Hermite Interpolation by Pythagorean Hodograph Curve

Aye Aye Mon, and Khaing Khaing Aye

***Abstract*—**Pythagorean Hodograph (PH) Curves admit the formulation of real-time CNC interpolators that are extremely accurate, flexible and robust. This paper states that Hermite interpolation by PH curve of degree 5. The construction of PH quintic Hermite interpolation by using C^1 Hermite boundary data gives four different solutions.

***Keywords*—**Pythagorean Hodograph Curve, Hermite Interpolation.

I. Introduction

THE Pythagorean Hodograph curves provide an elegant solution of various difficult problems occurring in applications, in particular in the context of CNC machining. An essential step of the application of PH curves is their construction from certain input data. Due to the special algebraic properties of PH curves, all constructions which are linear in the case of standard Bezier curves become nonlinear in the PH case. Pythagorean Hodograph curves provide exact solutions to a number of basic geometrical problems in computer-aided design (CAD) and computer-aided manufacturing (CAM). The PH quintic can inflect and can interpolate arbitrary first-order Hermite data (C^1Hermite data). This paper is based on approximate conversion of the planar curve into Pythagorean Hodograph (PH) spline curve. Based on theoretical results, the PH conversion algorithm can be analyzed. In the construction of PH quintic, three complex square roots must be computed, which corresponds to evaluating six real square roots. The PH spline are constructed via Hermite interpolation of C^1 boundary data. Also the approximation order of conversion procedure is analyzed. The C^1 Hermite Interpolation with PH quintic has approximation order four with respect to the original curve. By using the algorithm for PH curve, an analytical curve is approximated by PH quintic Hermite interpolants.

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II. Pythagorean Hodograph curve

The hodograph $r'(t) = \{x'(t), y'(t)\}$ of a polynomial curve is said to be Pythagorean if its components are members of a Pythagorean polynomial triple $(x'(t), y'(t), \sigma(t))$. And then, Pythagorean hodographs must be of the form

$$\begin{aligned} x'(t) &= w(t)[u^2(t) - v^2(t)], \\ y'(t) &= 2w(t)u(t)v(t) \end{aligned} \qquad (1)$$

A. Bernstein-Bezier Form of a Polynomial Curve

For most applications, PH curve is anticipated that the choice $w(t) = 1$ will be adopted and curves constructed from (relatively prime) polynomials $u(t)$ and $v(t)$ only. The corresponding Pythagorean hodograph curves are necessarily of odd degree. For finite arcs, the standard Bernstein-Bezier form of a polynomial curve is

$$r(t) = \sum_{k=0}^{n} P_k b_k^n(t), \text{ where } \quad b_k^n(t) = \binom{n}{k}(1-t)^{n-k} t^k \qquad (2)$$

The coefficients $\{P_k\}$ of $r(t)$ in this representation are known as the "control points" of the curve.

III. Construction of PH Quintic Hermite Interpolants

The first-order Hermite interpolation (C^1 Hermite Interpolation) problem is concerned with constructing a smooth curve, $r(t)$ for $t \in [0,1]$, with given end points and the derivatives, $r(0) = P_0$, $r'(0) = V_0$ and $r(1) = P_1$, $r'(1) = V_1$. As well known, there is a unique solution among the "ordinary" cubics. To obtain sufficient degrees of freedom within the PH curves, the PH quintics must be applied. It is convenient to use the complex representation, and assume that $P_0 = 0$ and $P_1 = 1$.The complex form plays a key role in simplifying the construction and shape analysis of planar PH curves.

The PH quintic in Bezier form is

$$r(t) = \sum_{i=0}^{5} P_i \binom{5}{i}(1-t)^5 t^i, \qquad (3)$$

The control points of $p(t) = r(t)$ are

$$P_1 = P_0 + \frac{w_0^2}{5},$$

$$P_2 = P_1 + \frac{w_0 w_1}{5},$$

$$P_3 = P_2 + \frac{(2w_1^2 + w_2 w_0)}{15},$$

$$P_4 = P_3 + \frac{w_1 w_2}{5},$$

$$P_5 = P_4 + \frac{w_2^2}{5},$$

where p_0 is an arbitrary constant and the control points have the form

$P_k = (x_k, y_k) = x_k + iy_k$ and $w_i = u_i + iv_i$.

The hodograph of the planar PH curve is $r'(t) = w^2(t)$. To define a PH quintic, a hodograph is chosen the square of a complex quadratic polynomial $w(t)$ expressed in Bernstein form,

$$r'(t) = [w_0(1-t)^2 + w_1 2(1-t)t + w_2 t^2]^2. \quad (4)$$

With the integration constant $r(0) = P_0$, to coefficients w_0, w_1, w_2 are determined by the Hermite interpolation conditions

$$r'(0) = V_0,$$

$$r'(1) = V_1,$$

$$\int_0^1 r'(t)dt = P_1 - P_0 = 1.$$

By using the above equations and Eq.4, the system of quadratic equations can be obtained as follows:

$$w_0^2 = V_0, \qquad w_2^2 = V_1, \quad (5)$$

and

$$w_0^2 + w_0 w_1 + \frac{2w_1^2 + w_2 w_0}{3} + w_1 w_2 + w_2^2 = 5. \quad (6)$$

This system has a simple formal solution. Eq.(5) gives two complex values of w_0, w_2 and Eq.(6) gives the complex value of w_1 as follows:

$w_0 = \pm\sqrt{V_0}$, $w_2 = \pm\sqrt{V_1}$, and

$$w_1 = \frac{-3(w_0 + w_2) \pm \sqrt{120 - 15(V_0 + V_1) + 10 w_0 w_2}}{4}.$$

By substituting the values of w_0 and w_2 into the value of equation w_1, eight different solutions are obtained.

Although there are 8 solutions, they define only 4 distinct PH quintics: if w_0, w_1, w_2 is a solution, so is $-w_0, -w_1, -w_2$, and it yields exactly the same curve. Empirically, one "good" interpolant is observed among the four distinct solutions. The others are typically exhibit undesired loops or extreme curvature variations. The good interpolant may be identified as the one that minimizes a global measure of shape, such as the absolute rotation index or elastic bending energy.

A. Absolute Value and Elastic Bending Energy

The absolute rotation Index and elastic bending energy are defined by the following equation respectively,

$$R_{abs} = \frac{1}{2\pi}\int_0^1 |K(t)||\, r'(t)\,|dt, \quad (7)$$

$$\varepsilon = \int_0^1 K^2(t)|r'(t)|dt, \quad (8)$$

which can be calculated through an analytic reduction of the integral. For PH quintics, this quantity lies in the range $0 \le R_{abs} \le 2$. To compute Eq.(7) it is convenient to express the hodograph $r'(t) = [w_0(1-t)^2 + w_1 2(1-t)t + w_2 t^2]^2$ in the form $r'(t) = k(t-a)^2(t-b)^2$, in terms of which the curvature can be expressed as

$$K(t) = \frac{\mathrm{Im}(\bar{r}'(t)r''(t))}{|r'(t)|^3} = \frac{2}{|k|}\frac{\beta|t-a|^2 + \alpha|t-b|^2}{|(t-a)(t-b)|^4}, \quad (9)$$

where $\alpha = \mathrm{Im}(a)$ and $\beta = \mathrm{Im}(b)$. The location of a and b in the complex plane relative to the interval [0,1] play a key role in determining the shape of PH quintics. The complex values a and b required to compute R_{abs} may then be written as

$$a = \frac{\alpha}{\alpha + 1} \quad \text{and} \quad \beta = \frac{\beta}{\beta + 1}$$

where α and β are the two roots z of the quadratic equation .

$$w_2 z^2 + 2w_1 z + w_0 = 0.$$

B. The four PH quintic Hermite Interpolants

There are 8 different values for w_0, w_1 and w_2. By substituting these values into $r(t)$, 4 distinct PH quintic Hermite interpolants are obtained. Because of it gives the same curve for w_0, w_1, w_2 and $-w_0, -w_1, -w_2$. The curve $r(t)$ is obtained by integrating Eq. (4). Therefore,

$$r(t) = [-\frac{1}{5}w_0^2 - \frac{1}{5}w_0w_1 - \frac{2}{15}w_1^2 - \frac{1}{15}w_0w_2](1-t)^5$$
$$+[-w_0w_1 - \frac{2}{3}w_1^2 - \frac{1}{3}w_0w_2 - w_1w_2]t(1-t)^4$$
$$+[-\frac{4}{3}w_1^2 - \frac{2}{3}w_0w_2 - 2w_1w_2]t^2(1-t)^3 + [-2w_1w_2]t^3(1-t)^2 + \frac{1}{5}w_2^2t^5.$$

The four PH quintic Hermite interpolants obtained from the given Hermite data $P_0 = 0,\ \ P_1 = 1,\ \ V_0 = 0.24 + i0.60$ and $V_1 = 0.38 + i0.52$ are as shown in Fig.1. In this Fig., it is mentioned together with the values for the bending energy and absolute rotation index. The derivatives have been scaled by a factor of 5 clarity.

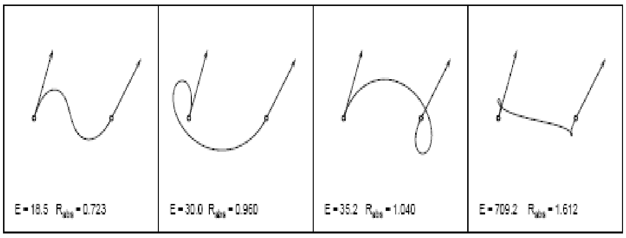


Fig. 1 The four distinct PH quintic Hermite interpolants

It is found that the first interpolant is smooth and the other interpolants have one or two loops. Thus the first interpolant is chosen as a good interpolant. Moreover, a good interpolant can be determined by the values of absolute rotation index and elastic bending energy.

The values of absolute rotation index and elastic bending energy are gotten by calculating Eq. (7) and Eq. (8).

The smallest values of the absolute rotation index and the elastic bending energy are found in the first interpolant. The greatest values of the absolute rotation index and the elastic bending energy are found in the fourth interpolant. Therefore, it is found that the good interpolant has the smallest values of the absolute rotation index and the elastic bending energy.

IV. Algorithms for PH Quintic Curve

An algorithmic construction of PH quintic interpolants to C^1 boundary data is used one of the four solutions discussed in the above section. The PH interpolation is applied to the conversion of arbitrary C^1 continuous curves into PH quintic splines.

A. C^1 Hermite Interpolation by PH Quintic Curves

The following algorithm is based on results from section III.B.

Algorithm 1 *Procedure* PH Quintic (P_0, V_0, P_1, V_1)

Input: End points P_0, P_1 and end point derivatives (velocity vectors) V_0, V_1. All these data are considered as complex numbers, by identifying the plane with the Argand diagram.

Output: PH quintic $p(\tau)$ defined over the interval [0, 1] and interpolating the input.

1. Transform the data to a certain canonical position,

$$\tilde{V}_0 = \frac{V_0}{P_1 - P_0}, \qquad \tilde{V}_1 = \frac{V_1}{P_1 - P_0}.$$

2. Compute the control points of the so-called preimage;

$$w_0 = \sqrt[+]{\tilde{V}_0}\ , \quad w_2 = \sqrt[+]{\tilde{V}_1},$$

$$w_1 = \frac{-3(w_0 + w_2) + \sqrt[+]{120 - 15(\tilde{V}_0 + \tilde{V}_1) + 10w_0w_2}}{4},$$

where $\sqrt[+]{\ }$ denotes square root with the positive real part.

3. Compute the control points of the hodograph (i.e., the first derivative vector) and transform it back to the original position:

$$h_0 = w_0^2(P_1 - P_0),$$
$$h_1 = w_0w_1(P_1 - P_0),$$
$$h_2 = (\frac{2}{3}w_1^2 + \frac{1}{3}w_0w_2)(P_1 - P_0),$$
$$h_3 = w_1w_3(P_1 - P_0),$$
$$h_4 = w_2^2(P_1 - P_0).$$

Compute the control points of the PH interpolant,

$$p_0 = P_0, \quad p_i = p_{i-1} + \frac{1}{5}h_{i-1} \quad for \quad i = 1,...,5.$$

and return the PH curve in Bernstein-Bezier representation

$$p(t) = \sum_{i=0}^{5} p_i \binom{5}{i} t^i(1-t)^{5-i}.$$

Remark

It can be verified by a direct computation that the curve $p(t)$ interpolates the input data and that it is a PH curve, i.e., its parametric speed is a (possibly piecewise) polynomial:

$$\|p'(t)\| = \|w(t)\|^2 |P_1 - P_0|, \qquad (11)$$

where $w(t) = w_0(1-t)^2 + 2w_1t(1-t) + w_2t^2$ is the so-called preimage. Algorithm PH Quintic fails for some cases of singular data. First of all the start point P_0 and the end point P_1 must be different because of the division in the step 1. Next, the function $\sqrt[+]{\ }$ is not defined on the line $R_0^- = \{\lambda + 0i : \lambda \in (-\infty, 0\}$. In order to compute w_0 and w_2 it is therefore necessary that the input tangent vectors V_0 and V_1 are non-zero and that they are not opposite to the difference vector $P_1 - P_0$. Finally we need

$$120 - 15(\tilde{V}_0 + \tilde{V}_1) + 10 w_0 w_2 \notin R_0^- . \qquad (12)$$

B. Conversion into PH Splines

The simplest non-adaptive algorithm for C^1 continuous curves is as follows:

Algorithm 2 Procedure PH splines $(c(t), \varepsilon)$

Input: A C^1 curve $c(t),\ t \in [0,1]$; prescribed error ε.

Output: PH spline $p^n = \{p_1^n, ..., p_n^n\}$ approximating $c(t)$.

1. Set $n = 1$.
2. Generate the sequence of PH interpolants:

$$p_i^n = PHQuintic\left(c(\frac{i-1}{n}), \frac{1}{n} c'(\frac{i-1}{n}), c(\frac{i}{n}), \frac{1}{n} c'(\frac{i}{n}) \right)$$

for $i = 1, ..., n$.

After a linear reparameterization join them into a spline p^n,

$$p^n(t) = p_i(nt - i + 1),\ t \in [\frac{i-1}{n}, \frac{i}{n}],\quad i = 1, ..., n$$

defined over the interval [0,1].

3. Evaluate the parametric distance between $c(t)$ and $p^n(t)$

$$\max_{t \in [0,1]} \| c(t) - p^n(t) \| = \max_{\substack{\tau \in [0,1] \\ i = 1, ..., n}} \left\| c(\frac{\tau + i - 1}{n}) - p_i^n(\tau) \right\|.$$

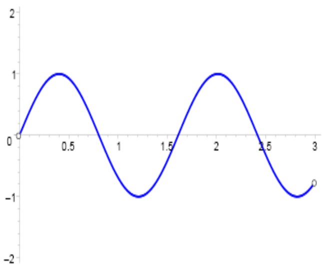
If it is greater than ε then set $n = 2^q$, $q = 0,1,....$ and GOTO (2). Otherwise STOP.

V. Approximation by PH Quintic Hermite Interpolants

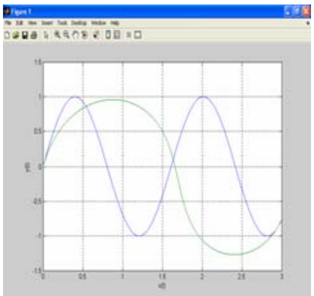
By using the procedure of PH Quintic (Algorithm 1) and PH Spline (Algorithm 2), the PH Quintic Hermite interpolant for the given curve can be constructed. By using these interpolants, we can approximate the given curve c(t). We now give a counter example as follows.

Example: Approximate the analytical curve $c(t) = (3t, \sin 11.7t),\ t \in [0,1]$ shown in Fig: 2(a) by the PH quintic Hermite interpolant.

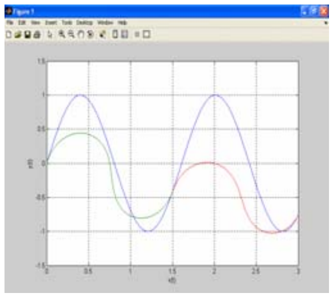
Firstly, the PH Hermite interpolant is constructed for the whole segment. And then, the piecewise PH interpolants can be obtained after splitting the parameter into 2, 4, 8 and 16 subintervals. The analytical curve $c(t)$ can be approximated by using these interpolants. These approximations are illustrated in the following figures. Fig: 2(a) is the original curve c(t). Fig: 2(b) shows the original curve and the approximation of the curve by using PH quintic Hermite interpolants in the whole segment. In this figure we can see that the error between the original curve and PH quintic Hermite interpolant is very large. Fig: 2(c), Fig 2(d), Fig. 2(e) show the approximation of the curve into the 2,4,8, subintervals respectively. After splitting the parameter into 16 subintervals shown in Fig: 2(f), the error from the original analytical curve c(t) is very small. These figures are showed by MATLAB program. Therefore, the more subintervals we use, the less approximation error can be found.



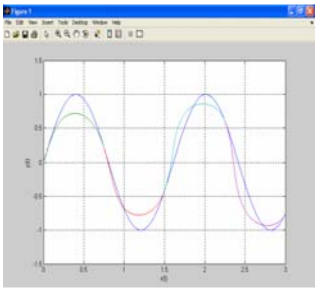
(a) The analytical curve $c(t) = (3t, \sin 11.7t),\ t \in [0,1]$



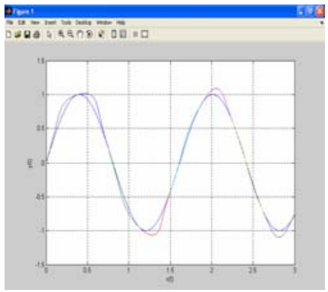
(b)



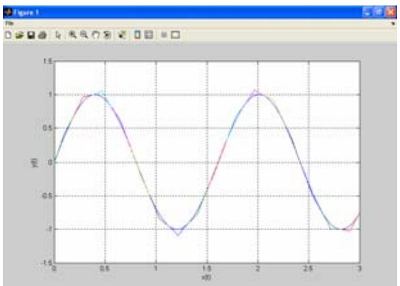
(c)



(d)



(e)



(f)

Fig. 2 Approximation by PH quintic curve

VI. Conclusion and Discussion

We have discussed the problem of C^1 Hermite Interpolartion with Pythagorean Hodograph curves. It has been shown, that this problem can be solved using curves of degrees 5. Based on these results, algorithms for converting arbitrary curves into PH spline form can be formulated.

Similarly, the problem of C^2 Hermite interpolation with PH curves can also be discussed. This problem can be solved by using PH curves of degree 9. In this case, we found that the error converges to zero very fastly. Therefore, by using the higher degree of PH curve, we can get the less error quickly. On the other side, we must deal with very large in data, so the computation will be very complex.

In general, it can be shown that the C^k Hermite interpolation always leads to three quadratic equations over C and can be solved by PH curves of degree 4k+1.

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