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RELATIONS BETWEEN CAYLEY GRAPH AND VERTEX-TRANSITIVE GRAPH

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Abstract

In this paper, we first express basic concepts of graph theory. Then we define vertex-transitive graph and Cayley graph with a given group by using generating set or nongenerating set. Finally, we prove that every Cayley graph is vertex-transitive graph and we also give an example that the converse of this theorem is false.

Keywords: graph, digraph, connected, vertex-transitive, group, order, Cayley graph, Cayley digraph, diameter of a graph.

1. Basic Concepts of Graph Theory

A **graph** $G = (V(G), E(G))$ with n vertices and m edges consists of a vertex set $V(G)=\{ v_1, v_2, \ldots, v_n \}$ and an edge set $E(G) = \{ e_1, e_2, \ldots, e_m\}$, where each edge consists of two (possibly equal) vertices called its endpoints. We write uv for an edge $e = (u, v)$. If $uv \in E(G)$, then u and v are **adjacent**. The ends of an edge are said to be **incident** with the edge. A **loop** is an edge whose endpoints are equal. **Parallel edges** or **multiple edges** are edges that have the same pair of endpoints. A **simple graph** is a graph having no loops or multiple edges. A graph is **finite** if its vertex set and edge set are finite. We adopt the convention that every graph mentioned in this paper is finite, unless explicity constructed otherwise. The **degree** of a vertex v of a graph G is the number of edges of G which are incident with v. A graph is said to be **regular (k-regular)** if all its vertices have the same degree (k). A three-regular graph is also called a **cubic graph**. A simple graph in which each pair of distinct vertices is joined by an edge is called a **complete graph**. A complete graph on n vertices is denoted by K_n. A sequence of distinct edges of the form$v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$ is called a **path of length r** from v_0 to v_r, denoted by (v_0, v_r)-path. The **distance** between two vertices u and v in a graph G is the length of the shortest path from u to v. The **diameter** of a graph is the maximum distance between two distinct vertices.

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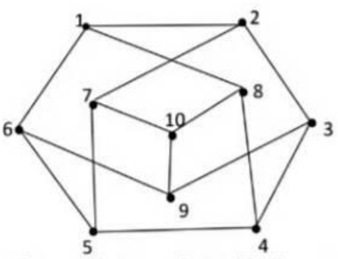


Figure 1.1: A graph G with diameter 2

A **subgraph** of a graph $G = (V(G), E(G))$ is a graph $Y = (V(Y), E(Y))$ with $V(Y) \subseteq V(G)$ and $E(Y) \subseteq E(G)$.Two vertices u and v of G are said to be **connected** if there is a (u, v)-path in G . A **connected graph** is a graph such that any two vertices are connected by a path, otherwise it is **disconnected**.

A **directed graph** (or **digraph**) $\vec{G} = (V(\vec{G}), E(\vec{G}))$ consists of a finite nonempty set $V(\vec{G})$, called **the set of vertices**, and set $E(\vec{G})$ of ordered pairs of (not necessarily distinct) vertices, called **the set of (directed) edges** or **arcs.** If $e = (u, v)$ or uv is a directed edge of $\vec{G}$, we say that e **joins** u to v, that u and v are **endpoints** of e(more specifically that u is the **tail** of e and v is the head of e). A digraph $\vec{G}$ is called **symmetric** if, whenever (u, v) is an arc of $\vec{G}$, then (v, u) is also. A digraph $\vec{G}$ is called **complete** if for every two distinct vertices u and v of $\vec{G}$, at least one of the arcs (u, v) and (v, u) is present in $\vec{G}$. A **complete symmetric digraph** of order n has both arcs (u, v) and (v, u) for every two distinct vertices u and v, denoted by $\vec{K}_n$.

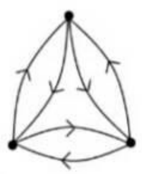


Figure 1.2: A complete symmetric digraph $\vec{K}_3$

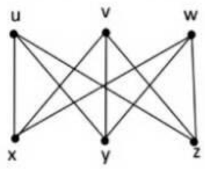
2. Vertex-Transitive Graph

Before defining a vertex-transitive graph, we express the definition of an automorphism of a graph which plays a crucial role in determining the vertex-transitive graph.

An **isomorphism** from graph G to graph G' is a bijection $\phi: V(G) \to V(G')$ such that $uv \in E(G)$ if and only if $\phi(u)\,\phi(v) \in E(G')$. We say "G is isomorphic to G'", written $G \cong G'$, if there is an isomorphism from G to G'.

The graphs G and G' drawn below are isomorphic by an isomorphism that maps u, v, w, x, y, z to l, m, n, p, q, r respectively.

G :



G′ :

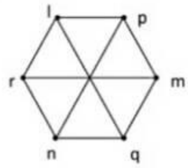


Figure 2.1: Isomorphic graphs G and G'

A **permutation** ϕ of $V(G)$ is a function from $V(G)$ into $V(G)$ that is both one to one and onto.

An **automorphism** of a (simple) graph G is a permutation ϕ of $V(G)$ which has the property that $uv \in E(G)$ if and only if $\phi(u)\,\phi(v) \in E(G)$, that is an isomorphism from G to G. The set of all automorphisms of a graph G forms a group under the operation of composition, which is called the **automorphism group**.

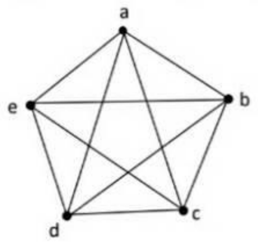


Figure 2.2: Complete graph K_5

The number of elements in X is called the **order** of X and it is denoted by O(X)or $|X|$.We define the **order of an element** x to be the least positive integer n such that $x^n = e$ and we denote it by O(x) or $|x|$.

A nonempty subset S of a group X is said to be a **subgroup** of X if, under the product in X, S itself forms a group.

If S is a subgroup of X, $a \in X$, then $aS = \{as | s \in S\}$. a S is called a **left coset** of S in X.

Let X be a group of permutation of a set A and $b \in A$, then the **stabilizer** of b (in X) is the subgroup $X_b = \{x \in X \mid x(b) = b\}$.

A group X is called a **cyclic group** if there exists an element $x \in X$, such that every element of X can be expressed as a power of x. In that case x is called **generator** of X.

Let $D_n = \{x^i y^j \mid i = 0, 1;\ j = 0, 1, ..., n-1;\ x^2 = e = y^n, x\,y = y^{-1}x\}$. Then D_n is a group, called the **dihedral group**, $(n \geq 3)$. $O(D_n) = 2n$. In fact, we can write D_n also as

$$D_n = \{y, y^2, ..., y^{n-1}, y^n, x\,y, x\,y^2, ..., x\,y^{n-1}, x \mid x^2 = e = y^n, x\,y = y^{-1}x\}.$$

Let X be a group. A subset $H \subseteq X$ is a **generating set** of X if every element of X is obtainable as the product (or sum) of elements of H.

For the group Z_n, a nonempty set of integers modulo n is a generating set if and only if its greatest common divisor (gcd) is 1. For instance, the set {4,7} generates Z_{24}, since gcd (4,7) = 1 but {6,9} does not generate Z_{24}, since gcd (6,9) = 3.

If A is a finite set {1,2, . . . ,n}, then the group of all permutations of A is the **symmetric group on n letters**, and is denoted by S_n. Note that S_n has n! elements.

In Figure 2.2, there is an automorphiosm ϕ of $V(K_5)$ such that $\phi(a) = b$, $\phi(b) = c$, $\phi(c) = d$, $\phi(d) = e$, $\phi(e) = a$.

For any two vertices u and v of G, there is an automorphism ϕ of G such that $\phi(u) = v$, we say that G is **vertex-transitive.**

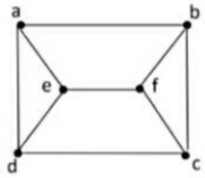


Figure 2.3: Vertex-transitive graph G

Now we interested in the structure of vertex-transitive graphs, in particular, Cayley graphs. First, we have to introduce some definitions of group theory.

3.Basic Definitions of Group Theory

A nonempty set of elements X is said to form a **group** if in X there is defined a binary operation, called the product and denoted by·, such that

(i) a, b $\in$X implies that a ·b $\in$ X (closed).

(ii) a, b, c $\in$ X implies that a · (b ·c) = (a· b) ·c (associative law).

(iii) There exists an element e$\in$ X such that a · e = e · a = a for all a $\in$ X (the existence of an identity element in X).

(iv) For every a $\in$ X there exists an element $a^{-1} \in$ X such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (the existence of inverses in X).

We usually write a b instead of a ·b.

A **finite group** is a group which has a finite number of elements, otherwise we call it an **infinite group**.

4. Cayley Graphs

Now we shall express the definitions of Cayley graphs and the constructions of Cayley graphs with their given groups.

4.1 Definitions

(i) Let X be a group and H a subset of X not containing the identity e. Then the **Cayley digraph** $\vec{C}$ has vertex set $V(\vec{C}) = X$ and arc set $E(\vec{C}) = \{(g, gh) \mid h \in H,\ g \in X\}$. We write $\vec{C} = \vec{C}(X, H)$.

(ii) Let $H = X - e$. Then the resulting Cayley digraph will be denoted by $\vec{K} = \vec{K}(X, H)$ and called the **complete Cayley digraph**.

(iii) Let X be a group and H a subset of X not containing the identity e such that $h \in H$ implies $h^{-1} \in H$ (that is, $H = H^{-1}$, where $H^{-1} = \{h^{-1} \mid h \in H\}$). Then the graph with vertex set $V(C) = X$ and edge set $E(C) = \{(g, gh) \mid h \in H, g \in X\}$ is called the **Cayley graph** C corresponding to X, H. We write $C = C(X, H)$. Equivalently, the Cayley graph $C = C(X, H)$ is the simple graph whose vertex set and edge set are defined as follows:

$$V(C) = X;\ E(C) = \left\{ (g, h) \mid g^{-1} h \in H, \text{ where } g \in X,\ h \in H \right\}.$$

(iv) When H is a set of generators for X the Cayley digraph and the Cayley graph will be referred to as the **basic Cayley digraph** and the **basic Cayley graph** respectively.

4.2 Examples

(i) Let X be the group Z_5, the set of integers modulo 5.

Let H be generating set $\{1\}$.

We can construct the Cayley digraph $\vec{C} = \vec{C}(Z_5, H)$ which has the vertex set $V(\vec{C}) = Z_5 = \{0, 1, 2, 3, 4\}$; and the arc set $E(\vec{C}) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$.

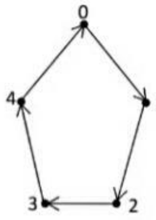


Figure 4.1: The Cayley digraph $\vec{C}(Z_5, \{1\})$

(ii) Let X be the cyclic group C_4 generated by a and $H = \{a, a^2, a^3\}$.

We can construct the complete Cayley digraph $\vec{K} = \vec{K}(C_4, H)$ which has the vertex set $V(\vec{K}) = C_4 = \{1, a, a^2, a^3\}$; and the arc set

$E(\vec{K}) = \{(1,a),(1,a^2),(1,a^3),(a,a^2),(a,a^3),(a,1),(a^2,a^3),(a^2,1),(a^2,a),(a^3,1),(a^3,a),(a^3,a^2)\}$.

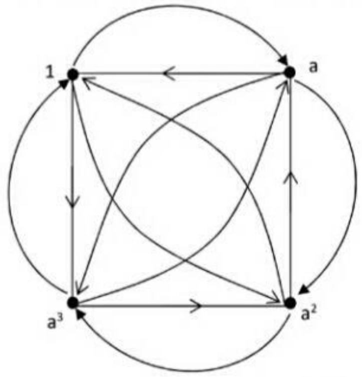


Figure4.2: The complete Cayley digraph $\vec{K}$ $(C_4, \{a, a^2, a^3\})$

(iii) Let X be the symmetric group $S_3 = \{1, (12), (13), (23), (123), (132)\}$.

Let $H = \{(12), (13), (23)\}$.

Then $H^{-1} = H$. We can construct the Cayley graph $C = C(S_3, H)$ which has

$V(C) = S_3$; $E(C) = \{(1,(12)), (1,(13)), (1,(23)), ((12), (132)), ((12), (123)),$

$((13), (132)), ((13), (123)), ((23), (132)), ((23), (123))\}$.

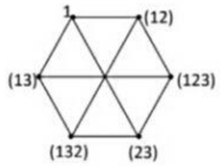


Figure4.3: The Cayley graph $C(S_3, \{(12), (13), (23)\})$.

4.3Theorem

The Cayley graph $C(X, H)$ is well-defined and is connected if and only if H is a set of generators for X.

Proof. See [7].

4.4 Examples

(i) Let X be the dihedral group $D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$, where $r^4 = s^2 = 1$, $sr = r^{-1}s$ and $H = \{ r, s \}$.

Then H is a generating set for D_4. We can construct the Cayley digraph $\vec{C} = \vec{C}(D_4, H)$ has the vertex set $V(\vec{C}) = D_4$ and the arc set $E(\vec{C}) = \{ (g, gh) \mid g \in D_4, h \in H \}$.

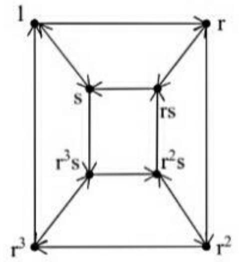


Figure 4.4: The connected Cayley digraph $\vec{C}(D_4, \{r, s\})$

(ii) Let X be $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let $H = \{2, 6\}$.

Since $H = H^{-1}$ and H is not a generating set, we can construct the disconnected Cayley graph C has vertex set $V(C) = Z_8$ and edge set $E(C) = \left\{ (g, gh) \mid g \in Z_8,\ h \in H \right\}$.

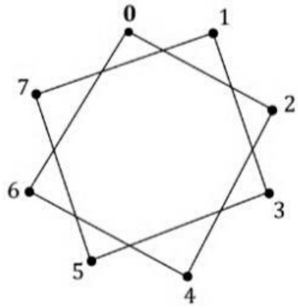


Figure 4.5: The disconnected Cayley graph $C(Z_8, H)$

In the above examples, we see that if the subset H is a generating set for the given group then the Cayley digraph is connected and if H is not a generating set then the Cayley graph is disconnected.

5. Relations between Cayley Graph and Vertex-Transitive Graph

In this section, we interested in a relation between Cayley graph and vertex-transitive graph.

5.1Theorem

Every Cayley graph $C(X, H)$ is vertex-transitive.

Proof.

For each g in X we define a permutation ϕ_g of $V(C) = X$ by the rule $\phi_g(h) = gh,\ h \in X$.

This permutation ϕ_g is an automorphism of C , for

$$\begin{aligned}(h,k) \in E(C) &\Rightarrow h^{-1}k \in H \\ &\Rightarrow (gh)^{-1}(gk) \in H \\ &\Rightarrow (\phi_g(h), \phi_g(k)) \in E(C)\end{aligned}$$

Now for any $h, k \in X$, $\phi_{kh^{-1}}(h) = (kh^{-1})h = k$.

Hence Cayley graph $C(X, H)$ is vertex-transitive.

5.2 Petersen graph

The **Petersen graph** P(5,2) is a cubic graph having a vertex set $V = \{u_0, \ldots, u_4, v_0, \ldots, v_4\}$ and an edge set $E = \{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+2}) \mid i = 0, \ldots, 4\}$ where all the subscripts are taken modulo 5. The **generalized Petersen graph** $P(n, k)(n \geq 5,\ 0 < k < n)$ is the cubic graph having a vertex set $\{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}$ and an edge set $\{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+2}) \mid i = 0, \ldots, n-1\}$ where all the subscripts are taken modulo n.

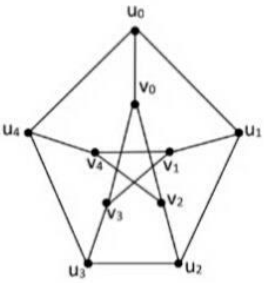


Figure5.1: The Petersen graph P(5, 2).

The following is an example of a vertex-transitive graph which is not a Cayley graph.

5.3 Example

The Petersen graph is vertex-transitive but it is not a Cayley graph.

Indeed, we can see the diameter of the Petersen graph is 2 and the diameter of a Cayley graph $C = C(X, H)$ is the smallest positive integer n such that $X = H \cup H^2 \cup ... \cup H^n$ where $H^2 = \{hk \mid h, k \in H\}$ and $H^i = H^{i-1}H$ for $i \geq 3$.

We now show that all the Cayley graphs of order 10 having degree 3 are of diameter greater than 2 and so none of them is the Petersen graph.

There are two groups of order 10. The first one is the cyclic group Z_{10} and the second one is the dihedral group D_5. The group operation here are additions and we replace H^{-1} by $-H$.

Case 1.

$X = Z_{10} = \{0, 1, ..., 9\}$.

Since $-H = H$ and $|H| = 3$, $5 \in H$ and H can only be one of the following four sets

$H_1 = \{1, 5, 9\}$, $\quad H_2 = \{2, 5, 8\}$,

$H_3 = \{3, 5, 7\}$, $\quad H_4 = \{4, 5, 6\}$.

Now $|H_i + H_i| = 5$ for each $i = 1, 2, 3, 4$.

Thus the diameter of C is greater than 2.

Case 2.

$X = D_5 = \{0, b, 2b, 3b, 4b, a, a+b, a+2b, a+3b, a+4b\}$ where $2a = 0$, $5b = 0$ and $b + a = a + 4b$.

In this case a, a + b, a + 2b, a + 3b and a + 4b are the only elements of order 2 in X.

Hence H can only be one of the following three types of sets

$H_1 = \{a + jb, b, 4b\}$, $\quad j = 0, 1, 2, 3, 4;$

$H_2 = \{a + jb, 2b, 3b\}$, $\quad j = 0, 1, 2, 3, 4;$

$H_3 = \{a + j_1b, a + j_2b, a + j_3b\}$, $\quad 0 \le j_1 < j_2 < j_3 \le 4$.

Now $|H_i + H_i| = 5$ for each $i = 1, 2, 3$.

Thus the diameter of C is greater than 2 also.

Petersen graph is a vertex-transitive graph but it is not a Cayley graph.

From the above example, we see that every vertex-transitive graph is not a Cayley graph. But every vertex-transitive graph can be constructed almost like a Cayley graph. This result will be shown in Theorem 5.5. We shall apply the following theorem to prove Theorem 5.5.

5.4 Theorem

Let S be a subgroup of a finite group X and let H be a subset of X such that $H^{-1} = H$ and $H \cap S = \varnothing$. If G is the graph having vertex set $V(G) = X/S$

(the set of all left cosets of S in X) and edge set $E(G)=\{(xS, yS) \mid x^{-1}y \in SHS\}$, then G is vertex-transitive.

Proof.

We first show that the graph G is well-defined.

Suppose that $(xS,\ yS) \in E(G)$ and $x_1S = xS,\ y_1S = yS$.

Then $x_1 = xs,\ y_1 = yk$ for some $s, k \in S$.

$$\begin{aligned} \text{Now}\quad x^{-1}y \in SHS &\Rightarrow (xs)^{-1}(yk) \in SHS \\ &\Rightarrow x_1^{-1}y_1 \in SHS \\ &\Rightarrow (x_1s,\ y_1s) \in E(G). \end{aligned}$$

Hence the graph G is well-defined.

Next, for each $g \in X$ we defined a permutation ϕ_g of $V(G) = X/S$ by the rule such that $\phi_g(xS) = gxS,\quad xS \in X/S$.

This permutation ϕ_g is an automorphism of G, for

$$\begin{aligned} (xS,\ yS) \in E(G) \quad &\Rightarrow x^{-1}y \in SHS \\ &\Rightarrow (gx)^{-1}(gy) \in SHS \\ &\Rightarrow (gxS,\ gyS) \in E(G) \\ &\Rightarrow \left(\phi_g(xS),\ \phi_g(yS)\right) \in E(G). \end{aligned}$$

Finally, for any $xS,\ yS \in X/S$, $\phi_{yx^{-1}}(xS) = yx^{-1}(xS) = yS$.

Hence the graph G is vertex-transitive.

The graph G constructed in above theorem is called the **group-coset graph** X/S generated by H and is denoted by $G(X/S, H)$.

5.5 Theorem

Let G be a vertex-transitive graph whose automorphism group is A. Let $H = A_b$ be stabilizer of $b \in V(G)$. Then G is isomorphic with the group-coset graph $G(A/H, S)$ where S is the set of all automorphism x of G such that $(b, x(b)) \in E(G)$.

Proof.

We can see that $S^{-1} = S$ and $S \cap H = \varnothing$.

We now show that $\phi : A/H \to G$ given by $\phi(xH) = x(b)$, where $xH \in A/H$, defines a map.

Suppose $xH = yH$.

Then $y = xh$ for some $y \in H$.

$\phi(yH) = y(b) = (xh)(b) = x(h(b)) = x(b) = \phi(xH)$.

We next show that ϕ is a graph isomorphism.

Suppose $\phi(xH) = \phi(yH)$.

Then $x(b) = y(b)$

$y^{-1}x(b) = b$

$y^{-1}x \in H$

$x \in yH$

$yH = xH$.

So ϕ is one to one.

Let c be a vertex of G.

Since G is vertex-transitive, there exists z in A such that $z(b) = c$

Thus $\phi(zH) = z(b) = c$.

So ϕ is onto.

Next $(xH, yH) \in E(G(A/H,S)) \quad \Leftrightarrow x^{-1}y \in HSH$

$\Leftrightarrow x^{-1}y = hzk \text{ for some } h, k \in H,\ z \in S$

$\Leftrightarrow h^{-1}x^{-1}yk^{-1} = z$

$\Leftrightarrow \left(b,\ h^{-1}x^{-1}yk^{-1}(b)\right) \in E(G)$

$\Leftrightarrow \left(b,\ x^{-1}y(b)\right) \in E(G)$

$\Leftrightarrow \left(x(b),\ y(b)\right) \in E(G)$

$\Leftrightarrow \left(\phi(xH),\ \phi(yH)\right) \in E(G).$

Thus G is isomorphic with the group-coset graph $G(A/H, S)$.

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