ENERGY LEVELS OF LAMBDA SINGLE-PARTICLE STATES IN $^{209}_{\Lambda}$Pb

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Abstract

We calculated numerically single particle energy levels of a neutron moving in a potential well for $^{209}_{\Lambda}$Pb by using Numerov method. Firstly, the outwards and inwards eigenfunctions are obtained from the recursive formulas. We solved radial part of Schrodinger equation with harmonic oscillator potential, central potential and Woods-Saxon potential to obtain single particle energy states of a neutron in $^{209}_{\Lambda}$Pb. The sub energy levels are obtained by using Woods-Saxon potential. It is observed that the energy levels depend on spin orbit potential.

I. INTRODUCTION

A hypernucleus is a nucleus which contains at least one hyperon in addition to the normal protons and neutrons. The first was discovered by Marian Danysz and Jerzy Pniewski in 1952 using the nuclear emulsion technique.

Protons and neutrons are made of up (u) and down (d) quarks. A Lambda (Λ) hyperon consists of one up, one down, and one strange (s) quark. Like neutrons the Λ-hyperons have no charge, but Λ-s are heavier than neutrons. As a hyperon does not have to obey the Pauli Exclusion principle with the neutrons and protons, it can enter deep inside a nucleus and occupy the same levels already filled with nucleons. This property of hyperons enabled us to view the deep-lying shell model structure of nuclei [1] that can not be seen in reactions with nucleons due to the Pauli blocking. The hyperons seem to act as glue inside a nucleus. It was found that if one replaces a neutron with a Λ-hyperon, it makes a bound $^{10}_{\Lambda}Li$ nucleus, while the normal ^{10}Li nucleus is known to be unbound [2]. This indicates that hypernuclei with large neutron-to-proton ratios could exist in a stable state, even though the corresponding normal neutron-rich nuclei could be unstable.

Although the nucleon-nucleon (NN) interaction is reasonably well known, the ΛN, $\Lambda\Lambda$N and $\Lambda\Lambda$ interactions are yet to be fully understood. One studies the Λ-hypernucleus to estimate the basic Λ-nucleon interaction. Experimentally about thirty-five hypernuclei with one Λ-hyperon and six hypernuclei with two Λ-hyperons have been found so far [3–6]. The Λ and $\Lambda\Lambda$- separation energies from hypernuclei provide a window to estimate the ΛN, $\Lambda\Lambda$N- interactions properties of Λs and nucleons.

There are many kinds of energies, such as single particle energy, potential energy, kinetic energy, separation energy, and Fermi energy. We summarized in the following picture that the occupation number as a function of kinetic energy.

Since the nucleus is a highly interactive system, although the temperature of the ground state of a nucleus should be absolute Zero, but the Fermi surface is not sharp but diffusive. The Fermi energy of a nucleon , which is the maximum kinetic energy of a nucleon, is approximately ~35 MeV. The potential energy is approximately ~ 50

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MeV per nucleon. There is an additional energy for proton due to Coulomb force, which is a Coulomb barrier. The separation energy is the difference between the potential energy and Fermi energy. The single particle energy is the energy for each single particle orbit. The binding energy for a nucleon is the energy requires to set that nucleon to be free, the energy difference between the single particle energy and the potential energy.

II. NUMERICAL CALCULATION

We solved numerically the radial part of Schroedinger's equation by using Numerov's method to obtain neutron single-particle energy. Numerov's method is a numerical method to solve ordinary differential equations of second order in which the first-order term does not appear. It is a fourth-order linear multistep method. The method is implicit, but can be made explicit if the differential equation is linear.

The Numerov method can be used to solve differential equation

$$\frac{d^2u}{dr^2} = -k(r)u(r) + S(r) \tag{1}$$

Three values of u_{n-1}, u_n, u_{n+1} taken at three equidistant points r_{n-1}, r_n, r_{n+1} are related as follows:

$$u_{n+1}(1+\frac{h^2}{12}k_{n+1}) = 2u_n(1-\frac{5h^2}{12}k_n) - u_{n-1}(1+\frac{h^2}{12}k_{n-1}) + \frac{h^2}{12}(S_{n+1}+10S_n+S_{n-1}) + 0(h^5) \tag{2}$$

Where $u_n = u(r_n)$, $k_n = k(r_n)$, $S_n = S(r_n)$ and $h = r_{n+1}-r_n$.

For nonlinear equations of the form

$$\frac{d^2u}{dr^2} = f(u,r) \tag{3}$$

the method gives

$$u_{n+1}-2u_n + u_{n-1} = \frac{h^2}{12}(f_{n+1}+10f_n+f_{n-1}) + 0(h^5) \tag{4}$$

This is an implicit linear multistep method which reduces to the explicit method given above if f is linear in u by setting f (u, r) = -k(r)u(r) + S(r). It achieves order-4 accuracy.

In numerical physics the method is used to find solutions of the unidimensional Schrodinger equation for arbitrary potentials. It is used to solve the radial equation for a spherically symmetric potential. After separating the variables and analytically solving the angular equation, we get the following equation of the radial function R(r):

$$\frac{d^2u(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right]u(r) = 0 \tag{5}$$

Where $u(r) = rR_{nl}$ is the reduced radial wave function.

A regular solution near the origin for $u(r) : u(r \to 0) \to r^{\ell+1}$

The asymptotic solution at $r \to \infty : u(r \to \infty) \to u(r) = e^{-\alpha r^2}$, α = constant

The Schrödinger Radial Equation can be written as follow:

$$\frac{d^2u(r)}{dr^2} + k(r)u(r) = 0 \tag{6}$$

$$\frac{d^2u(r)}{dr^2} = u_r'' = -k(r)u(r) \tag{7}$$

$k(r) = \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right]$ is the kernel equation.

Equation (5) can be solved by Numerov Algorithm as follow:

First we split the r range into N points according to $r_n = r_{n-1} + h$;then we write the wave function $u_n \equiv u(r_n) = u(r_{n-1}+h)$ and $k_n \equiv k(r_n) = k(r_{n-1}+h)$.

Expanding u(r) around r_n:

$$u(r) = u(r_n) + (r - r_n)u'(r_n) + \frac{(r - r_n)^2}{2!}u''(r_n) + \frac{(r - r_n)^3}{3!}u'''(r_n)$$
$$+ \frac{(r - r_n)^4}{4!}u^{iv}(r_n) + 0(h^5) \tag{8}$$

If we evenly discretize the space, we get a grid of r points, where $h = r_{n+1} - r_n$. By applying the above equations to this discreet space, we get a relation between the u_n and u_{n+1}:

$$u_{n+1} \equiv u(r_n + h) = u(r_n) + hu'(r_n) + \frac{h^2}{2!}u''(r_n) + \frac{h^3}{3!}u'''(r_n) + \frac{h^4}{4!}u^{iv}(r_n) + 0(h^5) \tag{9}$$

Since $u_n \equiv u(r_n)$,

$$u_{n+1} = u_n + hu_n' + \frac{h^2}{2}u_n'' + \frac{h^3}{6}u_n''' + \frac{h^4}{24}u_n^{iv} + 0(h^5) \tag{10}$$

Computationally, this amounts to taking a stepforward by an amount h. If we want to take a stepbackwards, we repleace every h with –h and get the expression for u_{n-1}:

$$u_{n-1} \equiv u(r_n - h) = u(r_n) - hu'(r_n) + \frac{h^2}{2!}u''(r_n) - \frac{h^3}{3!}u'''(r_n) + \frac{h^4}{4!}u^{iv}(r_n) + 0(h^5) \tag{11}$$

$$u_{n-1} = u_n - hu_n' + \frac{h^2}{2}u_n'' + \frac{h^3}{6}u_n''' + \frac{h^4}{24}u_n^{iv} + 0(h^5) \tag{12}$$

By summing the two equation (10) and equation (12),

$$u_{n+1} - 2u_n + u_{n-1} = h^2u_n'' + \frac{h^4}{12}u_n^{iv} + 0(h^5) \tag{13}$$

To get an expression for the u_n^{iv} factor, we simply have to differentiate $u_n'' = -k_n u_n$ twice and approximate it again in the same way we did this above:

$$u_n^{iv} = \frac{d^2}{dx^2}(-k_n u_n) \tag{14}$$

From equation (13),

$$u_{n+1} - 2u_n + u_{n-1} = h^2u_n'' \tag{15}$$

$$u_{n+1}'' - 2u_n'' + u_{n-1}'' = h^2u_n^{iv} \tag{16}$$

$$-u_{n+1}k_{n+1} + 2u_nk_n - u_{n-1}k_{n-1} = h^2u_n^{iv} \tag{17}$$

Substituting equation (17) into equation (13)

$$u_{n+1} - 2u_n + u_{n-1} = -h^2 u_n k_n + \frac{h^2}{12}(-u_{n+1}k_{n+1} + 2u_n k_n - u_{n-1}k_{n-1}) \quad (18)$$

$$u_{n+1}(1+\frac{h^2}{12}k_{n+1}) - 2u_n(1-\frac{5h^2}{r^2}k_n) + u_{n-1}(1+\frac{h^2}{12}k_{n-1}) = 0 \quad (19)$$

from equation (19), we obtained the following relations.

For forward recursive relation

$$u_n = \frac{2(1-\frac{5h^2}{12}k_{n-1})u_{n-1} - (1+\frac{h^2}{12}k_{n-2})u_{n-2}}{(1+\frac{h^2}{12}k_n)} \quad (20)$$

For backward recursive relation

$$u_{n-1} = \frac{2(1-\frac{5h^2}{12}k_n)u_n - (1+\frac{h^2}{12}k_{n+1})u_{n+1}}{(1+\frac{h^2}{12}k_{n-1})} \quad (21)$$

Therefore when we calculate our wavefunction using the backward-forward technique, we should note that the recursive formulas imply having knowledge of two initial values for each direction. It is also necessary to know the first derivative at the appropriate order. By subtraction from equation (20) to equation (21)

$$u_{n+1} - u_{n-1} = 2hu'_n + 2\frac{h^3}{6}u'''_n \quad (22)$$

$$u'_n = \frac{1}{2h}\left[(1+\frac{h^2}{6}k_{n+1})u_{n+1} - (1+\frac{h^2}{6}k_{n-1})u_{n-1}\right] \quad (23)$$

We calculated single neutron energy levels of $^{209}_{\Lambda}\mathrm{Pb}$ by using central potential and Wood's-Saxon potential as shown in Fig. (1), (2) and (3).

III. CALCULATION OF ENERGY LEVELS

Since both $u_{out}(r)$ and $u_{in}(r)$ satisfy a homogeneous equation, their normalization can always be chosen so that they are set to be equal at the r_c point. An energy eigenvalue is then signaled by the equality of derivatives at this point. At the matching point the eigenfunctions $u_{out}(r)$ and $u_{in}(r)$ and first derivatives $u'_{out}(r)$ and $u'_{in}(r)$ must all satisfy the continuity conditions:

$$(u_{out})_{rc} = (u_{in})_{rc} \qquad (u'_{out})_{rc} = (u'_{in})_{rc} \quad (24)$$

thus, we can write the corresponding condition for the logarithmic derivative at r_c as

$$\left[\frac{u'_{out}}{u_{out}}\right]_{rc} = \left[\frac{u'_{in}}{u_{in}}\right]_{rc} \quad (25)$$

and then we can define a Match (E) function at r_c whose zeros correspond to the energy eigenvalues as

$$\mathrm{Match(E)} = \left[\frac{u'_{out}}{u_{out}}\right]_{rc} - \left[\frac{u'_{in}}{u_{in}}\right]_{rc} \quad (26)$$

Therefore we proceed numerically in the following way: we set a trial energy range splitting this E range into N points, according to $E_n = E_{n-1} + \Delta E$, where ΔE is the

energy step. For each E_n we calculate their eigenfunctions u_{out} and u_{in} at the r_c point; and we build the Match(E) function here, looking for a change of sign in it (which implies a zero cross).

When we find the energy eigenvalue, the calculated inwards and outwards eigenfunctions will tend not to match at the r_c point. However we can look for a strategy to solve this problem: Denoting the outwards and inwards functions directly obtained from the recursive formulas as uo(r) and ui (r), respectively, the physical $u_{out}(r)$ and $u_{in}(r)$ eigenfunctions can be rewritten as

$u_{out}(r) = A\ uo(r) \qquad u_{in}(r) = B\ ui(r)$ (27)

A and B are constants. Their respective derivatives are

$$u'_{out}(r) = A\,uo'(r) \qquad u'_{in}(r) = B\,ui'(r) \tag{28}$$

$$(A\,uo(r))_{r_c} = (B\,ui(r))_{r_c} \qquad (A\,uo'(r))_{r_c} = (B\,ui'(r))_{r_c} \tag{29}$$

and performing the difference, we get

$$A = \left[\frac{ui - ui'}{uo - uo'}\right] B \equiv f_c B \tag{30}$$

where fc will be a scaling factor to be applied to $u_{out}(r)$. Therefore

$u_{out}(r) = B\ f_c\ uo(r) \qquad u_{in}(r) = B\ ui$ (31)

and B is a global factor that must be taken into account in the normalization process.

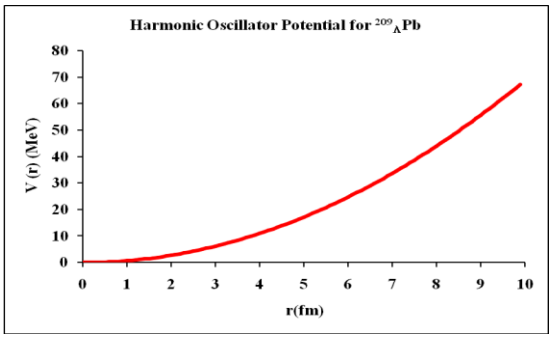

Fig. (1) Harmonic-Oscillator Potential for $^{209}_{\Lambda}$Pb

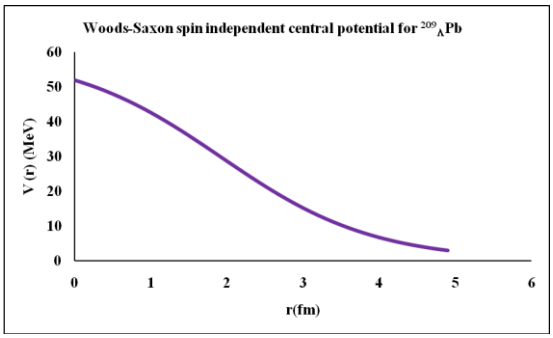

Fig. (2) Woods-Saxon spin independent central potential for $^{209}_{\Lambda}$Pb

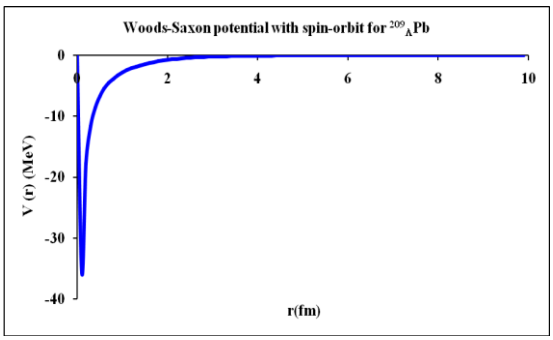

Fig. (3) Woods-Saxon potential with spin-orbit for $^{209}_{\Lambda}$Pb

IV. **RESULTS AND DISCUSSION**

We calculated wave functions for various state such as s, p, d and g states. The results are shown in Fig. (1), (2), (3) and (4). Fig.(1) shows lambda s-states wave functions

with harmonic oscillator potential for ${}^{209}_{\Lambda}\mathrm{Pb}$. Fig. (2) and (3) show lambda p and d-states wave functions with harmonic oscillator potential for ${}^{209}_{\Lambda}\mathrm{Pb}$. Fig. (4) shows neutron wave functions of 1s, 1p and 1d states with harmonic oscillator potential for ${}^{209}_{\Lambda}\mathrm{Pb}$. It is seen that all wave functions are finite.

The number of neutrons are allowed by the Pauli principle to occupy one of these levels. n is the radial quantum number (the number of times the radial wave function changes sign) and ℓ is the angular momentum quantum number represented in the spectroscopic notations s, p, d, f, g, h, i and j for ℓ=0, 1, 2, 3, 4, 5, 6 and 7, respectively. Each ℓ value can have $2\ell+1$ m states and each m state can contain a proton or neutron with spin up and spin down ($s_z=\pm 1/2$). The occupation number given by the Pauli principle is thus $No = 2(2\ell + 1)$.

The lambda single-particle states in ${}^{209}_{\Lambda}\mathrm{Pb}$ with three potential models, harmonic oscillator , Woods-Saxon without spin-orbit and Woods-Saxon with spin orbit are obtained by solving Schrödinger equation. The results are shown in Table (1). It is indicated the effect of the spin-orbit potential in splitting the states of a given ℓ value. The overall strength of the spin-orbit potential has been determined empirically. We compare our results with experimental results [7] and theoretical results [8].

Table (1) Comparison of theoretical results and experimental results of Lambda single-particle energies in ${}^{209}_{\Lambda}\mathrm{Pb}$

Sub Shell	Theoretical results [8] (MeV)			Experimental results [7] (MeV)	Our results (MeV)	
	O	A	F		Central Potential	Spin-Orbit Potential
1s	-23.1	-29.5	-26.5	-27.0	-25.39	-25.30
1p	-19.6	-25.7	-22.4	-22.0	-20.8	-20.89
1d	-14.5	-21.0	-17.5	-17.0	-15.39	-15.60
1f	-10.5	-15.7	-11.8	-12.0	-9.3	-9.59
1g	-5.1	-9.7	-5.6	-7.0	-2.79	-3.29

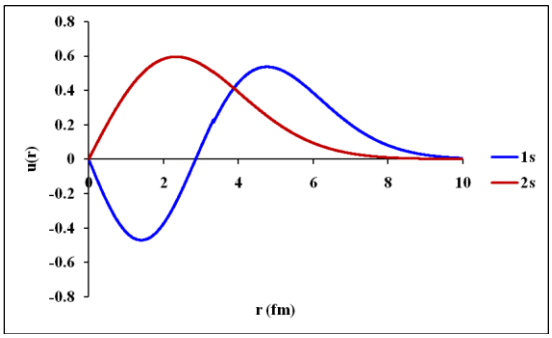

Fig. (1) Neutron s-states wave functions with HO potential for ${}^{209}_{\Lambda}\mathrm{Pb}$

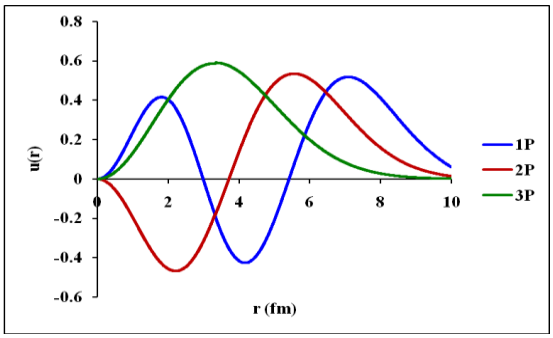

Fig. (2) Neutron p-states wave functions with HO potential for ${}^{209}_{\Lambda}\mathrm{Pb}$

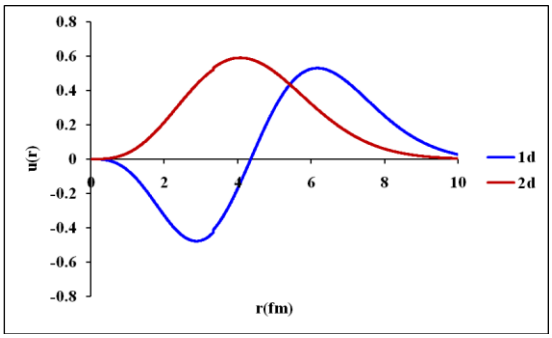

Fig. (3) Neutron d-states wave functions with HO potential for $^{209}_{\Lambda}$Pb

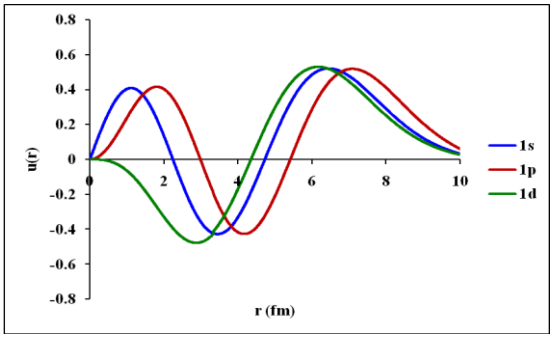

Fig. (4) Neutron wave functions of 1s, 1p and 1d state with HO potential for $^{209}_{\Lambda}$Pb

V. CONCLUSION

We calculated lambda single particle energy $^{209}_{\Lambda}$Pb in by using Numerov's method for central potential and spin-orbit potential. Our results are compared with experimental results [7] and other theoretical results [8]. A. Vidana et al., [8] are calculated Brueckner-Hartree-Fock method for Nijmegen various O, A and F which are divided according to effective mass of lambda $m^{*}_{\Lambda}/m_{\Lambda} = 0.78, 0.67$ and 0.81. Our results are a little different with those results and experimental results.

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