Hausdorff Distance between Subsets of a Metric Space

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Abstract

In this paper, after defining Hausdorff distance, the properties are described. Then, the space of closed and bounded subsets of a metric space endowed with the Hausdorff distance is presented.

Keywords: Hausdorff distance, metric space

Introduction

The Hausdorff distance gives the largest length of the set of all distances between each point of a set to the closest point of a second set. We will study the Hausdorff distance between two subsets of a metric space (see [Rudin, W., 1953]) and the space of closed and bounded (see [Rudin, W., 1953]) subsets of a metric space endowed with the Hausdorff distance.

Definitions 1

Let (X, d) be a metric space and A, B be nonempty subsets of X.

We define $d(a , B) = \inf_{b\in B} d(a, b)$. If $B = \varnothing$, then we define $d(a , B) = \infty$.

If $B \neq \varnothing$, then $\exists\, b \in B$ and so $d(a , B) = \inf_{b\in B} d(a , b) \leq d(a , b) < \infty$.

Thus $0 \leq d(a , B) < \infty$, if $B \neq \varnothing$. We also define $e(A, B) = \sup_{a\in A} d(a , B)$. Then $e(A, B)$ is called ***excess of A over B*** . The ***Hausdorff distance*** between two sets A and B is defined as $d_H(A , B) = \max \{e(A , B) , e(B , A) \}$.

Example

Let A and B set defined by $A=\{(x, y); 0 \leq x\leq 1, 0 \leq y \leq 1\}$ and let $B=\{(x, y); 3 \leq x \leq 5, 0 \leq y \leq 4\}$.

If $(a_1, a_2) \in A$, then $d ((a_1, a_2), B) =d((a_1, a_2), (3, a_2)) =3-a_1$.

Since $0 \leq a_1 \leq 1$, we find that $e (A, B) =3$.

If $(b_1, b_2) \in B$, then $d ((b_1, b_2), A) =d((b_1, b_2), (1, a_2))$ where $0 \leq a_2 \leq 1$, which varies our choice of (b_1, b_2). We find that

$e (B, A) = d((5, 4), (1, 1)) = 5$.

Therefore the Hausdorff distance is given by $d_H (A, B) = e (B, A) = 5$.

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Lemma 1

Let A, B , C be nonempty subsets of a metric space (X , d) . Then we can verify the followings:

(a) e(A , B) is not necessarily equal to e(B , A) .

(b) $0 \le e(A , B) \le \infty$ and $e(A , B) < \infty$, if X is bounded.

(c) e(A , A) = 0

(d) $e(A , B) \le e(A , C) + e(C , B)$

(e) If $N_r(A) = \{ x \in X \mid d(x , A) < r \}$,

then $e(A , B) = \inf_{r>0,\ A\subset N_r(B)} r$ and since the map $x \mapsto d(x , A)$ is continuous $N_r(A)$ is open.

(f) $d(a , B) = 0 \Rightarrow a \in \overline{B}$.

(g) $e(A , B) = 0 \Leftrightarrow A \subset \overline{B}$.

(h) If $A \subset B$ then e(A , B) = 0.

(i) $d_H(A , B) = 0 \Leftrightarrow \overline{A} = \overline{B}$

Proof:

(a) Let A = { 1 , 2 } and B = { 5 , 6 , 7 }

e(A , B) = sup { d(1 , B) , d(2 , B) } = sup { 4 , 3 } = 4

e(B , A) = sup { d(5 , A) , d(6 , A) , d(7 , A) } = sup { 3 , 4 , 5 } = 5 .

(b) It is obvious that $e(A , B) \ge 0$.

If X is bounded then $\exists M > 0$ such that $d(x , y) \le M , \forall x , y \in X$.

Then for any $x \in A$, $d(x , B) = \inf_{y\in B} d(x , y) \le M$, so that

$e(A , B) = \sup_{x\in A} d(x , B) \le M < \infty$.

(c) $A \subset \overline{A} \Rightarrow e(A , A) = 0$.

(d) Take any $\varepsilon > 0$ and $a \in A$. Since $d(a ,C) = \inf_{c\in C} d(a , c) < d(a , C) + \frac{\varepsilon}{2}$,

$\exists c \in C$ such that $d(a , c) < d(a , C) + \frac{\varepsilon}{2} \le \sup_{a\in A} d(a , C) + \frac{\varepsilon}{2} = e(A , C) + \frac{\varepsilon}{2}$.

Similarly, $\exists b \in B$ such that $d(c , b) < e(C , B) + \frac{\varepsilon}{2}$.

Then, $d(a , B) = \inf_{b\in B} d(a , b) \le d(a , b) \le d(a , c) + d(c , b) < e(A , C) + e(C , B) + \varepsilon$.

Thus $e(A, B) = \sup_{a \in A} d(a, B) \le e(A, C) + e(C, B) + \varepsilon$, and since ε is arbitrary it follows that $e(A, B) \le e(A, C) + e(C, B)$.

(e) Let $L = e(A, B) = \sup_{a \in A} d(a, B)$. Then for any $a \in A$, $d(a, B) \le L$.

So, $A \subset N_L(B)$ and $\inf_{r>0,\ A \subset N_r(B)} r \le L$. Suppose $\inf_{r>0,\ A \subset N_r(B)} r < r_0 < L$.

Then $\exists\, r > 0$ such that $A \subset N_r(B)$ and $r < r_0$. Then $A \subset N_{r_0}(B)$.

Thus, $x \in A \Rightarrow d(x, B) < r_0$ and so $e(A, B) = \sup_{x \in A} d(x, B) \le r_0 < L = e(A, B)$.

So, $\inf_{r>0,\ A \subset N_r(B)} r \ge L$ and $\inf_{r>0,\ A \subset N_r(B)} r = e(A, B)$.

(f) Suppose $d(a, B) = 0$. Then $\forall\, n \in Z^+$, $\inf_{b \in B} d(a, b) < \frac{1}{n}$.

So, $\exists\, b_n \in B$ such that $0 \le d(a, b_n) < \frac{1}{n} \to 0$. Thus, $b_n \to a$ and $a \in \overline{B}$.

(g) Suppose $e(A, B) = 0$. Then $\sup_{a \in A} d(a, B) = 0$, and $d(a, B) = 0$, $\forall a \in A$.

Thus $a \in A \Rightarrow d(a, B) = 0 \Rightarrow a \in \overline{B}$ which implies that $A \subset \overline{B}$.

On the other hand, suppose that $A \subset \overline{B}$.

Take any $a \in A$, and any $\varepsilon > 0$. Then since $a \in \overline{B}$, $\exists\, b_0 \in B$ such that $d(a, b_0) < \varepsilon$.

Thus $0 \le d(a, B) = \inf_{b \in B} d(a, b) \le d(a, b_0) < \varepsilon$, $\forall\, \varepsilon > 0$.

Thus $0 \le d(a, B) < \varepsilon$, $\forall\, \varepsilon > 0$ and so

$d(a, B) = 0$ which implies that $e(A, B) = \sup_{a \in A} d(a, B) = 0$.

(h) It is obvious.

(i) $d(A, B) = 0 \Leftrightarrow e(A, B) = 0$ and $e(B, A) = 0 \Leftrightarrow A \subset \overline{B}$ and $B \subset \overline{A}$

$\Leftrightarrow \overline{A} \subset \overline{B}$ and $\overline{B} \subset \overline{A} \Leftrightarrow \overline{A} = \overline{B}$.

Definitions 2

Let (X, d) be a metric space and A and B be nonempty subsets of X. We define $E_\varepsilon(A) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}$. $E_\varepsilon(A)$ is called the ***ε-expansion of A***. It is obvious that $E_\varepsilon(A) = \bigcup_{a \in A} B(a, \varepsilon)$ = union of all ε-balls around points in A. We also define

$H(A, B) = \inf\{ \varepsilon > 0 \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A) \}$,

$D(A, B) = \inf \{\varepsilon > 0 \mid A \subset E_\varepsilon(B) \text{ and } B \subset E_\varepsilon(A) \}$, and $\mathcal{CB}(X)$ as the ***space of nonempty closed and bounded subsets*** of X.

Lemma 2

Let (X, d) be a metric space and A, B , C be nonempty subsets of X. Then we can deduce that

(a) $E_\varepsilon(A) = N_\varepsilon(A)$.

(b) $H(A, B) = D(A, B)$.

(c) $d_H(A, B) = D(A, B)$.

(d) If $\varepsilon_1 \le \varepsilon_2$ then $N_{\varepsilon_1}(A) \subset N_{\varepsilon_2}(A)$.

(e) If $A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(C)$, then $A \subset N_{2\varepsilon}(C)$.

(f) If A is bounded then $N_\varepsilon(A)$ is bounded.

Proof:

(a) Suppose $x \in E_\varepsilon(A)$. Then $\exists\, a \in A$, such that $d(x , a) < \varepsilon$.

Then $d(x , A) = \inf_{a \in A} d(x , a) \le d(x , a) < \varepsilon$. So $x \in N_\varepsilon(A)$.

For the converse, assume that $x \in N_\varepsilon(A)$. Then $d(x , A) = \inf_{a \in A} d(x , a) < \varepsilon$.

So $\exists\, a \in A$ such that $d(x, a) < \varepsilon$. Hence $x \in E_\varepsilon(A)$.

(b) By definition, $H(A , B) = \inf\{\varepsilon > 0 \mid A\subset N_\varepsilon(B) \text{ and } B\subset N_\varepsilon(A) \}$.

From (a), $N_\varepsilon(A) = E_\varepsilon(A)$.

Hence $H(A , B) = \inf\{\varepsilon > 0 \mid A\subset N_\varepsilon(B) \text{ and } B\subset N_\varepsilon(A) \}$

$= \inf \{\varepsilon > 0 \mid A\subset E_\varepsilon(B) \text{ and } B\subset E_\varepsilon(A) \} = D(A , B)$.

By Lemma 1.(e), $e(A , B) = \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \}$.

Thus $d_H(A , B) = \max \{e(A , B) , e(B , A)\}$

$= \max\{ \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \} , \inf\{\varepsilon > 0 \mid B \subset N_\varepsilon(A) \} \}$.

Suppose $A \subset N_\delta(B)$ and $B \subset N_\delta(A)$.

Then $\inf \{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \} \le \delta$ and $\inf\{\varepsilon > 0 \mid B \subset N_\varepsilon(A) \} \le \delta$.

So $D(A , B) = \inf\{\varepsilon > 0 \mid B \subset N_\varepsilon(A) \text{ and } A \subset N_\varepsilon(B) \} \le \delta$.

Thus $D(A , B) \le \delta$, $\forall\, \delta > 0$ such that $A \subset N_\delta(B)$ and $B \subset N_\delta(A)$.

Taking infimum gives $D(A , B) \le d_H(A , B)$.

Suppose $D(A , B) < d_H(A , B)$.

Since $D(A , B) = \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}$, $\exists\, \varepsilon > 0$ such that

$A \subset N_\varepsilon(B)$, $B \subset N_\varepsilon(A)$ and $\varepsilon < d_H(A , B)$. Since $A\subset N_\varepsilon(B)$, $\forall\, a \in A$, $d(a , B) < \varepsilon$.

So $e(A , B) = \sup_{a\in A} d(a , B) \le \varepsilon$.

Similarly we can show that $e(B , A) \le \varepsilon$.

Hence $d_H(A , B) = \max\{e(A , B) , e(B , A)\} \le \varepsilon < d(A , B)$.

So $D(A , B) \ge d_H(A , B)$ and consequently $d_H(A , B) = D(A , B)$.

(d) $N_{\varepsilon_1}(A) = \{x\in X \mid d(x, A) < \varepsilon_1\}$. Let $x \in N_{\varepsilon_1}(A)$. Then $d(x, A) < \varepsilon_1$.

Since $\varepsilon_1 \le \varepsilon_2$, $d(x , A) < \varepsilon_2$ and so $x \in N_{\varepsilon_2}(A)$. Hence $N_{\varepsilon_1}(A) \subset N_{\varepsilon_2}(A)$.

(e) Let $a\in A$. Then $d(a, B) < \varepsilon$, since $a \in N_\varepsilon(B)$.

Since $d(a , B) = \inf_{y\in B} d(a , y)$, $\exists\, b \in B$ such that $d(a , b) < \varepsilon$.

Since $b \in N_\varepsilon(C)$, $d(b , C) < \varepsilon$ and so $\exists\, c\in C$ such that $d(b , c) < \varepsilon$.

Then $d(a , c) \le d(a , b) + d(b , c) < \varepsilon + \varepsilon = 2\varepsilon$.

Hence $d(a , C) = \inf_{z\in C} d(a , z) \le d(a , c) < 2\varepsilon$ and so $a \in N_{2\varepsilon}(C)$.

Hence $A \subset N_{2\varepsilon}(C)$.

(f) Suppose A is bounded. So $\exists a_0 \in A$ and $\lambda > 0$ such that $d(a , a_0) \le \lambda$, $\forall\, a \in A$.

Let $x\in N_\varepsilon(A)$. Then $d(x , A) < \varepsilon$.

Since $d(x , A) = \inf_{a\in A} d(x , a)$, $\exists\, a\in A$ such that $d(x , a) < \varepsilon$.

Then $d(x , a_0) \le d(x , a) + d(a , a_0) < \varepsilon + \lambda$.

Hence $d(x , a_0) \le \varepsilon + \lambda$, $\forall\, x \in N_\varepsilon(A)$ and so $N_\varepsilon(A)$ is bounded.

Lemma 3

Let X be a metric space and A, B, C be nonempty subsets of X. Then $e(A\cup B, C) \le \max(e(A, C), e(B, C))$.

Proof:

Take any $x \in A\cup B$.

If $x\in A$ then $d(x , C) \le e(A , C) = \sup_{a\in A} d(a , C) \le \max\{e(A , C) , e(B , C)\}$.

If $x \in B$ then $d(x, C) \le e(B, C) = \sup_{b \in B} d(b, C) \le \max\{e(A, C), e(B, C)$.

Hence $d(x, C) \le \max\{e(A, C), e(B, C)\}$, $\forall\, x \in A \cup B$.

Thus $e(A \cup B, C) = \sup_{x \in A \cup B} d(x, C) \le \max\{e(A, C), e(B, C)\}$. □

Lemma 4

Let X be a metric space and A, B, C, D be nonempty subsets of X. Then $e(A \cup B, C \cup D) \le \max\{e(A, C), e(B, D)\}$.

Proof:

Take any $x \in A \cup B$. Suppose $x \in A$. Consider $d(x, C \cup D)$. Take any $y \in C$.

Then, $y \in C \cup D$. Hence, $d(x, C \cup D) = \inf_{z \in C \cup D} d(x, z) \le d(x, y)$.

So, $d(x, C \cup D) \le d(x, y)$, $\forall\, y \in C$.

Hence, $d(x, C \cup D) \le \inf_{y \in C} d(x, y) = d(x, C) \le \sup_{x \in A} d(x, C)$

$= e(A, C) \le \max\{e(A, C), e(B, D)\}$.

Suppose $x \in B$ and consider $d(x, C \cup D)$. We will take any $y \in D$.

Then, $y \in C \cup D$. Hence $d(x, C \cup D) = \inf_{z \in C \cup D} d(x, z) \le d(x, y)$.

So, $d(x, C \cup D) \le d(x, y)$, $\forall\, y \in D$.

Hence, $d(x, C \cup D) \le \inf_{y \in D} d(x, y) = d(x, D) \le \sup_{x \in B} d(x, D)$

Then, $d(x, C \cup D) \le \max\{e(A, C), e(B, D)\}$, $\forall\, x \in A \cup B$.

So, $e(A \cup B, C \cup D) = \sup_{x \in A \cup B} d(x, C \cup D) \le \max\{e(A, C), e(B, D)\}$.

$= e(B, D) \le \max\{e(A, C), e(B, D)\}$.

Lemma 5

Let X be a metric space and A, B , C , D be nonempty subsets of X. Then $d_H(A \cup B, C \cup D) \le \max\{d_H(A, C), d_H(B, D)\}$.

Proof

By Lemma 4,

$e(A \cup B, C \cup D) \le \max\{e(A, C), e(B, D)\} \le \max\{d_H(A, C), d_H(B, D)\}$.

Similarly $e(C \cup D, A \cup B) \le \max\{e(C, A), e(D, B)\} \le \max\{d_H(A, C), d_H(B, D)\}$.

So $d_H(A \cup B , C \cup D) = \max\{ e(A \cup B, C \cup D) , e(C \cup D , A \cup B) \}$

$\leq \max \{d_H(A, C), d_H(B, D) \}$. □

Theorem 1

Let (X , d) be a metric space, and $\mathcal{CB}(X)$ be the collection of nonempty closed and bounded subsets of X. Then $(\mathcal{CB}(X) , d_H)$ is a metric space.

Proof:

Recall that $d_H(A , B) = \max\{e(A , B) , e(B , A)\}$.

By Lemma 1 (i), $d_H(A , B) = 0 \Leftrightarrow \overline{A} = \overline{B}$. Since A and B are closed sets, $A = \overline{A}$ and $B = \overline{B}$. Thus we conclude that $d_H(A , B) = 0 \Leftrightarrow A = B$.

To show triangle inequality, we take $A , B \in \mathcal{CB}(X)$.

Since A is bounded, $\exists\, x_1 \in X$ and $r_1 > 0$ such that $A \subset B(x_1 , r_1)$.

Since B is bounded, $\exists\, x_2 \in X$ and $r_2 > 0$ such that $B \subset B(x_2 , r_2)$.

Let $r = r_1 + r_2 + d(x_1 , x_2)$, and consider $B(x_1 , r)$. Then $a \in A \Rightarrow d(a , x_1) < r_1 < r$.

Also $b \in B \Rightarrow d(b, x_1) \leq d(b, x_2) + d(x_2, x_1) < r_2 + d(x_1, x_2) < r$.

Thus $A \cup B \subset B(x_1 , r)$.

If $a \in A$ and $b \in B$, then $d(a , b) \leq d(a , x_1) + d(x_1 , x_2) + d(x_2 , b) = r$.

Thus $d(a , b) < r \ \forall\, a \in A, b \in B$,

and so $e(A , B) = \sup_{a \in A} d(a , B) = \sup_{a \in A} \inf_{b \in B} d(a , b) \leq r < \infty$.

Similarly $e(B , A) < \infty$ and by Lemma 1 (b), $0 \leq d_H(A , B) < \infty$.

By Definition 2,

$d_H(A , B) = \max\{e(A , B) , e(B , A)\}= \max\{e(B , A) , e(A , B)\} = d_H(B , A), \forall\, A , B \in \mathcal{CB}(X)$.

By Lemma 1(d), $e(A , C) \leq e(A , B) + e(B , C)$.

Thus $e(A , C) \leq d_H(A , B) + d_H(B , C)$.

Similarly $e(C , A) \leq d_H(C , B) + d_H(B , A) = d_H(B , C) + d_H(A , B)$.

Thus $d_H(A , C) = \max \{e(A , C) , e(C , A)\} \leq d_H(A , B) + d_H(B , C)$. □

Now we will study the completeness of the space $\mathcal{CB}(X)$.

Theorem 2

If (X, d) is a complete metric space , then the space $(\mathcal{CB}(X), d_H)$ is also complete.

Proof:

Take any Cauchy sequence $\{D_k\}$ in $\mathcal{CB}(X)$, and any $\varepsilon > 0$.

Observe that D_k are closed and bounded subsets of X .

So $\exists\, N_1 \in Z^+$ with $N_1 > 1$, such that $j , k \geq N_1 \Rightarrow d_H(D_k , D_j) < \frac{\varepsilon}{2}$.

Similarly, $\exists N_2 \in Z^+$ with $N_2 > 2$ and $N_2 > N_1$ such that $j , k \geq N_2 \Rightarrow d_H(D_j , D_k) < \frac{\varepsilon}{2^i}$.

Similarly, $\exists N_i \in Z^+$ with $N_i > N_{i-1}$, and $N_i > i$ such that $j, k \geq N_i \Rightarrow d_H(D_k , D_j) < \frac{\varepsilon}{2^i}$.

So $j, k \geq N_i \Rightarrow \max\{e(D_k , D_j), e(D_j , D_k)\} < \frac{\varepsilon}{2^i} \Rightarrow e(D_k , D_j) < \frac{\varepsilon}{2^i}$ and $e(D_j , D_k) < \frac{\varepsilon}{2^i}$

$$\Rightarrow \inf\{\varepsilon > 0 \mid D_k \subset N_\varepsilon(D_j)\} < \frac{\varepsilon}{2^i} \text{ and } \inf\{\varepsilon > 0 \mid D_j \subset N_\varepsilon(D_k)\} < \frac{\varepsilon}{2^i} .$$

Suppose $j , k \geq N_i$. Then $\exists\, \delta_1 > 0$ such that $D_k \subset N_{\delta_1}(D_j)$ with $\delta_1 < \frac{\varepsilon}{2^i}$ and $\exists\, \delta_2 > 0$

such that $D_j \subset N_{\delta_2}(D_k)$ with $\delta_2 < \frac{\varepsilon}{2^i}$. But we know that $\varepsilon_1 \leq \varepsilon_2 \Rightarrow N_{\varepsilon_1}(D_j) \subset N_{\varepsilon_2}(D_j)$.

Hence, we have a strictly increasing sequence $\{N_i\}$ of positive integers with $N_i > i$ such that

$j , k \geq N_i \Rightarrow D_k \subset N_{\frac{\varepsilon}{2^i}}(D_j)$ and $D_j \subset N_{\frac{\varepsilon}{2^i}}(D_k)$. (1)

Now we claim that, $\forall$ pair (x, k) with $x \in D_k$ and $k \geq N_i$, $\exists\, y_j \in D_j$ for $j \geq k$ such that

$$d(x, y_j) < \frac{\varepsilon}{2^i} . \tag{2}$$

By (1), $x \in D_k \Rightarrow x \in N_{\frac{\varepsilon}{2^i}}(D_j) \Rightarrow d(x, D_j) < \frac{\varepsilon}{2^i}$.

Since $d(x, D_j) = \inf_{y \in D_j} d(x, y)$, $\exists y_j \in D_j$ such that $d(x, y_j) < \frac{\varepsilon}{2^i}$.

Hence, (2) is satisfied.

Now we will construct a Cauchy sequence $\{x_k\}$ with $x_k \in D_k$.

For $k < N_1$, we choose any $x_k \in D_k$. Suppose $N_1 \leq k , j \leq N_2$. Let $x_{N_1} \in D_{N_1}$.

By (2), $\exists x_j \in D_j$ with $d(x_j , x_{N_1}) < \frac{\varepsilon}{2}$, $x_k \in D_k$ with $d(x_k, x_{N_1}) < \frac{\varepsilon}{2}$, and also

$d(x_k, x_j) \le d(x_k, x_{N_1}) + d(x_{N_1} + x_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Hence $x_{N_2} \in D_{N_2}$ and $d(x_{N_1}, x_{N_2}) < \frac{\varepsilon}{2}$.

Suppose $N_2 \le j$, $k \le N_3$.

Applying (2) again, $\exists$ $x_j \in D_j$ with $d(x_j, x_{N_2}) < \frac{\varepsilon}{2^2}$, $\exists$ $x_k \in D_k$ with $d(x_k, x_{N_2}) < \frac{\varepsilon}{2^2}$,

and also $d(x_k, x_j) < \frac{\varepsilon}{2^2} + \frac{\varepsilon}{2^2} < \frac{\varepsilon}{2}$.

Hence we have a sequence $\{x_k\}$ with $x_k \in D_k$ and for $N_i \le j, k$ $\exists$ $x_j \in D_j$ with $d(x_j, x_{N_i}) < \frac{\varepsilon}{2^i}$, $\exists$ $x_k \in D_k$ with $d(x_k, x_{N_i}) < \frac{\varepsilon}{2^i}$, and also $d(x_k, x_j) < \frac{\varepsilon}{2^i} + \frac{\varepsilon}{2^i} < \frac{\varepsilon}{2^{i-1}}$.

It is obvious that this sequence is a Cauchy sequence and so it has a limit point say x_0.

Let F be the set of such limit points of sequences $\{x_k\}$ with $x_k \in D_k$.

i.e., $F = \liminf_{n\to\infty} D_n$. Then F is closed and we have proved that $F \ne \varnothing$.

Take any $x \in D_k$, with $k \ge N_1$ and $j \ge k$.

Then $d(x, x_j) \le d(x, x_{N_i}) + d(x_{N_i}, x_{N_2}) + \ldots + d(x_{N_{j-1}}, x_{N_j}) + d(x_{N_j}, x_j)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^j} \le \varepsilon\left(\frac{1}{2} + \frac{1}{2^2} + \ldots\right) = \varepsilon .$$

Since $x_k \to x_0$, $\exists$ $N \in Z^+$ with $N \ge N_1$ such that $k \ge N \Rightarrow d(x_0, x_k) < \varepsilon$.

Suppose $k \ge N$. Then $d(x, x_0) \le d(x, x_j) + d(x_j, x_0) < \varepsilon + \varepsilon = 2\varepsilon$.

Hence $d(x, F) = \inf_{y\in F} d(x, y) \le d(x, x_0) < 2\varepsilon$ and so $x \in N_{2\varepsilon}(F)$.

Hence $D_k \subset N_{2\varepsilon}(F)$.

Now we will show that $F \subset N_{2\varepsilon}(D_N)$.

Take any $y \in F$. Then $\exists$ $y_n \in D_n$ such that $y_n \to y$.

Thus for sufficiently large n, $d(y_n, y) < \varepsilon/2$.

We also have $n, m \ge N_1 \Rightarrow d_H(D_n, D_m) < \varepsilon/2$.

Hence $n \ge N \Rightarrow D_n \subset N_\varepsilon(D_N)$ and $D_N \subset N_\varepsilon(D_n)$ since $N \ge N_1$.

Then $d(y, X_N) \le d(y, y_n) + d(y_n, X_N) < \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow y \in N_\varepsilon(D_N)$.

Hence $F \subset N_\varepsilon(D_N)$.

But $n \ge N \Rightarrow D_N \subset N_\varepsilon(D_n)$. Thus $F \subset N_{2\varepsilon}(D_n)$.

Hence $n \geq N \Rightarrow D_n \subset N_{2\varepsilon}(F)$ and $F \subset N_{2\varepsilon}(D_n) \Rightarrow d_H(D_n, F) < 2\varepsilon$.

Hence $D_n \to F$ and so $(\mathcal{CB}(X), d_H)$ is complete. □

Conclusion

The Hausdorff distance is a measure that assigns a nonnegative real number as the distance between sets. Given a metric (X, d), we found the Hausdorff distance and defined the Hausdorff metric $(CB(X), d_H)$ on the space of nonempty subsets of X. Finally, we proved that if (X, d) is complete, then $(CB(X), d_H)$ is complete.

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