CALCULATION OF DIFFERENTIAL CROSS SECTION FOR $\pi^+ + n \rightarrow \Lambda + \mathrm{K}^+$ REACTION

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Abstract

We calculated the differential cross section for $\pi^+ + n \rightarrow \Lambda + \mathrm{K}^+$ by using Fermi Golden Rule . Firstly, the total transition rate is the transition probability per unit time. This formula is of great practical importance. It is called Fermi's golden rule. Firstly, we formulated the transition rate from initial state to finial state. We calculated threshold energy of above reaction and momentum of product particle lambda. Threshold energy is obtained 758.43MeV. We calculated various differential cross section for various incident energy. It is seen that the differential cross section is largest at incident energy 812.105 MeV. This value of differential cross section is $9.51\times10^{-4} b/Sr$.

1. INTRODUCTION

Quantum mechanics cannot predict the occurrence of an event with certainty. Suppose a single photon of radiation strikes the metal surface. Then, it is impossible to predict whether it will be absorbed. If it is absorbed it is difficult to know exactly where and when it is absorbed. However, if a beam containing many photons strikes the metal surface, it is possible to predict from the intensity of radiation used, the average number of photons absorbed in a certain region. Thus, in this case, quantum mechanics appears to determine only the probability of an event. This behavior is true not only for the photoelectric effect but also for all quantum processes. This means that probability plays a fundamental role in quantum mechanics.

The wave function ψ which represents the wave nature of the particle can be regarded as a measure of the presence of the particle. The particle cannot be expected to find in the region of space where $\psi \neq 0$. This suggests that represents, in some manner, the probability of finding the particle somewhere in the region of space. However, ψ is a complex function where as probability is real and positive. Probability ranges from zero (0) to one (1). A probability of 1 means certainty where as probability of 0 means non-occurrence.

2. CALCULATION OF TRANSITION RATE

2.1 Time Dependent Potential

Only few problems in quantum mechanics with either time independent (or) time-dependent Hamiltonians can be solved exactly. Time-dependent Hamiltonian is split into two parts, namely,

$$\mathrm{H} = \mathrm{H}_0 + \mathrm{V}(\mathrm{r},\mathrm{t}) \qquad (1)$$

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Where H_0 does not contain time explicity. The problem V(t) = 0 is assumed to be solved in the sense that the energy c eigenvalues E_n defined by.

$$H_0 |n\rangle = E_n |n\rangle \tag{2}$$

At any time (t > 0), Theschrodinger equation for a state ket

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_s = H |\alpha, t_0, t\rangle_s \tag{3}$$

To transform from Schrodinger picture to interaction picture,

$$|\alpha, t_0, t\rangle_I = e^{iH_0 t/\hbar} |\alpha, t_0, t\rangle_s \tag{4}$$

We define observables in the interaction picture as

$$\begin{aligned} A_I &= e^{iH_0 t/\hbar} |\alpha, t_0 = 0, t\rangle_s \\ V_I &= e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} \end{aligned} \tag{5}$$

Where V is the time-dependent potential in the Schrodinger picture. The connection between the Schrodinger picture and the Heisenbeng picture:

$$\begin{aligned} |\alpha\rangle_H &= e^{iHt/\hbar} |\alpha, t_0, t\rangle_s \\ A_H &= e^{iHt/\hbar} A_s e^{-iHt/\hbar} \end{aligned} \tag{6}$$

We now derive the fundamental differential equation that characterizes the time evolution of a state ket in the interaction picture.

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = i\hbar \frac{\partial}{\partial t} \left\{ e^{iH_0 t/\hbar} |\alpha, t_0, t\rangle_s \right\} \tag{7}$$

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = -H_0 e^{iH_0 t/\hbar} |\alpha, t_0, t\rangle_s + e^{iH_0 t/\hbar} i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_s \tag{8}$$

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = -H_0 e^{iH_0 t/\hbar} |\alpha, t_0, t\rangle_s + e^{iH_0 t/\hbar} (H_0 + V_s(t) |\alpha, t_0, t\rangle_s \tag{9}$$

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = -H_0 e^{iH_0 t/\hbar} |\alpha, t_0, t\rangle_s + H_0 e^{iH_0 t/\hbar} |\alpha, t_{0,t}\rangle_s + e^{iH_0 t/\hbar} V_s(t) |\alpha, t_0, t\rangle_s \tag{10}$$

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = e^{iH_0 t/\hbar} V_s(t) |\alpha, t_0, t\rangle_s \tag{11}$$

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = e^{iH_0 t/\hbar} V_s(t) e^{-iH_0 t/\hbar} e^{iH_0 t/\hbar} |\alpha, t_0, t\rangle_s \quad (12)\; i\hbar \frac{\partial}{\partial t} |\alpha, t_0, t\rangle_I = V_I(t) |\alpha, t_0, t\rangle_I \tag{13}$$

Which is a Schrodinger like equation with the total H replaced by V_I. In other words, $|\alpha, t_0, t\rangle_I$ would be a ket fixed in time if V(t) are absent.

In the interaction picture, we continue using $|n\rangle$ as our base $|n\rangle$ kets. Thus, we expend as follow.

$$|\alpha, t_0, t\rangle_I = \Sigma_n C_n(t) |n\rangle \tag{14}$$

Multiplying both sides of eqn: (12) by $\langle n|$ from the left, we obtain

$$i\hbar \frac{\partial}{\partial t} \langle n | \alpha, t_0, t\rangle_I = \langle n | V_I | \alpha, t_0, t\rangle_I \tag{15}$$

Since $|n\rangle$ kets are complete orthonormal set,

$$i\hbar \frac{\partial}{\partial t} \langle n | \alpha, t_0, t\rangle_I = \langle n | V_I | \Sigma_m |m\rangle \langle m | \alpha, t_0, t\rangle_I = \Sigma_m \langle n | V_I | m\rangle \langle m | \alpha, t_0, t\rangle_I \tag{16}$$

This can also be written using

$$\langle n \mid e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar} \mid m \rangle = V_{nm}(t) e^{i(E_n - E_m)t/\hbar} \text{ and}$$

$$C_n(t) = \langle n \mid \alpha, t_0, t \rangle_I \tag{17}$$

$$i\hbar \frac{d}{dt} C_n(t) = \Sigma_m V_{nm} e^{i(E_n - E_m)t/\hbar} C_m(t) \tag{18}$$

This is the basic coupled differential equation that must be solved to obtain the probability of finding $| n \rangle$.

2.2 Time Dependent Perturbation Theory

We must be content with approximate solution obtained by perturbation expansion :

$$c_n(t) = c_n^{(0)} + c_n^{(1)} + c_n^{(2)} + \ldots \tag{19}$$

Where $c_n^{(1)}$, $c_n^{(2)}$, … signify amplitudes of first order, second order , and so on in the strength parameter of the time dependent potential. The time evolution operator in the interaction picture is defined by

$$\left| \alpha, t_0, t \right\rangle_I = U_I(t, t_0) \left| \alpha, t_0, t_0 \right\rangle_I \tag{20}$$

Where, $U_I(t, t_0) =$ time-evolution operator

In the interaction picture, differential equation for the state ket is

$$i\hbar \frac{\partial}{\partial t} \left| \alpha, t_0, t \right\rangle_I = V_I(t) \left| \alpha, t_0, t \right\rangle_I \tag{21}$$

we expand the above equation as follow,

$$i\hbar \frac{\partial}{\partial t} \left\{ U_I(t, t_0) \left| \alpha, t_0 \right\rangle_I \right\} = V_I(t) U_I(t, t_0) \left| \alpha, t_0 \right\rangle_I \tag{22}$$

$$i\hbar \frac{\partial}{\partial t} U_I(t, t_0) = V_I(t) U_I(t, t_0) \tag{23}$$

We must solve this operator differential equation subject to the initial condition

$$U_I(t, t_0)\Big|_{t=t_0} = 1$$

First, let us note that the differential equation together with the initial condition is equivalent to the following integral equation.

$$i\hbar \int_{t_0}^{t} dU_I(t, t_0) = \int_{t_0}^{t} V_I(t') U_I(t', t_0) dt' \tag{24}$$

By integrating both sides, we obtain the followingequation,

$$U_I(t, t_0) - 1 = -\frac{i}{\hbar} \int_{t_0}^{t} V_I(t') U_I(t', t_0) dt' \tag{25}$$

We can obtain an approximated solution to this equation by iteration.

$$U_I(t', t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t'} V_I(t'') U_I(t'', t_0) dt'' \tag{26}$$

$$U_I(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} V_I(t') dt' \left\{ 1 - \frac{i}{\hbar} \int_{t_0}^{t'} V_I(t'') U_I(t'', t_0) dt'' \right\} \tag{27}$$

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\,dt' + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_I(t') V_I(t'') + \ldots + \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \ldots$$
$$\times \int_{t_0}^{t^{(n-1)}} dt^{(n)} V_I(t') V_I(t'') \ldots V_I(t^{(n)}) + \ldots \quad (28)$$

This series is known as Dyson series.

2.3 Transition probability

Once $U_I(t,t_0)$ is given, we can predict the time development of any state ket .If the initial state at t = o is one of energy eigen states of H_0.Then to obtain the initial state ketat alater time t is

$$\begin{aligned} |i,t_o,t\rangle_I &= U_I(t,o)|i\rangle \\ &= \Sigma_n |n\rangle\langle n|U_I(t,o)|i\rangle \\ &= \Sigma_n \langle n|U_I(t,0)|i\rangle|n\rangle \end{aligned} \quad (29)$$

Comparing the above expression with the expansion, $|i,t_0,t\rangle_I = \Sigma_n c_n(t)|n\rangle$,

Then, $c_n(t) = \langle n|U_I(t,0)|i\rangle$. (30)

Then, time evolution operator in the Schrodinger picture U(t ,o) and the interaction picture U_I(t ,0) are related to each other as

$$\begin{aligned} |i,t_0,t\rangle_I &= e^{iH_0t/\hbar}|i,t_0,t\rangle_S \\ &= e^{iH_0t/\hbar}U(t_0,t)|i,t_0,t_0\rangle_S \\ &= e^{iH_0t/\hbar}U(t_0,t)e^{-iH_0t/\hbar}|i,t_0,t_0\rangle_I \end{aligned} \quad (31)$$

The time evolution operator in the interaction picture is defined by

$$U_I(t_0,t) = e^{iH_0t/\hbar}U(t_0,t)e^{-iH_0t/\hbar} \quad (32)$$

$$|i,t_0,t\rangle_I = U_I(t_0,t)|i,t_0,t_0\rangle_I \quad (33)$$

Now, the matrix element of $U_I(t, t_0)$ is taken between the energy eigen state of H_0.

$$\begin{aligned} \langle n|U_I(t_0,t)|i\rangle &= \langle n|e^{iH_0t/\hbar}U(t_0,t)e^{-iH_0t/\hbar}|i\rangle \\ &= e^{(E_n-E_i)t/\hbar}\langle n|U(t,t_0)|i\rangle \end{aligned} \quad (34)$$

Where , $\langle n|U(t,t_0)|i\rangle$ is defined to be the transition amplitude. For transition probability,

$$|\langle n|U_I(t,t_0)|i\rangle|^2 = |C_n(t)|^2 \quad (35)$$

The matrix elements of U_I(t,t_0) are taken between initial and final states that are not energy eigenstates. We can also expend C_n(t) as in

$$\begin{aligned} C_n(t) &= \langle n|U_I(t,t_0)|i\rangle \\ &= \langle n|1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\,dt' + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_I(t') V_I(t'') + \ldots|i\rangle \\ &= \langle n|1|i\rangle - \frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle\,dt' + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t'}\langle n|V_I(t')V_I(t'')|i\rangle\,dt'dt'' + \ldots. \end{aligned} \quad (36)$$

Comparing the expression of both sides of Eqn.(32) with Eqn.(25), become as follow;

$$c_n^{(0)}(t) = \langle n|1|i\rangle = \delta_{ni} \quad (37)$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' \tag{38}$$

$$c_n^{(2)}(t) = \left(\frac{i}{\hbar}\right)^2\int_{t_0}^{t}\int_{t_0}^{t'}\langle n|V_I(t')V_I(t'')|i\rangle dt'dt'' \tag{39}$$

By using potential in interaction picture, $V_I(t') = e^{iH_0t/\hbar}V(t')e^{-iH_0t/\hbar}$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}e^{i\omega_{ni}t'}V_{ni}(t')dt' \tag{40}$$

$$c_n^{(2)}(t) = \left(\frac{i}{\hbar}\right)^2\sum_m\int_{t_0}^{t}\int_{t_0}^{t'}e^{i\omega_{nm}t'}V_{nm}(t')e^{i\omega_{ni}t''}V_{mi}(t'')dt'dt'' \tag{41}$$

We have used, $e^{i(E_n-E_i)t/\hbar} = e^{i\omega_{ni}t}$.

The transition probability for $|i\rangle \to |n\rangle$ state with $i \neq n$ is obtained by

$$|c_n(t)|^2 = \left|c_n^{(1)}(t) + c_n^{(2)} + ...\right|^2$$

$$\rho((i \to n)) = \left|c_n^{(1)}(t) + c_n^{(2)}(t) + ...\right|^2 \tag{42}$$

2.4 Constant Perturbation

At t=0,a constant perturbation term is

$$V(t) = \begin{cases} 0, & \text{for } t < 0 \\ V & (\text{independent of } t),\ \text{for } t > 0 \end{cases}$$

V(t) may be dependent on space coordinates , spin of isospin etc.

Suppose at t=0, we have only initial state $|i\rangle$, $H_0|i\rangle = E_i|i\rangle$. With t_0 taken to be zero, we obtain

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{0}^{t}e^{i\omega_{ni}t'}V_{ni}(t')dt' \tag{43}$$

The probability for finding $|n\rangle$ state at time t (t=0), the first order is

$$\left|c_n^{(1)}\right|^2 = \frac{4|V_{ni}|^2}{|E_n - E_i|^2}\frac{\sin^2(\omega_{ni}t)}{2} \tag{44}$$

The probability of finding $|n\rangle$ depends not only on $|V_{ni}|^2$ but also on the energy difference $(E_n - E_i)$. If the final state is continuous,there are many states with $E \approx E_n$. The transition probability is summed over final states with $E_n \approx E_i$: $\sum_{n,E_n\approx E_i}\left|c_n^{(1)}\right|^2$.

The density of final states is defined as the number of states within energy interval $(E, E+dE)$ as $\rho(E)dE$.

$$\sum_{n,E_n\approx E_i}\left|c_n^{(1)}\right|^2 = \int dE_n\rho(E_n)\left|c_n^{(1)}\right|^2$$

$$= 4\int \sin^2\left(\frac{(E_n - E_i)t}{2\hbar}\right)\frac{|V_{ni}|^2}{|E_n - E_i|^2}\rho(E_n)dE_n \tag{45}$$

At $t \to \alpha$, we can take advantage of

$$\lim_{t\to\infty}\frac{1}{\left|E_n-E_i\right|^2}\sin^2\left(\frac{\left(E_n-E_i\right)t}{2\hbar}\right)=\frac{\pi t}{2\hbar}\delta\left(E_n-E_i\right). \tag{46}$$

It is now possible to take the advantage of $\left|V_{ni}\right|^2$ outside the integral sign and perform the integration with the δ - function:

$$\sum_n\left|c_n^{(1)}\right|^2=\frac{2\pi}{\hbar}\left|V_{ni}\right|^2\rho\left(E_n\right)t \tag{47}$$

Thus the total transition probability is proportional to t for large values of t. The total transition rate is the transition probability per unit time. It is defined by

$$\frac{d}{dt}\left(\sum_n\left|c_n^{(1)}\right|^2\right)=w_{i\to[n]} \tag{48}$$

Where, $[n]$ stands for a group of final states with energy similar to, we obtained

$$w_{i\to[n]}=\frac{2\pi}{\hbar}\left|V_{ni}\right|^2\rho\left(E_n\right)_{E_n\square E_i} \tag{49}$$

This formula valid the first order time-dependent perturbation theory. It is of great practical importance, it is also called Fermi's golden rule.

2.5 Calculation of Differential Cross Section for $\pi^+ + n \to \Lambda + K^+$ Reaction

The transition from initial state to finial state is,

$$\frac{\sqrt{2E_\pi}\sqrt{2E_K}}{(\hbar c)^2}T_{fi}=\left[\vec{K}_K+K_\Lambda\left|\vec{K}_\pi\right.\right]\left[\vec{\tilde{q}}'\left|t\right|\vec{\tilde{q}}\right] \tag{50}$$

$$\frac{\sqrt{2E_\pi}\sqrt{2E_K}}{(\hbar c)^2}T_{fi}=\left(\frac{2\pi}{L}\right)^3\delta(\vec{K}_\pi-\vec{K}_K-\vec{K}_\Lambda)\left(\frac{2\pi}{L}\right)^3\left\langle\vec{\tilde{q}}'\left|t\right|\vec{\tilde{q}}\right\rangle \tag{51}$$

$$\vec{\tilde{q}}=\frac{M_n\vec{K}_\pi}{m_k+M_n} \tag{52}$$

$$\vec{\tilde{q}}=\frac{M_\Lambda\vec{K}_K-m_\pi\vec{K}_\Lambda}{m_k+m_\Lambda} \tag{53}$$

The differential is defined as the transition rate by incident flux ,

$$d^6\sigma=\frac{E_\pi L^3}{\hbar K_\pi c^2}\times\frac{2\pi}{\hbar}\delta(E_f-E_i)d\vec{K}_K d\vec{K}_\Lambda\times\frac{(\hbar c)^4}{4E_\pi E_K}\times\left(\frac{2\pi}{L}\right)^6\delta(0)\delta(\vec{K}_\pi-\vec{K}_K-\vec{K}_\Lambda)\cdot\left|\left\langle\vec{\tilde{q}}'\left|t\right|\vec{\tilde{q}}\right\rangle\right|^2 \tag{54}$$

When we explained equation (51), we get the following equation as

$$\frac{d^2\sigma^{elem}}{d^2\Omega_1}=\int_0^\infty K_K^2 dK_K\frac{\left|\left\langle\vec{\tilde{q}}'\left|t\right|\vec{\tilde{q}}\right\rangle\right|^2}{\hbar K_\pi c^2}\frac{(\hbar c)^4}{4E_K}\frac{(2\pi)^4}{\hbar}\delta(E_f-E_i) \tag{55}$$

The total energy of initial state is

$$E_i=E_\pi+M_n c^2 \tag{56}$$

The total energy of finial state is

$$E_f=\sqrt{(\hbar cK_K)^2+m_\pi^2c^4}+\sqrt{(\hbar c)^2(K_\pi^2+K_K^2-2K_\pi K_K\cos\theta_1+M_\Lambda^2c^4)} \tag{57}$$

$$\frac{dE_f}{dK_K}=(\hbar c)^2\left[\frac{K_K}{E_K}+\frac{K_K-K_\pi\cos\theta_1}{E_\Lambda}\right] \tag{58}$$

When we substitute the equation (58) in the equation (51) , we get the following equation as

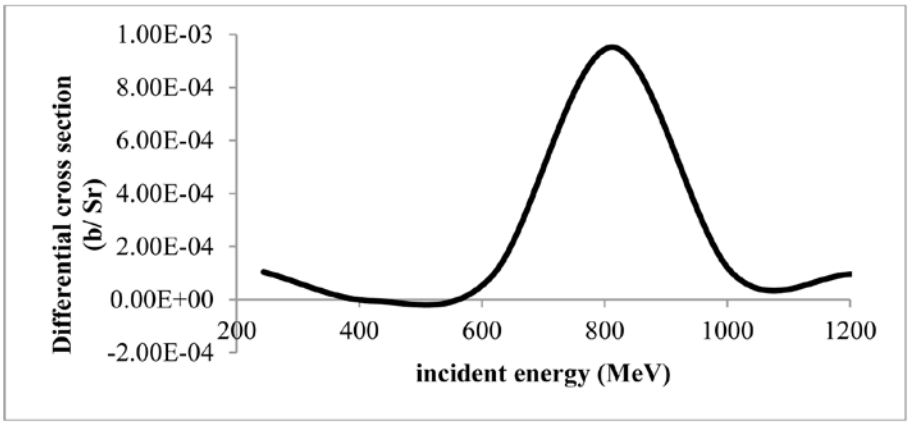
$$\frac{d^2\sigma^{elem}}{d^2\Omega_1} = K_K^{\ 2} \frac{\left|\langle t\rangle_{av}\right|}{\hbar K_\pi c^2} \frac{(\hbar c)^4}{4E_K} \times \frac{(2\pi)^4}{\hbar} \times \frac{1}{(\hbar c)^2\left[\frac{K_K}{E_K} + \frac{K_K - K_\pi \cos\theta_1}{E_\Lambda}\right]} \quad (59)$$

We solved above equation by using FORTRAN CODE to obtain differential cross section.

3. RESULT AND DISCUSSION

We formulated the transition rate independent of time. The total transition rate is the transition probability per unit time. This formula is of great practical importance. It is called Fermi's golden rule. The transition rate is as a cross section. The total rate or cross section is obtained by summing over all possible relevant states, which involves integrating over magnitudes all solid angles in the final state. This formula has been applied to such a wide variety of quantum phenomena that it was called the Golden Rule by Enrico Fermi.

The differential cross section is transition rate by incident flux. Therefore we calculated the differential cross section by using Fermi Golden Rule. Firstly, we calculated threshold energy of $\pi^+ + n \rightarrow \Lambda + K^+$ reaction and momentum of product particle lambda. Threshold energy is obtained 758.43 MeV and the momentum of product particle lambda is 430.94 MeV/c. Finally, we calculated the differential cross section by using above energy and momentum. We calculated various differential cross section for various incident energy. The results are shown in the following figure. It is seen that differential cross section is largest at incident energy 812.105 MeV. This value of differential cross section is $9.51\times10^{-4} b/Sr$.



Figure(3.1) differential cross section at various incident energy

4. CONCLUSION

We formulated the transition rate independent of time. This formula is of great practical importance. This formula has been applied to such a wide variety of quantum phenomena that it was called the Golden Rule by Enrico Fermi.

This rule is used not only the scattering of a single particle in a fixed potential but also the nuclear reaction.

The differential cross section is largest at incident energy 812.105 MeV. This value of differential cross section is $9.51\times10^{-4} b/Sr$. Therefore, this reaction is strong interaction because incident energy is high energy and strangeness is conserved.

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