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Ohn Mar Myint

Abstract

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Babuska's Theorem

Let Σ and V be two real Hilbert spaces with scalar products $\langle .,.\rangle_\Sigma$ and $\langle .,.\rangle_V$ respectively. Let $c(\tau, v)$ be a bilinear form on $\Sigma \times V$ such that

$$|c(\tau, v)| \le \alpha \|\tau\|_\Sigma \|v\|_V \qquad \forall \tau \in \Sigma, \forall v \in V, \tag{1.1}$$

$$\inf_{\substack{\tau\in\Sigma \\ \|\tau\|=1}} \sup_{\substack{v\in V \\ \|v\|\le 1}} c(\tau, v) \ge \gamma > 0, \tag{1.2}$$

$$\sup_{\tau\in\Sigma} c(\tau, v) > 0 \ \forall v \in V, v \ne 0, \tag{1.3}$$

where $\alpha < \infty$. Then, for a functional $g \in V^*$, there exists a unique element $\tau_g \in \Sigma$ such that

$$c(\tau_g, v) = \langle g, v\rangle_{V^* \times V}, \quad \forall v \in V, \tag{1.4}$$

$$\|\tau_g\|_\Sigma \le \frac{1}{\gamma}\|g\|. \tag{1.5}$$

Proof : The proof consists of four steps.

Step 1: From (1.1) for every $\tau \in \Sigma$,

$$\phi_\tau(v) = c(\tau, v)$$

is a linear functional on V with the norm

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$$\|\varphi_\tau\|_V = \sup_{\|v\|\leq 1} |c(\tau, v)| \leq \alpha \|\tau\|_\Sigma \,. \tag{1.6}$$

Thus we may write, by Riesz theorem, $v_1 \in V$ and

$$\langle v_1, v \rangle_V = c(\tau, v),$$

i.e., there exists a mapping $\mathcal{R}$ of Σ to V such that

$$\langle \mathcal{R}(\tau), v \rangle_V = c(\tau, v), \tag{1.7}$$

and

$$\|\mathcal{R}\| \leq \alpha \,. \tag{1.8}$$

We see that $\mathcal{R}$ is linear and continuous.

Step 2 : Now we will show that $\mathcal{R}(\Sigma)$ is a closed set in V. We have to show that $\mathcal{R}(\Sigma) = \overline{\mathcal{R}(\Sigma)}$

Let $v \in \overline{\mathcal{R}(\Sigma)}$. Therefore $v_n \in \mathcal{R}(\Sigma)$ such that $v_n \to v$.

There exist $\tau_n \in \Sigma$ such that $v_n = \mathcal{R}(\tau_n)$.

$\{v_n\}$ is a cauchy sequence in $\mathcal{R}(\Sigma)$.

$\|\mathcal{R}(\tau_m) - \mathcal{R}(\tau_n)\|_V = \|v_m - v_n\|_V$

Since $\|\tau_m - \tau_n\|_\Sigma \leq \frac{1}{\gamma}\|\mathcal{R}(\tau_m - \tau_n)\| = \frac{1}{\gamma}\|v_m - v_n\|$.

$\{\tau_n\}$ is a cauchy sequence in Σ which is complete.

So there exist $\tau \in \Sigma$ such that $\tau_n \to \tau$.

Thus $\mathcal{R}(\tau_n) \to \mathcal{R}(\tau) = v$.

Assume that the form a is symmetric and let

$$U_g = \{ \tau \in \Sigma \mid b(\tau, v) = \langle g, v \rangle_{V^* \times V} \quad \forall v \in V \}. \tag{2.3}$$

The problem defined by (2.1) and (2.2) can be viewed as a constrained optimization problem :

Find $\sigma \in U_g$ such that σ is a point at which

$$\min_{\tau \in U_g} \left(\frac{1}{2} a(\tau, \tau) - \langle f, \tau \rangle_{\Sigma^* \times \Sigma} \right) \tag{2.4}$$

is attained.

Introducing a Lagrangian multiplier and calculating using variational method, we see that (2.1) and (2.2) will be satisfied if this minimum occurs, when

$$a(\sigma, \tau) - \langle f, \tau \rangle = -b(\tau, u) \, \forall \tau \in \Sigma. \tag{2.5}$$

The minimization problem thus yields the system of equations defined by (2.1) and (2.2). The first component σ of this problem will be an optimal solution of (2.4).

Now we return to the weak formulation defined by (2.1) and (2.2). For a weak solution of these equations to exist and if so, to be unique, certain criteria have to be satisfied. The bilinear forms a and b induce linear continuous operators $\mathcal{A} : \Sigma \to \Sigma^*$ and $\mathcal{B} : \Sigma \to V^*$, defined by

$$\langle \mathcal{A} \sigma, \tau \rangle_{\Sigma^* \times \Sigma} = a(\sigma, \tau) \quad \forall \tau \in \Sigma, \tag{2.6}$$

$$\langle \mathcal{B} \tau, v \rangle_{V^* \times V} = b(\tau, v) \quad \forall v \in V. \tag{2.7}$$

We also define $\mathcal{B}'$ the adjoint operator of $\mathcal{B}$, by

$$\langle \mathcal{B}' v, \tau \rangle_{\Sigma^* \times \Sigma} = b(\tau, v) \quad \forall \tau \in \Sigma. \tag{2.8}$$

Then (2.1) and (2.2) are equivalent to

$$\mathcal{A}\sigma + \mathcal{B}' u = f, \tag{2.9}$$

$$\mathcal{B}\sigma = g. \tag{2.10}$$

With reference to these operators, we are able to give necessary and sufficient conditions for a solution to exist.

Theorem 2.1 (Babuska-Brezzi Theorem)

A solution (σ, u) to the problem defined by (2.1) and (2.2) exists provided that

(i) The operator $\mathcal{A}$ is coercive on the kernel of $\mathcal{B}$. That is, there exists a positive constant α such that

$$a(\tau, \tau) \geq \alpha \|\tau\|_{\Sigma}^{2} \quad \forall \tau \in U_0$$

where

$$U_0 = \{\tau \in \Sigma \mid b(\tau, v) = 0 \quad \forall v \in V\}.$$

(ii) Given $v \in V$ there exists a non-zero $\tau \in \Sigma$ and a positive constant δ such that

$$|b(\tau, v)| \geq \delta \|\tau\|_{\Sigma} \|v\|_{V}.$$

When a solution σ exists, it will be unique. Similarly, u will be unique up to an element in the kernel of $\mathcal{B}'$.

See: Ronges, M.E., for proof

Theorem 2.2

Let $g \in R(\mathcal{B})$ = range of $\mathcal{B}$ and let the bilinear form $a(.,.)$ be coercive on Ker $\mathcal{B}$, that is, there exists α_0 such that

$$a(\tau_0, \tau_0) \geq \alpha_0 \|\tau_0\|_{\Sigma}^{2} \quad \forall \tau_0 \in \operatorname{Ker} \mathcal{B}. \tag{2.11}$$

Then there exists a unique solution $\sigma \in \Sigma$ of

$$a(\sigma, \tau_0) = \langle f, \tau_0 \rangle_{\Sigma' \times \Sigma} \quad \forall \tau_0 \in \operatorname{Ker} \mathcal{B}, \tag{2.12}$$

and

$$\mathcal{B}\sigma = g. \tag{2.13}$$

Proof

The condition $g \in R(\mathcal{B})$ is of course necessary. Let us suppose it is satisfied: [illegible] with $\mathcal{B}\sigma_g = g$.

By setting $\sigma = \sigma_0 - \sigma_g$ and taking $\tau = \tau_0 \in \text{Ker}\,\mathcal{B}$, we write (2.1) in the form

$$a(\sigma_0 + \sigma_g, \tau_0) - b(\tau_0, u) = \langle f, \tau_0 \rangle \quad \forall\, \sigma_0 \in \text{Ker}\,\mathcal{B},\ \tau_0 \in \text{Ker}\,\mathcal{B},$$

$$a(\sigma_0, \tau_0) = \langle f, \tau_0 \rangle - a\ (\sigma_g, \tau_0).$$

A sufficient condition for the existence and uniqueness of σ_0 is therefore the coercivity condition (2.11). There remains to check that $\sigma = \sigma_0 + \sigma_g$ does not depend on the choice of σ_g. Indeed if we had two solutions of (2.12) and (2.13) say σ_1 and σ_2, we would have $\sigma_1 - \sigma_2 \in \text{Ker}\,\mathcal{B}$ and from (2.12)

$$a(\sigma_1 - \sigma_2, \tau_0) = 0, \quad \forall\, \tau_0 \in \text{Ker}\,\mathcal{B}.$$

and this implies $\sigma_1 - \sigma_2 = 0$ by condition (2.11).

Mixed Variational Principle

Definition 3.1

Let $\mathbf{u}(x) \in \mathbb{R}^d$, the gradient of a vector function $\mathbf{u}$ is defined as

$$\textbf{grad u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_d} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_d} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_d}{\partial x_1} & \frac{\partial u_d}{\partial x_2} & \cdots & \frac{\partial u_d}{\partial x_d} \end{bmatrix}_{d\times d} = \begin{bmatrix} (\text{grad } u_1)^T \\ (\text{grad } u_2)^T \\ \vdots \\ (\text{grad } u_d)^T \end{bmatrix}_{d\times d}$$

Notation The set of $d \times d$ matrices will be denoted as

$$M^{d\times d} = \left\{ \tau = [\tau_{ij}]_{d\times d} \,\middle|\, \tau_{ij} \in \mathbb{R}\ ;\ i, j = 1, 2, \ldots, d \right\}.$$

We denote $S^{d\times d}$ as the space of real symmetric $d \times d$ matrices.

Definition 3.2

The divergence of a matrix function $\tau \in M^{d\times d}$ is defined as

$$\operatorname{div}\ \tau = \begin{bmatrix} \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \ldots + \frac{\partial \tau_{1d}}{\partial x_d} \\ \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \ldots + \frac{\partial \tau_{2d}}{\partial x_d} \\ \vdots \qquad \vdots \qquad \vdots \\ \frac{\partial \tau_{d1}}{\partial x_1} + \frac{\partial \tau_{d2}}{\partial x_2} + \ldots + \frac{\partial \tau_{dd}}{\partial x_d} \end{bmatrix}_{d\times 1} = \begin{bmatrix} \operatorname{div}\ \tau_1 \\ \operatorname{div}\ \tau_2 \\ \vdots \\ \operatorname{div}\ \tau_d \end{bmatrix}_{d\times 1}.$$

where $\tau_i = [\tau_{i1}, \tau_{i2}, \ldots, \tau_{id}]^T, \quad 1 \le i \le d.$

Definition 3.4

We define the inner product between vectors and the inner product between matrices by :

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{d} u_i v_i \qquad \text{for } \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^{d} \sigma_{ij}\ \tau_{ij} \qquad \text{for } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{R}^{d\times d}.$$

Definition 3.5

We define the spaces $\left(L^2(\Omega)\right)^{d\times d}$ by

$$\left(L^2(\Omega)\right)^{d\times d} = \left\{\tau \in M^{d\times d} \,\middle|\, \tau_{ij} \in L^2(\Omega),\ i, j = 1, 2, \ldots, d\right\}$$

with the norm

$$\|\boldsymbol{\tau}\|_{(L^2(\Omega))^{d\times d}} = \left[\sum_{i,j=1}^{d} \|\tau_{ij}\|^2_{L^2(\Omega)}\right]^{1/2},$$

with norm

$$\|\mathbf{u}\|_{H(\mathrm{div},\Omega)} = \left[\|\mathbf{u}\|^2_{(L^2(\Omega))^d} + \|\mathrm{div}\ \mathbf{u}\|^2_{L^2(\Omega)} \right]^{1/2} .$$

Definition 3.7

The space $H(\mathbf{div}, \Omega, S^{d\times d})$ is defined by

$$H(\mathbf{div}, \Omega, S^{d\times d}) = \left\{ \tau \in S^{d\times d} \mid \tau \in \left(L^2(\Omega)\right)^{d\times d}, \mathbf{div}\ \tau \in (L^2(\Omega))^d \right\}$$

with norm

$$\|\tau\|_{H(\mathbf{div},\Omega,S^{d\times d})} = \left[\|\tau\|^2_{(L^2(\Omega))^{d\times d}} + \|\mathbf{div}\ \tau\|^2_{L^2(\Omega)} \right]^{1/2} .$$

Note

$H(\mathrm{div}, \Omega)$ and $H(\mathbf{div}, \Omega, S^{d\times d})$ are Hilbert spaces.

We begin with the most classical example, the system of linear elasticity. The equations of linear elasticity consists of the constitutive equation

$$\mathbb{K}\, \boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega \tag{3.1}$$

and the equilibrium equation

$$\mathbf{div}\ \boldsymbol{\sigma} = \mathbf{g} \quad \text{in } \Omega. \tag{3.2}$$

Here Ω denotes the region in the two-dimensional space, , occupied by the elastic body, $\mathbf{u} : \Omega \to \mathbb{R}^2$ denotes the displacement field.

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\mathbf{grad u} + (\mathbf{grad\ u})^T \right)$$

denotes the corresponding infinitesimal strain tensor; **g** denotes the imposed volume load, and $\boldsymbol{\sigma} : \Omega \to S^{2\times 2}$ denotes the stress field. The material properties are determined by the compliance tensor $\mathbb{K}$ which is a positive definite symmetric operator from $S^{2\times 2}$ ... equally well ...

To determine a unique solution, we supplement the elasticity equation by the boundary condition

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0 . \qquad (3.4)$$

Mixed methods for the elasticity problem are mostly based on the following mixed variational principle which is a form of the Hellinger-Reissner principle:

The solution $(\boldsymbol{\sigma}, \mathbf{u})$ of elasticity problem can be characterized as the unique critical point of the functional

$$\mathscr{J}(\tau, v) = \int_\Omega \left(\frac{1}{2} \quad \boldsymbol{\tau} : \boldsymbol{\tau} + \mathbf{div}\, \mathbb{K}\mathbf{v} - \mathbf{g}.\mathbf{v}\right) dx - \int_{\partial\Omega} \mathbf{u}_0.(\boldsymbol{\tau}\,\mathbf{n})\, ds \qquad (3.5)$$

over the space of all symmetric tensor fields $\boldsymbol{\tau}$ and all vector fields v.

Indeed, if we set the first variation of $\mathscr{J}$ with respect to $\boldsymbol{\tau}$ equal to zero, we get

$$\mathscr{J}(\sigma+\lambda\tau, \mathbf{u}) = \int_\Omega \frac{1}{2}\mathbb{K}.(\boldsymbol{\sigma}+\lambda\boldsymbol{\tau}):(\boldsymbol{\sigma}+\lambda\boldsymbol{\tau})dx + \int_\Omega \mathbf{div}\,(\boldsymbol{\sigma}+\lambda\boldsymbol{\tau}).\mathbf{u}\,dx$$

$$- \int_\Omega \mathbf{g}.\mathbf{u}\,\mathbf{dx} - \int_{\partial\Omega} \mathbf{u}_0.(\boldsymbol{\sigma}+\lambda\boldsymbol{\tau})\,\mathbf{n}\mathbf{ds}$$

$$= \int_\Omega \left(\frac{1}{2}\mathbb{K}(\boldsymbol{\sigma}:\boldsymbol{\sigma}+\lambda\boldsymbol{\sigma}:\boldsymbol{\tau}+\lambda\boldsymbol{\tau}:\boldsymbol{\sigma}+\lambda^2\boldsymbol{\tau}:\boldsymbol{\tau})\right)dx$$

$$+ \int_\Omega (\mathbf{div}\,\boldsymbol{\sigma}.\mathbf{u} + \lambda\,\mathbf{div}\,\boldsymbol{\tau}.\mathbf{u})\,dx$$

$$+ - \int_\Omega \mathbf{g}.\mathbf{u}\,dx - \int_{\partial\Omega} \mathbf{u}_0.\boldsymbol{\sigma}\mathbf{n}\mathbf{ds} - \int_{\partial\Omega} \lambda\,\mathbf{u}_0.\boldsymbol{\tau}\,\mathbf{n}\mathbf{ds}$$

$$= \mathscr{J}(\boldsymbol{\sigma},\mathbf{u}) + \lambda \int \mathbb{K}\ \boldsymbol{\sigma}:\boldsymbol{\tau}\,dx + \frac{\lambda^2}{2}\int \mathbb{K}\ \boldsymbol{\tau}:\boldsymbol{\tau}\,dx + \lambda\int_\Omega \mathrm{div}\,\boldsymbol{\tau}.\mathbf{u}\,dx$$

$$\lim_{\lambda\to 0}\frac{\mathscr{I}(\boldsymbol{\sigma}+\lambda\boldsymbol{\tau},\mathbf{u})-\mathscr{I}(\boldsymbol{\sigma},\mathbf{u})}{\lambda}=\int_\Omega \mathbb{K}\boldsymbol{\sigma}:\boldsymbol{\tau}\,dx+\int_\Omega \mathbf{div}\,\boldsymbol{\tau}\,.\,\mathbf{u}\,dx-\int_{\partial\Omega}\mathbf{u}_0\,.\,\boldsymbol{\tau}\mathbf{n}ds=0$$

So

$$\int_\Omega \mathbb{K}\,\boldsymbol{\sigma}:\boldsymbol{\tau}\,dx+\int_\Omega \mathbf{div}\,\boldsymbol{\tau}\,.\,\mathbf{u}\,dx=\int_{\partial\Omega}\mathbf{u}_0\,.\,\boldsymbol{\tau}\mathbf{n}ds\,. \tag{3.6}$$

But

$$\int_\Omega \mathbf{div}\,\boldsymbol{\tau}\,.\,\mathbf{u}\,dx=-\int_\Omega \boldsymbol{\tau}:\boldsymbol{\varepsilon}(\mathbf{u})\,dx+\int_{\partial\Omega}\boldsymbol{\tau}\mathbf{n}\,.\,\mathbf{u}\,ds\,.$$

Thus

$$\int_\Omega \mathbb{K}\boldsymbol{\sigma}:\boldsymbol{\tau}\,dx-\int_\Omega \boldsymbol{\tau}:\boldsymbol{\varepsilon}(\mathbf{u})\,dx+\int_{\partial\Omega}\boldsymbol{\tau}\mathbf{n}\,.\,\mathbf{u}\,ds=\int_{\partial\Omega}\mathbf{u}_0\,.\,\boldsymbol{\tau}\,\mathbf{n}\,ds\,.$$

From this, we obtain the constitutive equation and displacement boundary condition. Taking the variation of $\mathscr{I}$ with respect to **v**, we get

$$\mathscr{I}(\boldsymbol{\sigma},\mathbf{u}+\theta\mathbf{v})=\int_\Omega\left(\frac{1}{2}\,\mathbb{K}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\mathbf{div}\,\boldsymbol{\sigma}\,.\,(\mathbf{u}+\theta\mathbf{v})-\mathbf{g}\,.\,(\mathbf{u}+\theta\mathbf{v})\right)dx$$

$$-\int_{\partial\Omega}\mathbf{u}_0\,.\,\boldsymbol{\sigma}\,\mathbf{n}\,ds$$

$$=\frac{1}{2}\int_\Omega \boldsymbol{\sigma}:\boldsymbol{\sigma}\mathbb{K}dx+\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\,.\,\mathbf{u}\,dx+\theta\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\,.\,\mathbf{v}\,dx$$

$$-\int_\Omega \mathbf{g}.\mathbf{u}\,dx-\theta\int_\Omega \mathbf{g}.\mathbf{v}\,dx-\int_{\partial\Omega}\mathbf{u}_0\,.\,\boldsymbol{\sigma}\,\mathbf{n}\,ds$$

$$=\mathscr{I}(\boldsymbol{\sigma},\mathbf{u})+\theta\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}.\mathbf{v}\,dx-\theta\int_\Omega \mathbf{g}\,.\,\mathbf{v}\,dx$$

and consequently

$$\int_{\Omega} \operatorname{div} \sigma \cdot v \, dx = \int_{\Omega} g \cdot v \, dx, \tag{3.7}$$

... we obtain the equilibrium equation div σ = g . To make the variational principle precise, we state over what space of functions τ and v to vary. The appropriate choice for τ is the subspace of H (div, Ω, $S^{2\times2}$) and for v the space $(L^2(\Omega))^2$.

Mixed Problem

A key point, which is characteristic of mixed variational principle, is that the pair (σ, u) is not an extreme point of the Hellinger-Reissner functional. It is a saddle point, In fact

$$\mathcal{J}(\sigma, v) \le \mathcal{J}(\sigma, u) \le \mathcal{J}(\tau, u) \tag{4.1}$$

for all $\tau \in$ H (div, Ω, $S^{2\times2}$) and all v $\in$ $(L^2(\Omega))^2$. It follows from this saddle point condition that

$$\sup_{v \in (L^2(\Omega))^2} \; \inf_{\tau \in H(\operatorname{div},\Omega,S^{2\times2})} \mathcal{J}(\tau, v) = \mathcal{J}(\sigma, u) \tag{4.2}$$

and

$$\inf_{\tau \in H(\operatorname{div},\Omega,S^{2\times2})} \; \sup_{v \in (L^2(\Omega))^2} \mathcal{J}(\tau, v) = \mathcal{J}(\sigma, u). \tag{4.3}$$

Now, because u satisfies the constitutive equation, $\varepsilon(u)$ is square integrable. It follows from Korn's inequality that the gradient of u is square integrable, i.e., $u \in (H^1(\Omega))^2$.

Mixed variational principles for linear elasticity can be written as:

Mixed variational principle : Among all tensor fields and all vector fields, the stress and displacement fields give the unique critical point of the Hellinger-Reissner functional $\mathcal{J}$. The critical point is a saddle point. That is, $\sigma \in$ H (div, Ω, $S^{2\times2}$), $u \in (L^2(\Omega))^2$, and

$$\sup_{v} \mathcal{J}(\sigma, v) = \mathcal{J}(\sigma, u) = \inf_{\tau \in H(\operatorname{div},\Omega,S^{2\times2})} \mathcal{J}(\tau, u)$$

$$\int_{\Omega} (\mathbb{K}\boldsymbol{\sigma} : \boldsymbol{\tau} + \mathbf{div}\,\boldsymbol{\tau} : \mathbf{u} + \mathbf{div}\,\boldsymbol{\sigma} : \mathbf{v})\,dx = \int_{\Omega} \mathbf{g} \cdot \mathbf{v}\,dx + \int_{\partial\Omega} u_0 \cdot \tau n\,ds$$

for all $\boldsymbol{\tau} \in H(\mathbf{div}, \Omega, S^{2\times 2})$ and all $\mathbf{v} \in (L^2(\Omega))^2$.

A Problem in Elasticity

For isotopic elastic materials the stress–strain l ıear relation reduces to Hooke's Law:

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{I} + 2\,\mu\,\boldsymbol{\varepsilon}(\mathbf{u}), \tag{5.1}$$

where **I** refers to the identity tensor and λ and μ denote Lamè elasticity constants, describing the incompressibility and the rigidity of the material respectively. For homogeneous media these are constants. For nearly incompressible materials, such as rubber, $\lambda >> \mu$.

The combination of the equation of motion in $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{g}$, Hooke's Law for a homogeneous medium and the strain-displacement relation gives the following set of equations:

$$\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{g}, \tag{5.2}$$

$$\boldsymbol{\sigma} = \lambda\,\mathbf{div}\,\mathbf{u}\,\mathbf{I} + \mu\,(\mathbf{grad}\;\mathbf{u} + (\mathbf{grad}\;\mathbf{u})^T). \tag{5.3}$$

Consider the linear elasticity problem, defined by the system of equations (5.2) and (5.3) on open, bounded domain $\Omega \subset \mathbb{R}^d$. In addition to these equations a set of boundary conditions must be enforced to complete the system. For simplicity, we will consider Dirichlet conditions for the displacement in the following, thus enforcing $u|_{\partial\Omega} = 0$. Boundary conditions for the stress, for instance specification of $\boldsymbol{\sigma} \cdot \mathbf{n}$ all or parts of the boundary conditions could be applied. In this section, we aim to derive an appropriate variational formulation of the resulting system of equations.

A simple calculation, based on (5.1) reduced to d = 2, show that

$$\mathrm{tr}\,\boldsymbol{\sigma} = 2\,(\lambda + \mu)\,\mathrm{tr}\,\boldsymbol{\varepsilon}. \tag{5.4}$$

This observation enables the inversion of (5.1) resulting in the stress-strain relation in

If we let $\gamma = \frac{1}{2}$ rot $\mathbf{u}$, then by definition, the strain ε can be decomposed in terms of the displacement $\mathbf{u}$ and its rotation γ, as displayed in

$$\varepsilon = \mathbf{grad\, u} - \gamma \quad \mathbf{J} \tag{5.6}$$

where $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

In order to obtain a weak formulation of (5.5), we multiply it by a test function $\boldsymbol{\tau} \in H\left(\mathbf{div}, \Omega, S^{2\times 2}\right)$, and integrate over Ω. Thus, the symmetry requirement on the stress tensor is enforced through a restriction of the space of test functions. Now we calculate

$$\int_\Omega \left(\frac{1}{2\mu} \boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{\lambda}{4\mu(\lambda+\mu)} \operatorname{tr} \boldsymbol{\sigma}\, \mathbf{I} : \boldsymbol{\tau} - \varepsilon : \boldsymbol{\tau} \right) dx = 0$$

to get

$$\int_\Omega \left(\frac{1}{2\mu} \boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{\lambda}{4\mu(\lambda+\mu)} \operatorname{tr} \boldsymbol{\sigma} \operatorname{tr} \boldsymbol{\tau} \right) dx - \left\langle \boldsymbol{\tau}, \mathbf{grad\, u} \right\rangle = 0 \tag{5.7}$$

where $\boldsymbol{\tau} : \mathbf{J} = -\tau_{12} + \tau_{21}$, since $\boldsymbol{\tau}$ is symmetric. For brevity, we denote:

$$\left\langle\left\langle \mathcal{D}\sigma, \tau \right\rangle\right\rangle = \int_\Omega \left(\frac{1}{2\mu} \sigma : \tau - \frac{\lambda}{4\mu(\lambda+\mu)} \operatorname{tr} \sigma \operatorname{tr} \tau \right) dx .$$

Integration by parts on the displacement term of (5.7), applying Dirichlet boundary conditions, we have

$$\left\langle \boldsymbol{\tau}, \mathbf{grad\, u} \right\rangle = -\left\langle \mathbf{div}\, \boldsymbol{\tau}, \mathbf{u} \right\rangle .$$

Thus (5.6) becomes,

$$\left\langle\left\langle \mathcal{D}\sigma, \tau \right\rangle\right\rangle + \left\langle \operatorname{div} \tau, \mathbf{u} \right\rangle = 0 \qquad \text{for } \tau \in H\left(\operatorname{div}, \Omega, S^{2\times 2}\right).$$

... of (5.2), we

Given $\mathbf{g} \in \left(L^2(\Omega)\right)^2$, find $(\boldsymbol{\sigma}, \mathbf{u}) \in H\left(\mathbf{div}, \Omega, S^{2\times 2}\right) \times \left(L^2(\Omega)\right)^2$, such that

$$\left\langle\left\langle \mathcal{D}\sigma, \tau\right\rangle\right\rangle + \left\langle \mathbf{div}\, \tau, \mathbf{u}\right\rangle = 0 \quad \forall\ \tau \in H\left(\mathbf{div}, \Omega, S^{2\times 2}\right) \qquad (5.8)$$

$$\left\langle \mathbf{div}\, \sigma, \mathbf{v}\right\rangle = \langle g, v\rangle \qquad \forall\, v \in \left(L^2(\Omega)\right)^2 \qquad (5.9)$$

hold. As $H\left(\mathbf{div}, \Omega, S^{2\times 2}\right)$ and $\left(L^2(\Omega)\right)^2$ are Hilbert spaces ensuring that form $\left\langle\left\langle \mathcal{D}\sigma, \tau\right\rangle\right\rangle$ and $\left\langle \mathbf{div}\, \tau, \mathbf{u}\right\rangle$ are continuous, we observe that this set of equations fits into the framework for saddle point problems.

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