Optimal Solutions of Balanced Assignment Problems by Hungarian Method

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Abstract

Firstly, this paper introduces the mathematical formulation concerning the assignment problem. Then, this research paper also expresses the Hungarian method and explained how to solve the minimum and maximum balanced assignment problems with that method.

Keywords: Assignment problem, Hungarian method, resources, jobs, minimization and maximization optimality.

Introduction

It is assumed that assignment is a vital part when relegating employments to specialists. One main characteristic of assignment problems is that one career or job is assigned towards one machine or project.

Mathematical Formulation

Let n jobs and n persons with different skills be available. If c_{ij} is the cost of doing j^{th} work by i^{th} person, then the cost matrix is given below as in Table 1.

Table 1

	Jobs				
Persons	1	2	3	… j	… n
1	c_{11}	c_{12}	c_{13}	… c_{1j}	… c_{1n}
2	c_{21}	c_{22}	c_{23}	… c_{2j}	… c_{2n}
.	.	.	.	. . .	. . .
.	.	.	.	. . .	. . .
.	.	.	.	. . .	. . .
i	c_{i1}	c_{i2}	c_{i3}	. . c_{ij}	. . c_{in}
.	.	.	.	. . .	. . .
.	.	.	.	. . .	. . .
.	.	.	.	. . .	. . .
n	c_{n1}	c_{n2}	c_{n3}	. . c_{nj}	. . c_{nn}

To minimize z (Cost) = $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$,

where i = 1, 2, …, n, j = 1, 2, …, n,

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$$x_{ij} = \begin{cases} 1; & \text{if } i^{th} \text{ person is assigned } j^{th} \text{ work,} \\ 0; & \text{if } i^{th} \text{ person is not assigned the } j^{th} \text{ work,} \end{cases}$$

with the restrictions,

(i) $\sum_{i=1}^{n} x_{ij} = 1$; $j = 1, 2, ..., n$, i.e., i^{th} person will do only one work,

(ii) $\sum_{j=1}^{n} x_{ij} = 1$; $j = 1, 2, ..., n$, i.e., j^{th} person will be done only one person.

Theorem

Every element of any row (or column) of the cost matrix $[c_{ij}]$ in an assignment problem adds a constant (or subtract) then an assignment that also minimizes the total cost for the new matrix will minimize the total cost matrix.

Proof: See (Sharma, 2005) ■

Theorem

If all are $c_{ij} \geq 0$ and there exists a solution $x_{ij} = X_{ij}$ such that $\sum_{i} c_{ij} x_{ij} = 0$, then this solution is an optimal solution, i.e., minimizes z.

Proof: See (Sharma, 2005). ■

The Hungarian Method

The steps involved in the Hungarian method are expressed below.

Step I (a) Subtract the minimum entry of each row from all the entries of the respective row in the cost matrix.

(b) Then, subtract the minimum entry of each column from all the entries of the relevant column.

Step II (a) After that examine the rows one by one until a row containing exactly one zero is found. Then an experimental assignment indicated by ‘ ’ is marked to that zero and crossed all the zeros in the column in which the assignment is made.

(b) Now an identical procedure is applied successively to columns by starting with column 1 and we examine all columns until we can find a column containing exactly one zero. Later make an experimental assignment in that situation and cross other zeros in the row in which the assignment was completed.

Continue these succeeding operations on rows and columns till all zeros have either been assigned or crossed-out.

Now, there are two possibilities:

(i) Either all the zeros are assigned or crossed out, i.e., we get the required assignment. This completes the second step.

(ii) Afterward this step, we can get two situations.

(a) Total assigned zero's = n. (n = number of rows and columns). The assignment is optimal.

(b) Total assigned zero's < n. Use step III and onwards.

Step III Draw of minimum lines to cover zeros. So as to cover all the zeros at least once, we may take on the following procedure.

(a) Marks (√) to all rows in which the assignment has not been done.

(b) See the position of zero in marked (√) row and then mark (√) to the corresponding column.

(c) See the marked (√) column and find the position of assigned zero's and then mark (√) to the corresponding rows which are not marked till now.

(d) Repeat the procedure (ii) and (iii) till the completion of marking.

(e) Draw the lines through unmarked rows and marked columns.

Step IV At that moment, select the smallest element from the uncovered elements.

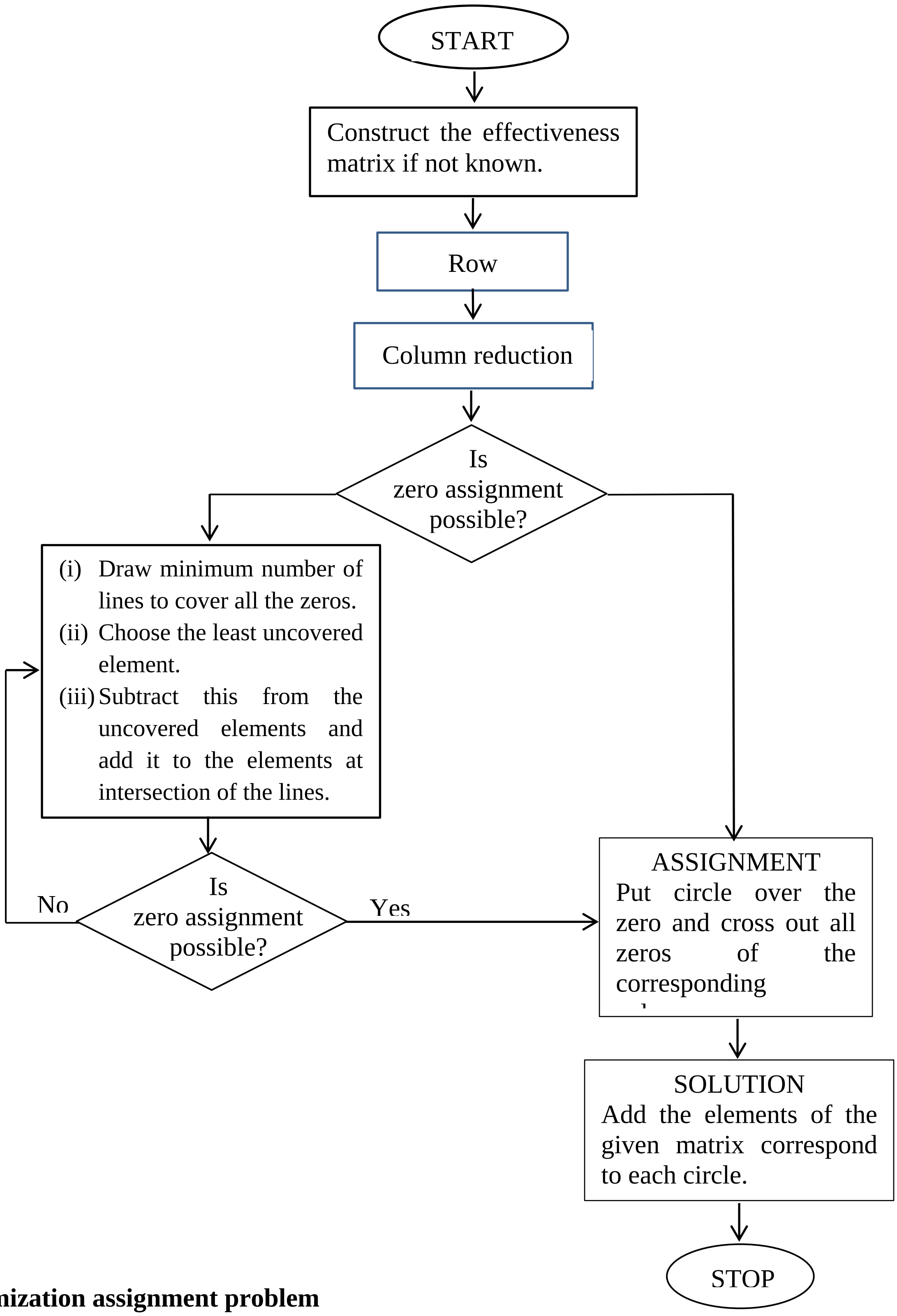
(i) Subtract this smallest element from all those elements which are not covered.

(ii) Add this smallest element to all those elements which are at the intersection of two lines.

Step V Thus we have increased the number of zeros. Finally, modify the matrix with the aid of step II and find the necessary assignment.

This procedure will be more clear by the following examples.

Flow chart for assignment problem



Minimization assignment problem

In this problem, a taxi hire company has one taxi at each of five depots a, b, c, d and e. A customer wants a taxi in every township, namely A, B, C, D and E. Distances (in kms) between depots and towns are given in the following distance matrix is as shown in Table 2.

We examine how taxis should be assigned to customers so as to minimize the distance travelled.

Table 2

Towns	Depots a	b	c	d	e
A	140	110	155	170	180
B	115	100	110	140	155
C	120	90	135	150	165
D	30	30	60	60	90
E	35	15	50	60	85

Table 3

Towns	Depots a	b	c	d	e
A	30	0	45	60	70
B	15	0	10	40	55
C	30	0	45	60	75
D	0	0	30	30	60
E	20	0	35	45	70

Afterward, we follow steps 1 and 2 of the assignment algorithm by subtracting the smallest number from every number in that row to give Table 3. Similarly the smallest number in each column is subtracted from every number in that column as shown in Table 4.

Table 4

Towns	Depots a	b	c	d	e	
A	30	0	35	30	15	√
B	~~15~~	~~0~~	~~0~~	~~10~~	~~0~~	
C	30	0	35	30	20	√
D	~~0~~	~~0~~	~~20~~	~~0~~	~~5~~	
E	20	0	25	15	15	√
		√				

Table 5

Towns	Depots a	b	c	d	e
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	5
D	0	15	20	0	5
E	5	0	10	0	0

From Table 4, we see that all the zeros are either assigned or crossed out, but the total number of assignment, is 4 < 5, i.e., (number of towns to be assigned to machine). Therefore, we have to follow step III as in Table 4.

Then, from Table 5 we have got five assignments as required by the problem. The assignment is as follow: A→e, B→c, C→b, D→a, E→d.

As a result, the minimum distance = 180 + 110 + 90 + 30 + 60 = 470 km.

Maximization assignment problem

Next problem is the British Navy needs to assign four ships to patrol four sectors of the North Sea. Illegal fishing boats are to be on the outlook in some areas and enemy submarines are watched in other sectors, so the commander rates each ship of its profitable efficiency in respectively sector. These relative efficiencies are illustrated in Table 6. On the basis of the ratings as shown in Table 6, the commander wishes to decide the patrol assignments producing the maximum total efficiencies.

Table 6 Efficiencies of British Ships in Patrol sectors

Ships	Sectors A	B	C	D
1	20	60	50	55
2	60	30	80	75
3	80	100	90	80
4	65	80	75	70

Table 7 Opportunity costs of British Ships

Ships	Sectors A	B	C	D
1	80	40	50	45
2	40	70	20	25
3	20	0	10	20
4	35	20	25	30

By means of exchanging we can see the maximizing efficiency Table 6 into a minimization opportunity cost Table 7. And then, this is done by subtracting each rating from 100 that the largest rating in the entire table. The resulting opportunity costs are given in Table 7.

Subsequently, we follow steps 1 and 2 of the assignment procedure. The smallest number is subtracted from every number in that row to give Table 8; and then the smallest number in each column is subtracted from every number in that column as shown in Table 9.

Table 8 Row opportunity costs for the British Navy problem

Ships	Sectors A	B	C	D
1	40	0	10	5
2	20	50	0	5
3	20	0	10	20
4	15	0	5	10

Table 9 Total opportunity costs for the British Navy problem

Ships	Sectors A	B	C	D
1	25	0	10	0
2	5	50	0	0
3	5	0	10	15
4	0	0	5	5

In Table 9 all the zeros are either crossed out or assigned. Besides total assigned zeros = 4 (i.e., the number of rows or columns).

Hence an optimal assignment can be made. The optimal assignment is ship 1 towards sector D, ship 2 towards sector C, ship 3 towards sector B, and ship 4 towards sector A.

The overall efficiency, computed from the original efficiency data Table 10, can now be shown: Table 10.

Assignments	Efficiency
Ship 1 to Sector D	55
Ship 2 to Sector C	80
Ship 3 to Sector B	100
Ship 4 to Sector A	65
Total Efficiency	300

Conclusion

At the end of this research paper we will be able to formulae special linear programming problems using the assignment model and know Hungarian method to find the proper assignment. Moreover, we can employ Hungarian method to assign workers and machines so as to maximum profit on minimum investment in real life situations.

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