Study of Two-Nucleon Bound State using Phenomenological Potentials

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Abstract

The bound state properties for two-nucleon system are studied using phenomenological potentials. Lippmann-Schwinger equation is solved to study the binding energy. Firstly, Yukawa potential is used. Then Yamaguchi potential is used to study binding energy numerically and analytically. Our results of binding energy with Yukawa and Yamaguchi potential are -2.221968MeV and -2.191987MeV respectively. In studying the binding energy of two-nucleon stated with Yukawa potential, it gives the result which agrees with the experimental value. But, binding energy with Yamaguchi potential is slightly less than the experimental value.

Keywords: Lippmann-Schwinger equation, Yukawa potential, Yamaguchi potential

Introduction

Diplon, deuton, deuteron (Van Orden, 2001): under different names, the nucleus of deuterium, or diplogen, has been the subject of intense studies since its discovery in 1932. As the only two-nucleon bound state, its properties have continuously been viewed as important in nuclear theory as the hydrogen atom is in atomic theory. The existence of the first isotope of hydrogen was suggested in 1931 by Birge and Menzel (Birge R.T., Menzel D.H, 1931). The stable isotope was discovered by Urey and collaborators (Urey H.C. and Murphy G. N., 1932) a few months later.

The standard approach in conventional nuclear theory is to treat the two-nucleon interaction in a phenomenological manner, that is, assume it to be derivable from a potential, and then set up a reasonable form for it in agreement with general theoretical considerations and experimental data. Most experimental data providing direct information on the potential concern the two-nucleon systems: neutron-proton (n-p), proton-proton (p-p), and neutron-neutron (n-n). The n-p system produces a bound state, the ${}^{2}_{1}H$ nucleus, whereas no bound state exists in the p-p and n-n systems. The study of the nuclear two-body problem provides the requisite information on two-nucleon forces. In this paper we will present how to handle the two–nucleon bound state with phenomenological potentials.

Two-body bound system

In studying the two-body system in momentum space we will start with time-independent Schrödinger equation.

$$\hat{H}|\psi\rangle = E\,|\psi\rangle \qquad (1)$$

$$\left(\hat{H}_0 + \hat{V}\right)|\psi\rangle = E\,|\psi\rangle \qquad (2)$$

where, $\hat{H}_0$ is the kinetic energy operator and $\hat{V}$ is the potential energy operator. The kinetic energy operator has the property

$$\hat{H}_0|p\rangle = \frac{\hat{p}^2}{2\mu}|p\rangle = \frac{p^2}{2\mu}|p\rangle \qquad (3)$$

The solution of Schrödinger equation can be written as

$$|\psi\rangle = \frac{1}{E-\hat{H}_0}\,\hat{V}\,|\psi\rangle \qquad (4)$$

This is the Lippmann-Schwinger equation in ket form and then we project it into momentum space as

$$\langle p\ell m|\psi\rangle = \left\langle p\ell m\left|\frac{1}{E-\hat{H}_0}\,\hat{V}\,\right|\psi\right\rangle \qquad (5)$$

where, μ is reduced mass, p is the relative momentum between two particles, ℓ is the relative orbital angular momentum and m is the magnetic quantum number.

When operator $\hat{H}_0$ operates on, $\langle p\ell m|$ the above equation becomes

$$\langle p\ell m|\psi\rangle = \left\langle p\ell m\left|\frac{1}{E-\frac{p^2}{2\mu}}\,\hat{V}\,\right|\psi\right\rangle \qquad (6)$$

After inserting the completeness relation,

$$\langle p\ell m|\psi\rangle = \frac{1}{E-\frac{p^2}{2\mu}}\sum_{\ell' m'}\int p'^2\,dp'\langle p\ell m|\,V\,|p'\ell' m'\rangle$$

$$\langle p'\ell' m'|\psi\rangle \qquad (7)$$

For spherically symmetric potential, we can write as

$$\psi(p) = \frac{1}{E-\frac{p^2}{2\mu}}\sum_{\ell'}\int_0^\infty p'^2 dp'\,V_{\ell\ell'}(pp')\psi_{\ell'}(p') \qquad (8)$$

This is known as the Lippmann-Schwinger equation for two particles in potential well.

To write a FORTRAN code, we need to transform this integral equation to discrete form by using Gauss-Legendre integration method which is

$$\int f(x)dx = \sum_{i=1}^{n} f(x_i)W_i \qquad (9)$$

where, x_i is gauss point and w_j is gauss weight.

In discrete form, the Lippmann-Schwinger equation becomes

$$\psi_\ell(p_i) = \frac{1}{E-\frac{p_i^2}{2\mu}} \sum_{\ell'} \sum_j p_j^2 \, V_\ell(p_i, p_j) \psi_\ell(p_j) \quad (10)$$

The above equation can be solved numerically by using the FORTRAN 90 code.

The iterative method is applied to find the eigen value in our calculation so the kernel of Eq. (10) is constructed as a function of energy as

$$K_{\ell,\ell',i,j}(E) = \frac{1}{E-\frac{p_i^2}{2\mu}} \sum_{\ell'} \sum_j p_j^2 \, w_j V_\ell(p_i, p_j) \quad (11)$$

As a compact form for pure S-wave, Eq. (10) becomes

$$\psi(p_i) = K_{ij}(E)\psi(p_j) \quad (12)$$

Arbitrary value which is function of E is introduced as

$$\eta(E)\psi(p_i) = K_{ij}(E)\psi(p_j) \quad (13)$$

When $\eta(E)$ is equal to 1, one can see Eq. (12) and Eq. (13) are the same. Therefore our background idea is that we will find the energy E which can give $\eta(E)$ value 1.

Potential matrix elements in momentum space

The phenomenological potentials for two-body system are mostly represented in configuration space. So we must have to transform these potentials from configuration space to momentum space.
The potential matrix elements can be written as

$$V_{\ell\ell'}(p,p') = \langle p\ell m|V|p'\ell' m'\rangle \quad (14)$$

The potential matrix elements in configuration space and momentum space are related as follows.

$$\langle p\ell m|V|p'\ell' m'\rangle = \int r^2 dr \int r'^2 dr' \sum_{\ell'' m''} \sum_{\ell''' m'''} \langle p\ell m|r\ell'' m''\rangle \langle r\ell'' m''|V|r'\ell''' m'''\rangle\langle r'\ell''' m'''|p'\ell' m'\rangle \quad (15)$$

where,

$$\langle p\ell m|r\ell' m'\rangle = \sqrt{\frac{2}{\pi}}\, j_\ell(pr) i^\ell \delta_{\ell\ell'}\delta_{mm'} \quad (16)$$

For the spherically symmetric potential and if the tensor force is ignored,

$$\langle r\ell m|V|r'\ell' m'\rangle = V_\ell(r)\frac{\delta(r-r')}{rr'}\delta_{\ell\ell'}\delta_{mm'}$$

Therefore, the transformed potential matrix element becomes

$$V_\ell(p.p') = \frac{2}{\pi}\int r^2 dr\, j_\ell(pr) i^{2\ell} j_\ell(p'r) V_\ell(r) \quad (17)$$

For the particular case $\ell = 0$,

$$V(p,p') = \frac{2}{\pi pp'} \int dr \left(\frac{e^{ipr} - e^{-ipr}}{2i}\right) \left(\frac{e^{ip'r} - e^{-ip'r}}{2i}\right) V(r) \quad (18)$$

Again, Eq. (18) can be rearranged and the result is

$$V(p,p') = \frac{1}{2\pi pp'} \int [e^{i(p+p')r} - e^{i(p-p')r} - e^{-i(p-p')r} + e^{-i(p+p')r}]\, V(r) dr \quad (19)$$

We define A = p + p' and B = p - p' and we get

$$V(p,p') = \frac{1}{2\pi pp'} \int_0^\infty [e^{iAr} - e^{iBr} - e^{-iBr} + e^{-iAr}]\, V(r) dr \quad (20)$$

This is the transformation of potential matrix elements from configuration space to momentum space for S-wave.

Yukawa potential in momentum space

The Yukawa potential in coordinate space is

$$V(r) = V_0 \frac{e^{-br}}{r} \quad (21)$$

where, V_0 is the potential strength or the depth parameter and b is the range parameter. To write a FORTRAN code, we transform the Yukawa potential into momentum space by using the Eq. (20) as

$$V_0(p,p') = \frac{V_0}{2\pi pp'} ln\left[\frac{b^2+A^2}{b^2+B^2}\right] \quad (22)$$

Observable quantities of bound system

After solving the Eq. (13) numerically, we will obtain the ground state energy and the corresponding wave function. The normalized wave function is

$$\psi_\ell^{nol}(p) = \frac{1}{\sqrt{\sum_\ell \int \psi_\ell^*(p)\psi_\ell(p)p^2 dp}}\, \psi_\ell^{un-nol}(p) \quad (23)$$

To check the quality of wave function, we will calculate $K.E$ and $P.E$ by using normalized wave function for S-wave only such that

$$K.E = \frac{1}{2\mu}\int \psi_0^*(p)\psi_0(p)p^4 dp \quad (24)$$

$$P.E = \int p^2\, dp \int p'^2 dp'\, \psi_0^*(p) V(p,p') \psi_0(p') \quad (25)$$

By adding kinetic energy and potential energy, we may obtain the binding energy as

$$B.E = K.E + P.E \quad (26)$$

Next to calculate the root mean square distance of two-body system, we will transform the wave function from momentum space to coordinate space.

The wave function in coordinate space is

$$\psi_{\ell m}(r) = \langle r\ell m|\psi\rangle \quad (27)$$

We insert the completeness relation and we get

$$\psi_{\ell m}(r) = \sum_{\ell' m'} \int p^2\, dp\, \langle r\ell m|p'\ell' m'\rangle \psi_{\ell' m'}(p) \quad (28)$$

Using the Eq. (16), the above equation becomes

$$\psi_{\ell m}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp\, j_\ell(pr) i^{-\ell} \psi_{\ell' m'}(p) \quad (29)$$

Then, the wave function μ_ℓ can be defined as

$$u_\ell = r\psi_\ell(r) \quad (30)$$

For normalized s-wave function in coordinate space,

$$\int u_0^2(r) dr = 1 \quad (31)$$

$$\langle r^2 \rangle = \int r^2 u_0^2(r) dr \quad (32)$$

From this equation we can calculate the root mean square distance of two-body system.

$$R_{rms} = \frac{1}{2}\{\int r^2 [u_0^2(r)] dr\}^{1/2} \quad (33)$$

Yamaguchi potential in momentum space

The Yamaguchi potential is generally of the following form as

$$V = |g\rangle \lambda \langle g| \quad (34)$$

When we project it on momentum space,

$$\langle p|V|p'\rangle = \langle p|g\rangle \lambda \langle g|p'\rangle \quad (35)$$

In usual form as

$$V(p,p') = \lambda g(p) g(p') \quad (36)$$

where, g (p) is a function dependent on momentum and for a single channel it has the following form as

$$g(p) = \frac{1}{\beta^2 + p^2} \quad (37)$$

where, λ and β are constants.

Analytical calculation for two-body bound state

The two-body bound state with Yamaguchi potential can be solved analytically. The Lippmann Schwinger equation for two-body system in a single channel which can be written with Yamaguchi potential as

$$\psi(p) = \frac{1}{E - \frac{p^2}{\mu}} \int_0^\infty p'^2\, dp' \lambda g(p) g(p') \psi(p') \quad (38)$$

The analytical solution of wave function can be written as,

$$\psi(p) = C \frac{1}{E - \frac{p^2}{\mu}} g(p) \quad (39)$$

where, $C = \lambda \int_0^\infty p'^2\, dp' g(p') \psi(p')$ and one can write easily as

$$C = \lambda \int_0^\infty p'^2\, dp' g(p') C \frac{1}{E - \frac{p'^2}{\mu}} g(p') \quad (40)$$

We set E as

$$E = -\frac{\alpha^2}{\mu} \quad (41)$$

$$C = \lambda \int_0^\infty p'^2\, dp' g^2(p') C \frac{1}{-\frac{\alpha^2}{\mu} - \frac{p'^2}{\mu}} \quad (42)$$

$$C = -\frac{1}{2}\lambda\mu C \int_{-\infty}^\infty p'^2\, dp' \frac{1}{(p'^2+\beta^2)^2} \frac{1}{(p'^2+\alpha^2)} \quad (43)$$

The above equation can be written as

$$C = \frac{1}{4\beta}\lambda\mu C \frac{\partial}{\partial\beta} \int_{-\infty}^\infty p'^2\, dp' \frac{1}{(p'^2+\beta^2)} \frac{1}{(p'^2+\alpha^2)} \quad (44)$$

The pole appear at $p' = \pm i\beta$ and $p' = \pm i\alpha$. Therefore Residue theorem is used as,

$$Res(+i\beta) = \frac{i\beta}{2(\alpha^2 - \beta^2)} \quad (45)$$

$$Res(+i\alpha) = -\frac{i\alpha}{2(\alpha^2 - \beta^2)} \quad (46)$$

$$\int_{-\infty}^\infty p'^2\, dp' \frac{1}{(p'^2+\beta^2)} \frac{1}{(p'^2+\alpha^2)} = 2\pi i \big(Res(+i\beta) + Res(+i\alpha)\big) \quad (47)$$

$$\int_{-\infty}^\infty p'^2\, dp' \frac{1}{(p'^2+\beta^2)} \frac{1}{(p'^2+\alpha^2)} = \frac{\pi}{\alpha+\beta} \quad (48)$$

The Eq. (44) becomes as

$$C = \lambda\mu C \frac{1}{4\beta} \frac{\partial}{\partial\beta} \frac{\pi}{\alpha+\beta} \quad (49)$$

It can be rearranged as

$$C = -\frac{\lambda\mu\pi}{4\beta(\alpha+\beta)^2} C \quad (50)$$

The intermediate step is

$$1 = -\frac{\lambda\mu\pi}{4\beta(\alpha+\beta)^2} \quad (51)$$

Finally we get

$$\alpha = -\beta + \sqrt{-\frac{\lambda\mu\pi}{4\beta}} \quad (52)$$

The value of α is calculated by using the above equation in which $\beta =$ 1.13 fm^{-1}, $\lambda =$ -0.5592 fm^{-2} and the mass of nucleon is taken as 938.903 MeV, but the calculation are working in momentum space so it is needed to change MeV to fm^{-1} by dividing ħc = 197.3286 MeVfm. If the values of α has been known, then the binding energy in fm^{-1} can be obtained with the help of Eq. (41) and multiplied by $\hbar c$. The binding energy will be expressed in MeV unit. By inserting the values of β, λ and μ in Eq. (52), the value of α is 0.229892 fm^{-2}. Then this value is inserted in Eq. (41) and the value of binding energy for two-nucleon state is -0.011107 fm^{-1} (-2.191844) MeV.

Results and discussion

In studying two-body bound state, we have used the iterative method to find binding energy of deuteron. The detail discussion on iterative method and energy search program will appear in the next coming up papers. We solved the Lippmann-

Schwinger equation in integral form. To write a FORTRAN code, we transformed this integral equation to discrete form by using Gauss-Legendre integration method. We define the integral range that the initial point is p_0, the midpoint is p_{mid}, the maximum point is $p_{max,}$ and the number of grid point is Np. We let p_0=0.0fm^{-1}, p_{mid}=5fm^{-1}, p_{max}=10fm^{-1}and the number of grid point Np=20 for our program.

The Eq. (13) is solved by using iterative method. If 'E' is the ground state energy, the η value will be the physical eigen value 1. Since energy eigen value is not known, we put an arbitrary value for initial wave function and initial energy, 'E'. First, we initialized the wave function arbitrarily that is all elements of wave function to be one. For the first iteration, the new wave function was obtained. This new wave function was used as initial wave function in next iteration. We find the the convergence of η value.

Again, we find the values of η by varying input energies from -0.1MeV to -3.0MeV as shown in Table (1) and characteristic graph of η and E is shown in Fig. (1). From this table we have programmed for energy search. The η value will be 1 at certain energy between -2.1MeV and -2.3MeV. Finally, converged binding energy of deuteron is -2.221968 MeV at Np=70. In this calculation, two ranges Yukawa type potential with V_{a0} = -557.0MeV fm, μ_a =1.5fm^{-1}, V_{r0} = +1271.002 MeV fm and μ_r =3.1fm^{-1} have been used. When the Lippmann-Schwinger equation has been solved by using iterative method, the binding energy of deuteron in the ground state and the corresponding un-normalized wave function in momentum space are obtained. Then, we confirm our numerical wave function in momentum space by reproducing the binding energy. Therefore, the kinetic and potential energies are calculated by using our normalized numerical wave function and then the binding energy of deuteron is checked by adding these values. The two results are exactly equal at -2.221968MeV as shown in Table (2).

The binding energy of deuteron with Yamaguchi potential are also calculated numerically as well as analytically. In calculating numerically, we choose p_0=0.0fm^{-1}, p_{mid}=5fm^{-1}, p_{max}=10fm^{-1} and the binding energy is converged to 2.180709MeV at Np=60. Then we increase p_{max} by 10fm^{-1} until we reach to p_{max}=80.0 fm^{-1}. Our numerical value does not change anymore. Finally, we get the data set p_0=0.0fm^{-1}, p_{mid}=5fm^{-1}, p_{max}=50fm^{-1} and Np=60 giving numerical stability of the binding energy at -2.191987MeV as shown in Table (3) which agrees with our analytical result -2.191844MeV up to three decimal places.

Then, the numerical wave function in momentum space is transformed into coordinate space. This wave function is used in calculating root mean square radius. The result of root mean square radius with Yamaguchi potential is about 2.04fm. The graphs of wave function in momentum and configuration spaces are shown in Fig. (2) and Fig. (3).

Conclusion

In this paper the two-nucleon bound state equation in momentum space has been presented. Then the transformation of potential from coordinate space to momentum space has been expressed. In this case it is focused on Yukawa potential only.

It is found that the binding energy of two-nucleon system with Yukawa type potential numerically. Our result of binding energy with Yukawa type potential is -2.221968MeV and value of root mean square radius is 1.992fm. With Yamaguchi potential, the binding energy of two-nucleon system are calculated numerically and analytically. These two results agree well up to three decimal places.

The experimental values of binding energy and root mean square radius for deuteron are -2.224575 MeV and 1.971 fm. In studying the binding energy of two-nucleon stated with Yukawa potential, it gives the result which agrees with the experimental value. But, the binding energy with Yamaguchi potential is slightly less than the experimental value. As a result, our result of the root mean square radius, 2.04 fm is slightly greater than the experimental value. To reproduce the experimental value with Yamaguchi potential, one can adjust the value of λ as a pleasurable numerical exercise.

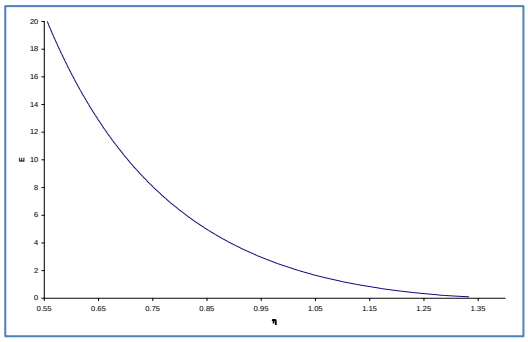

Figure 1. Characteristic graph of η and E.

Table 1. The physical eigen value $\eta(E)$ as a function of E

E (MeV)	η (E)
-0.1	1.33350271
-0.5	1.20925279
-1.0	1.12553967
-1.5	1.06593639
-2.0	1.01918327
-2.5	0.98055972
-3.0	0.94757186

Table 2. Average values of K.E, P.E and B.E with Yukawa potential.

KE (MeV)	10.712495
PE (MeV)	-12.934463
BE (MeV)	-2.221968
BE (MeV) from our iterative method	-2.221968

Table (3) Average values of K.E, P.E and B.E with Yamaguchi potential.

KE (MeV)	10.773908
PE (MeV)	-12.965895
BE (MeV)	-2.191987
BE (MeV) from our iterative method	-2.191987

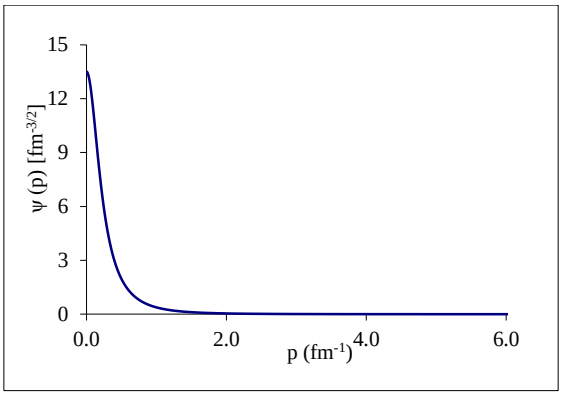

Figure 2. The wave function of deuteron in momentum space.

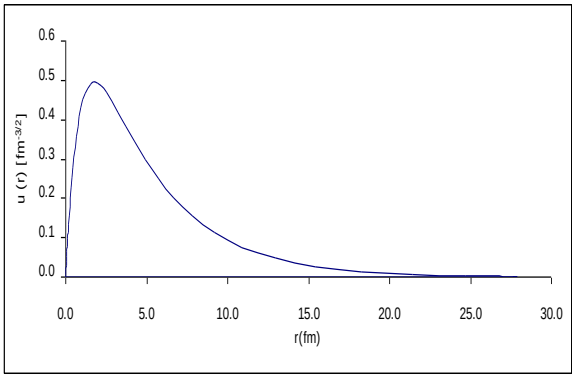

Figure 3. The wave function of deuteron in coordinate space.

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