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Application of Finite Element Method to Fluid Flow

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Abstract- **Applications of the finite element method to a restricted class of problems in potential flow have required the availability of an associated variational principle. Incompressible flow under prescribed pressure fields and compressible flow in which the continuity equation is implicitly satisfied and the fluid density is known as a function of time. As such, they do not represent completely general models of general fluid flow or of the Navier-Stokes equations. It is the purpose herein to present brief derivations of the finite element equations describing a discrete model of compressible and incompressible Stokesian fluids.**

Keywords—Linearized Compressible Flow, Finite Element Method, Fluid Flow

I. Introduction

Let us consider isothermal motion of an arbitrary fluid. If the continuity equation and the principle of balance of linear momentum is satisfied, then a global form of the law of conservation of energy can be written [6]

$$\int_v \rho \frac{Dv_i}{Dt} v_i dv + \int_v t_{ij} d_{ij} dv = \Omega \tag{1}$$

in which ρ = the mass density; v_i = component of velocity field; t_{ij}= Cauchy stress tensor; d_{ij}= the rate of deformation tensor and Ω = the mechanical power of external forces;

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_m v_{i,m} \tag{2}$$

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{3}$$

$$\Omega = \int_v \mathrm{F_i v_i dv} + \int_A \mathrm{S_i v_i dA} \tag{4}$$

in which F_i and S_i are body and surface forces; and the comma denotes partial differentiation with respect to a fixed system of spatial Cartesian coordinates x_i. In addition to equation (1), it is required that the continuity equation

$$\frac{\partial \rho}{\partial t} + (\rho u_k)_{,k} = 0 \tag{5}$$

must be satisfied at every point in the continuum.

II. Finite Element Method of Fluid Flow

Now construct a finite element model of the region R through which the fluid flows. This consists of a collection of a finite number of connected subregions R_e called finite elements, which are generally assumed to be of some relatively simple geometric shape. A number of nodal points are identified in and on the boundaries of each element, so that the whole assembly is viewed as being connected together at various boundary nodes. A typical finite element is isolated and the flow of fluid through it independent of the other elements is considered.

Let ρ and v_i denote the density and velocity fields associated with a typical element e. Proceed by constructing local approximations of these fields over the element which are uniquely determined by the values of ρ and v_i at the node points of the element:

$$\rho_{(e)} = \psi^N(x)\rho_N^{(e)} \ and\, v_{i(e)} = \psi^N(x) v_{Ni}^{(e)} \tag{6}$$

in which $\rho_N^{(e)}$ and $v_{Ni}^{(e)}$ are the values of the local fields $\rho_{(e)}$ and $v_{i(e)}$ at node N of element e. The repeated nodal indices N in Equation (6) are to be summed from 1 to N_e, in which N_e is the total number of nodes of element e. The local interpolation functions $\psi^N(x)$ are generally selected so that ρ and v_i are continuous across interelement boundaries once the elements have been connected to form the complete discrete model. Procedures for connecting elements together and applying boundary conditions are well-documented and will not be described herein. The functions $\psi^N(x)$ also have the properties

$$\psi^N(x_M) = \delta_M^N = \sum_{N=1}^{N_e} \psi^N(x) = 1$$

in which δ_M^N is the Kronecker delta; and $x_M = x_{Mi}$ is the coordinates of node M of the element. Substituting equation (6) into equation (1) and simplifying

$$v_{Ni}(a^{MQN}\rho_M \dot{v}_{Qi} + b_m^{MQRN}\rho_M v_{Qm} v_{Ri} + \int_v t_{ij}\psi_j^N dv - p_i^N) = 0 \tag{7}$$

is obtained in which Ve is the volume of the element p_i^N is the component of generalized force at node N; and a^{MQN}, b_m^{MQRN} are multidimensional arrays:

$$a^{MQN} = \int_{Ve} \psi^M(x)\psi^Q(x)\psi^N(x)dv \tag{8}$$

$$b_m^{MQRN} = \int_{Ve} \psi^M(x)\psi^Q(x)\psi_{,m}^R(x)\psi^N(x)dv \tag{9}$$

$$p_i^N = \int_{Ve} F_i\psi^N(x)dv + \int_{Ae} S_i\psi^N(x)dA \tag{10}$$

where M, N, Q, R=1,2,……, Ne and i, j, m=1,2,3 and all repeated indices are summed.

As equation (1) and consequently equation (7), must hold for arbitrary continuous velocity fields, the term in parentheses in equation (7) must vanish. Thus, for the equations of motion for a typical fluid element the system of nonlinear equations is obtained:

$$a^{MQN}\rho_M \dot{v}_{Qi} + b_m^{MQRN}\rho_M v_{QM} v_{Ri} + \int_{Ve} t_{ij}\psi_{,j}^N dv = p_i^N \qquad (11)$$

This result applies to arbitrary fluids as the form of the constitutive equation for stress is, as yet, unspecified. By following a similar procedure, a finite element model of equation (2), the continuity equation, is also obtained;

$$c^{NM}\dot{\rho}_M + d_k^{NMR}\rho_R v_{Mk} = 0 \qquad (12)$$

in which $c^{MN} = \int_{Ve} \psi^M(x)\psi^N(x)dv \qquad (13)$

$$d_k^{NMR} = \int_{Ve} \psi^N(x)\left[\psi^M(x)\psi^R(x)\right]_{,k} dv \qquad (14)$$

in which n_k = the components of a unit vector normal to the bounding surface area of the element Ae.

III. Compressible Stokesian Fluids

For adiabatic of compressible Stokesian fluids, the stress tensor t_{ij} is of the form

$$t_{ij} = \left[-\pi(\rho^{-1},\theta) + \lambda_v d_{kk}\right]\delta_{ij} + 2\mu d_{ij} \qquad (15)$$

in which π is the thermodynamic pressure which must be given by an equation of state for the fluid; θ is the absolute temperature; λ_v and μ_v and are the dilatational and shear viscosities, respectively.

The tensor d_{ij} for the finite element is obtained in terms of the nodal velocities by introducing equation (6) into equation (3). If equation (15) is then incorporated into equation (11), the finite equations for compressible Stokesian fluids are obtained:

$$a^{MQN}\rho_M \dot{v}_{Qi} + b_m^{MQRN}\rho_M v_{Qm} v_{Ri} + \int\left[(\lambda_v \psi_{,k}^M v_{Mk} - \pi)\delta_{ij} + 2\mu_v \psi_{,j}^M v_{Mj}\right]\psi_{,j}^N dv = p_i^N \qquad (16)$$

The finite element analogue of the continuity equation (12) must be added.

IV. Invompressible Stokesian Fluids

In the case of incompressible fluids, π becomes the hydrostatics pressure $\overline{p}$; the density ρ is a constant; and the incompressibility condition

$$d_{kk} = 0 \qquad (17)$$

must be satisfied. Then equation (15) reduces to

$$t_{ij} = -\overline{p}\delta_{ij} + 2\mu_v d_{ij} \qquad (18)$$

and the equation of motion for an element becomes

$$m^{NM}\dot{v}_{Mi} + e_m^{NMR} v_{Mm} v_{Ri} + \int_{Ve} \psi_{,j}^N (2\mu_v \psi_{,j}^M v_{Mi} - p\delta_{ij})dv = p_i^N \qquad (19)$$

in which m^{NM} and e_m^{NMR} are the mass and convected mass matrices for the element

$$m^{NM} = \int_{Ve} \rho\psi^N(x)\psi^M(x)dv \qquad (20)$$

$$e_m^{NMR} = \int_{Ve} \rho\psi^N(x)\psi^M(x)\psi_{,m}^R(x)dv \qquad (21)$$

Although equation (12) is now implicity satisfied, equation (19) represents $3Ne$ nonlinear differential equations in the $3Ne+1$ unknown nodal velocities v_{Ni} and the uniform element hydrostatic pressure p. To complete the system, an additional equation is needed. This is furnished by the incompressibility condition equation (17), which, for the finite element, is satisfied in an average sense by

$$\int_{Ve} d_{kk}dv = v_{Nk}\int_{Ve} \psi_{,k}^N(x)dv = 0 \qquad (22)$$

Equations (19) and (22) complete the description of motion of a finite element of an incompressible Stokesian fluid.

V. Example of One Dimensional Forms

Although a detailed exploration of results obtained using these equations is not within the scope presented herein, it is informative to examine the forms of nonlinear equations for a sample one-dimensional case. Consider the case of one-dimensional compressible flow through a typical finite element of unit length. In this case, if x is a local coordinate, a first approximation is

$$V = a_1 + a_2 x = \begin{bmatrix}1 & x\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = f(x)^t[a]$$

If nodal velocity v_1 and v_2 are needed at $x=0$ and $x=1$. Then

$$\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = G[a].$$

Therefore $[a] = G^{-1}[V]$,

$$V = f(x)^t G^{-1}[V] = (1-x)v_1 + xv_2.$$

Then shape functions are

$$\psi^1(x) = 1-x \quad and \quad \psi^2(x) = x$$

and (7) becomes

$$\frac{3}{12}v_1\dot{v}_1\rho_1 + \frac{1}{12}v_1\dot{v}_1\rho_2 + \frac{1}{12}v_1\dot{v}_2\rho_1 + \frac{1}{12}v_1\dot{v}_2\rho_2 + \frac{1}{12}v_2\dot{v}_1\rho_1 + \frac{1}{12}v_2\dot{v}_1\rho_2 + \frac{1}{12}v_2\dot{v}_2\rho_1 + \frac{1}{4}v_2\dot{v}_2\rho_2 - \frac{3}{12}v_1^3\rho_1 - \frac{1}{12}v_1^3\rho_2 - \frac{1}{12}v_1^2 v_2\rho_1 - \frac{1}{12}v_1^2 v_2\rho_2 + \frac{1}{12}v_1 v_2^2\rho_1 + \frac{1}{12}v_1 v_2^2\rho_2 + \frac{1}{12}v_2^3\rho_1 + \frac{1}{4}v_2^3\rho_2 + \int_0^1 t_{ij}(-v_1+v_2)dx = 0$$

and continuity equation becomes

$$\rho(\dot{v}_{i-1}+4\dot{v}_i+\dot{v}_{i+1})+\rho(v_{i+1}^2+v_{i-1}^2)+\rho(v_{i+1}+v_{i-1})v_i+ \\ 6(\lambda_N+2\mu_N)(v_{i+1}-v_{i-1})-6\pi_i=0 \quad (33)$$

while the continuity equation becomes

$$v_{i+1}-v_{i-1}=0 \quad (34)$$

This, in this case, is identical to the central difference approximation of the incompressibility condition equation (17).

VI. Conclusion

For compressible and incompressible Stokesian fluids, finite element equation describing the motion of the fluids may be developed without resorting to variational principles by considering energy balances over an element. These involve systems of non-linear ordinary differential equations in the nodal velocities V_{Ni} and, for compressible fluids, the nodal densities ρ_N. As the interpolation functions used in the local approximations can generally be designed so that the velocity and density converge in the mean to continuous velocity and density distributions, and since the formulation amounts to enforcing exact satisfaction of the principle of conservation of energy for each element, it is reasonable to expect that the proposed finite element model is a very good approximation of the continuum.

In the case of compressible fluids, a finite element model of the continuity equation must be derived to supplement the equations of motion. For incompressible flows, an incompressibility condition involving the nodal velocities must be added, both to ensure incompressibility in an average sense over the element and to compute element hydrostatic pressure if they are not specified a priori.

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References

[1] Connor, J.J. and Brebbia, C.A. 1976. "Finite element techniques for fluid flow." Newnes-Sutterworths.

[2] Leonard, J. W. 1969. " Linearized compressible flow by the finite element method." Buffalo, New York.

[3] Oden, J. T. and Somogyi, D. 1969. "Finite element applications in fluid dynamics." ACSE, Vol. 95, No.EM3, pp.821-826.

[4] Oden, J. T. 1970, "Finite element formulation of problems of irreversible thermodynamics and finite deformation of nonlinear continua." Edited by R.H. Gallagher, et al, University of Alabama Press.

[5] Oden, J. T. 1970. "Finite element analogue of Navier- Stokes equation." ASCE, Vol.96, No.EM4, pp. 529-534.

[6] Oden,J. T.1970. "Finite elements of non-linear continua." McGraw-Hill.

[7] Strang, G. and Fix, G. J. 1973. "An analysis of the finite element method." Prentice Hall, Englewood Cliffs, N. J..

[8] Thomspon, E.G., Lee, G. C. and Lin, F.S.1969. "Finite element method for incompressible slow viscous flow with a free surface." Lowa State University Press, Vol. 5, pp. 93-111.

[9] Tong, P. 1970. "The finite element method for fluid flow." Edited by R. H. Gallagher, et al., University of Alabama Press.

[10] Zienkiewic, O. C., Kelly, D. W. and Bettess, P. 1977. "The coupling of the finite element method and boundary solution procedures." Int. J. num. Meth. Engng. 11, pp. 355-75.