = \mathbf{A_f^{-1} b_f}) .$$

(ii) A computational algorithm for solving the system L in time linearly proportional to the desired number of correct digits of the solution **y**, and which is amenable to an efficient implementation.

A computational problem **f** is said to be L-reducible if there is a corresponding rule C_f, not necessarily unique. The E-method is applicable in all L-reducible problems: the computational algorithm remains invariant while the particular correspondence rule, no more complex than the assignment of values, characterizes the problem.

The choice of a linear system as the target of correspondence stems from an observation that the expansion of an n^{th} order determinant has the form of a sum of n! terms, each term being a product of n factors. Since the solution of a linear system L appears as the ratio of the corresponding determinants, there is an obvious potential to represent and accordingly evaluate certain general arithmetic expressions, rational functions in particular, as the ratios of determinants in expanded form.

The exposition of the E-method in this section closely follows the order in which the fundamental ideas were developed. Thus the problem of evaluating rational functions, which alone is of sufficient importance, will be used to introduce and demonstrate the correspondence part of the E-method. Its correspondence rule, C_R, will be defined in the next section while the computational algorithm will be given in Section 2.2 after discussing in some detail what appears to be the generic problem for the E-method.

2.2 Definition of Correspondence Rule

A simple way of establishing the correspondence C_R between the coefficients and the argument of a given rational function $\mathbf{R}_{\mu,\nu}$ **(x)** and a system L of simultaneous linear equations, such that the value of $\mathbf{R}_{\mu,\nu}$ is computed as the first component of the solution vector **y**, is described as the correspondence problem of the E-method.

Let $\mathbf{R}_{\mu,\nu}$ **(x)** be a real- valued rational function:

$$\mathbf{R}_{\mu,\nu}(\mathbf{x}) = \frac{P_\mu(x)}{Q_\nu(x)} = \frac{p_\mu x^\mu + p_{\mu-1} x^{\mu-1} + \dots + p_0}{q_\nu x^\nu + q_{\nu-1} x^{\nu-1} + \dots + q_1 x + 1} \quad (1)$$

$$= \frac{\sum_{i=0}^{\mu} p_i x^i}{\sum_{i=0}^{\nu} q_i x^i} \ .$$

Without loss of generality it is assumed that $q_0 = 1$.

Let

$$\mathbf{A}(x)\ \mathbf{y} = \mathbf{b} \quad (2)$$

be a nonhomogeneous system of n simultaneous linear equations,

with $\mathbf{A}(x) = (a_{ij})_{n \times n}$ - the nonsingular system coefficient matrix, (3)

$\mathbf{y} = [y_1, y_2, \dots, y_n]$ - the solution vector and (4)

$\mathbf{b} = [b_1, b_2, \dots, b_n]$ - the right-hand side vector. (5)

$$L : \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots a_{1j} \\ a_{21} & a_{22} & a_{23}\ldots\ldots a_{2j} \\ : & : & : \quad\quad : \\ : & : & : \quad\quad : \\ a_{i1} & a_{i2} & a_{i3}\ldots\ldots a_{ij} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ : \\ : \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix} . \quad (6)$$

Let D(x) denote the determinant of **A(**x**)**:

$$D(x) = \det \mathbf{A}(x). \quad (7)$$

Similarly,

$$D_j(x) = \det (\mathbf{a_1},\mathbf{a_2}, \ldots,\mathbf{a_{j-1}}, \mathbf{b}, \mathbf{a_{j+1}}, \ldots,\mathbf{a_n}) , \quad (8)$$

where $\mathbf{a_j} = (a_{1j}, a_{2j}, \ldots, a_{ij})^t$ is the j-th column vector. (9)

2.3 Theorem

If max(μ,ν) < n-1 and the coefficients a_{ij} ‘s, b_i’s of the system (10) are put into correspondence with the coefficients p_i's, q_i's and the argument x according to the following rule C_R :

$$A_{ij} = \begin{cases} 1 & \text{for } i = j; \\ q_{i-1} & \text{for } j = 1 \text{ and } i = 2, 3, \ldots,\nu+1; \\ -x & \text{for } j = i+1 \text{ and } i = 1,2,\ldots, n-1; \\ 0 & \text{otherwise}; \end{cases} \quad (10)$$

$$b_i = \begin{cases} p_{i-1} & \text{for } i = 1, 2, \ldots, \mu+1; \\ 0 & \text{otherwise} . \end{cases} \quad (11)$$

Thus

$$\begin{bmatrix} 1 & -x & 0 & & \ldots & 0 \\ q_1 & 1 & -x & 0 & \ldots & 0 \\ q_2 & 0 & 1 & -x & \ldots & 0 \\ : & : & : & : & : & : \\ : & : & : & : & & : \\ q_\nu & & & & : & \\ : & & & & . & 0 \\ : & & & & 1 & -x \\ 0 & & & \ldots. & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ . \\ . \\ : \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ : \\ . \\ p_\mu \\ : \\ : \\ 0 \end{bmatrix} ,$$

then

$$y_1(x) = \frac{D_1(x)}{D(x)} = \frac{P_\mu(x)}{Q_\nu(x)} = R_{\mu,\nu}(x)\ . \tag{12}$$

Proof:

By the Laplace expansion of the determinants

$$D(x) = \sum_{i=1}^{n} a_{i1} c_{i1}(x) \tag{13}$$

and

$$D_1(x) = \sum_{i=1}^{n} b_i\, c_{i1}(x), \tag{14}$$

where

$C_{i1}(x) = (-1)^{i+1} \det A_{i1}(x)$ is the cofactor of the element a_{i1}, and $\det A_{i1}(x)$ is its corresponding minor, defined in the usual way. In general,

$$c_{i1}(x) = \begin{cases} \prod_{k=2}^{n} a_{kk} & \text{for } i = 1; \\ (-1)^{i+1} \left[\prod_{k=1}^{i-1} a_{k,k+1} \right] \left[\prod_{k=i+1}^{n} a_{kk} \right] & \text{for } i = 2, 3, \ldots, n\ . \end{cases} \tag{15}$$

In particular,

$$c_{i1}(x) = \begin{cases} 1 & \text{for } i = 1; \\ (-x)^{i-1} & \text{for } i = 2, 3, \ldots, n\ . \end{cases} \tag{16}$$

And, since

$$(-1)^{i+1}[-x]^{i-1} = x^{i-1}, \tag{17}$$

it immediately follows that

$$\begin{aligned} D(x) &= \sum_{i=1}^{n} a_{i1} c_{i1}(x) \\ &= 1 + \sum_{i=2}^{\nu+1} q_{i-1} x^{i-1} \\ &= 1 + \textstyle\sum_{i=1}^{\nu} q_i x^i \\ &= Q_\nu(x)\ . \end{aligned} \tag{18}$$

And

$$D_1(x) = \sum_{i=1}^{n} b_i\, c_{i1}\ (x) \qquad (19)$$

$$= \sum_{i=1}^{\mu+1} p_{i-1}\ x^i$$

$$= \sum_{i=0}^{\mu} p_i\ x^i$$

$$= p_\mu\ (x).$$

Therefore,

$$y_1(x) = \frac{D_1(x)}{D(x)} = \frac{P_\mu(x)}{Q_\nu(x)} = R_{\mu,\nu}\,(x)\ . \qquad \square$$

Theorem 2.3 establishes the correspondence rule C_R so that the E-method can be applied to evaluate a given rational function $R_{\mu,\nu}\,(x)$.

Figure 1 illustrates how the system L: **A**(x)**y** = **b** appears after an initialization has been performed according to the correspondence rule C_R.

$$\begin{bmatrix} 1 & -x & 0 & & \dots & 0 \\ q_1 & 1 & -x & 0 & \dots & 0 \\ q_2 & 0 & 1 & -x & \dots & 0 \\ : & : & : & : & : & : \\ : & : & : & : & & : \\ q_v & & & & : & \\ : & & & & . & 0 \\ : & & & & 1 & -x \\ 0 & & & \dots. & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ . \\ . \\ : \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ : \\ . \\ p_\mu \\ : \\ : \\ 0 \end{bmatrix} .$$

Figure 1

It can be noted that the correspondence rule C_R is degenerate in a sense that only one of the n generated components y_i, i = 1, .., n, namely, y_1 is of interest.

2.4 Example

Let $f(x) = R_{2,3}(x) = \dfrac{p_2x^2+p_1x+p_0}{q_3x^3+q_2x^2+q_1x+1}$ be a real-valued rational function.

Then Cp: $R_{2,3}(x) \rightarrow L$ specifies

$$L: \begin{bmatrix} 1 & -x & & \\ q_1 & 1 & -x & \\ q_2 & & 1 & -x \\ q_3 & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ 0 \end{bmatrix},$$

so that $y_1 = R_{2,3}(x)$.

3. Evaluation of polynomials

The E-method, introduced by M.D. Ercegovac in [4], allows efficient evaluation of polynomials and certain rational functions on simple. Here we concentrate on the evaluation of polynomials assuming radix- 2 arithmetic.

3.1 Theorem

Consider the evaluation of $p_\mu(x) = p_\mu x^\mu + p_{\mu-1}x^{\mu-1} + \ldots + p_0$. One can easily show that $p_\mu(x)$ is equal to y_0, where $[y_0, y_1, \ldots, y_n]^t$ is the solution of the following linear system.

$$\mathbf{L:}\ \begin{bmatrix} 1 & -x & 0 & & \ldots & 0 \\ 0 & 1 & -x & 0 & \ldots & 0 \\ 0 & 0 & 1 & -x & \ldots & 0 \\ & & . & . & . & . \\ \vdots & \vdots & \vdots & & \vdots & . \\ . & & & & \vdots & \\ \vdots & & & & . & 0 \\ 0 & & & & 1 & -x \\ 0 & & & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ . \\ . \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ . \\ . \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

Then

$$y_0(x) = p_\mu(x).$$

Proof:

Seen [4].

□

3.2 Example

Evaluation of a simple polynomial $P_3(x) = 3x^3 + 4x^2 + 2x + 1$, for x = 2.

In this example we express three methods. Method (i) is simple and using substitution method. Method (ii) is combining of E-method and multiplication of matrices or Gaussian elimination backward method. Method (iii) is combining of E-Method and Microsoft Excel Computer program. Comparing the three computing method we get the same solution. We occurred that combining of E- Method and Microsoft Excel Computer program is simple and easy for evaluation of polynomial function.

Method (i).

Let $P_3(x) = 3x^3 + 4x^2 + 2x + 1$, for x = 2.

We used substitution method,

$P_3(2) = 3(2^3) + 4(2^2) + 2(1) + 1 = 45.$

Solution of polynomial $P_3(x) = 45$, when x = 2.

Method (ii).

We get linear system L: from polynomial $P_3(x) = 3x^3 + 4x^2 + 2x + 1$, by Correspondences Rule from E- method,

$$L: \begin{bmatrix} 1 & -2 & & \\ & 1 & -2 & \\ & & 1 & -2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

Then, we use multiplication of matrices method,

$$y_0 - 2\,y_1 = 1$$

$$y_0 = 1 + 2\,y_1$$

$$y_1 - 2\,y_2 = 2$$

$$y_1 = 2 + 2\,y_2$$

$$y_2 - 2\,y_3 = 4$$

$$y_2 = 4 + 2\,y_3$$

$$y_3 = 3$$

$$y_2 = 10$$

$$y_1 = 22$$

$$y_0 = 45$$

Solution of linear system L : $y_0 = 45$, by E- method.

Method (iii).

We apply to computing of polynomial using Microsoft Excel program.

Step 1: Polynomial function change to linear system L: $\mathbf{A}(x)\ \mathbf{y} = \mathbf{b}$ by Correspondence Rule from E- method.

Step 2: Using MINVERSE function from Microsoft Excel program to get $\mathbf{A}^{-1}(x)$.

Step 3: Using MMULT function from Microsoft Excel program to get solution of system L.

So, solution of polynomial $P_3(x) = 3x^3 + 4x^2 + 2x + 1$ is 45 when x = 2 by Correspondence Rule as shown in Figure 4.

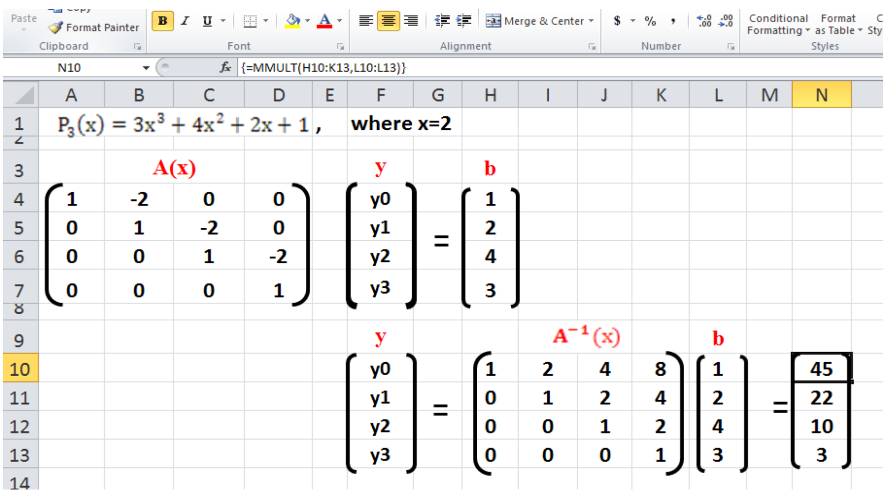

Figure 2.

3.3 Example

Evaluation of a polynomial as an approximation to 2^x, $x \in [0, 1]$, with a precision of 7 decimal digits for x = 0.5. The coefficients of $P_5(x)$ are from [6] : In this example Correspondence Rule from E- method and Microsoft Excel program are used for evaluation of 2^x with x = 0.5.

Assume, $2^x \approx P_5(x)$, for $x = 0.5$,

$$P_5(x) = P_5 x^5 + P_4 x^4 + P_3 x^3 + P_2 x^2 + P_1 x + P_0 ,$$

where.

$P_0 = 0.999999925,$

$P_1 = 0.693153073,$

$P_2 = 0.2401536l7,$

$P_3 = 0.558263130 \times 10^{-1},$

$P_4 = 0.898934003 \times 10^{-2},$

$P_5 = 0.187757667 \times 10^{-2},$

By Correspondence Rule,

$$\underbrace{\begin{pmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{A(x)}} \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} 0.99999992500 \\ 0.69315307300 \\ 0.24015361700 \\ 0.05582631300 \\ 0.00898934003 \\ 0.00187757667 \end{pmatrix}}_{\mathbf{b}},$$

then, using MINVERSE and MMULT function from Microsoft Excel program we get solution of system as following:

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} 1 & 0.5 & 0.25 & 0.125 & 0.0625 & 0.03125 \\ 0 & 1 & 0.5 & 0.25 & 0.125 & 0.0625 \\ 0 & 0 & 1 & 0.5 & 0.25 & 0.125 \\ 0 & 0 & 0 & 1 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{A^{-1}(x)}} \underbrace{\begin{pmatrix} 0.99999992500 \\ 0.69315307300 \\ 0.24015361700 \\ 0.05582631300 \\ 0.00898934003 \\ 0.00187757667 \end{pmatrix}}_{\mathbf{b}} = \begin{pmatrix} 1.41421366 \\ 0.82842747 \\ 0.27054880 \\ 0.06079037 \\ 0.00992812 \\ 0.00187757 \end{pmatrix},$$

Therefore, $2^x = y_0 = 1.41421366$ where $x = 0.5$ by Correspondence Rule from E- method.

Above Example 3.3 and other method have the same solution with 6 decimal place as shown in Table 1.

Sr	Method	2^x where $x = 0.5, (\sqrt{2})$
1	Combine Technique	1.41421366
2	E- Method	1.414213657
3	Newton's Respon	1.414213522
4	Calculator Machine	1.414213562

Table 1.

3.4 Example

In this example Correspondence Rule (E- method) and Microsoft Excel program are used for evaluation of sinh(x), x ϵ [0,1/8]. As a general example of the E-method we present the evaluation of $R_{3,4}$ as an approximation to sinh(x), x ϵ [0,1/8], with a precision of 13 decimal digits for x = 0.1019734533301 . The coefficients are taken from [6], before normalizing q_0 to 1, they appear as follows:

$P_0 = 0.0$,

$P_1 = 0.53538901456087786 * 10^3$,

$P_2 = 0.0$,

$P_3 = 0.5646207450687849 * 10^2$,

$q_0 = 0.5353890145608794*10^3$,

$q_1 = 0.0$,

$q_2 = -0.3276943311233*10^2$,

$q_3 = -0.0$,

$q_4 = 1.0$,

For x = 0.1019734533301,

Combination of Correspondence Rule and Microsoft Excel program are used in example. The complete evaluation is illustrated in Figure 3 and Figure 4 on Microsoft Excel program. We get the solution y_0 = 0.102150278, the solution of the system sinh(x) when x = 0.1019734533301.

	A	B	C	D	E	F	G	H	I
1	$\mathrm{Sinh}(x) \approx R_{3,4}$								
2									
3			A(x)				y		b
4	535.3890456	-0.1019734533301	0	0	0		y0		0.000000000000
5	0	1	-0.1019734533301	0	0		y1		535.389014560877
6	-32.7694331123	0	1	-0.1019734533301	0		y2	=	0.000000000000
7	0	0	0	1	-0.1019734533301		y3		56.462074506878
8	0.100000000000	0	0	0	1		y4		0.000000000000
9									

Figure 3.

y		$A^{-1}(x)$					b		
y0		0.00186899	0.000190587	1.94349E-05	1.98184E-06	2.02095E-07	0.000000000000		0.1021502878
y1		0.006245242	1.000636849	0.102038395	0.010405208	0.001061055	535.389014560877		536.3174760745
y2	=	0.061243804	0.006245242	1.000636849	0.102038395	0.010405208	0.000000000000	=	9.1049335227
y3		-1.90587E-05	-1.94349E-06	-1.98184E-07	0.99999998	0.101973451	56.462074506878		56.4610328451
y4		-0.000186899	-1.90587E-05	-1.94349E-06	-1.98184E-07	0.99999998	0.000000000000		-0.0102150288

Figure 4.

Conclusion

In Section1 and Section 2, we expressed E-method for evaluation of rational and polynomial functions with examples. We discovered new combined technique using combination of Correspondence Rule from E-method and Microsoft Excel program for evaluation of polynomial and rational functions with approximation. Example 3.3 and Example 3.4 are used as a combined technique, these examples ware visible distinctly and simple for checking solution with other evaluation methods.

We compared the solutions of Example 3.3 in Table (1) with other evaluation method. Combine technique had the same solution with 6 decimal places. So combine technique was easy way method than other evaluation method, but this technique was semi computerize technique.

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