A Qualitative Analysis of Some Magic Graphs

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Abstract

In this paper, edge-magic labeling of graph is presented. Some examples of magic graph and antimagic graph are described theorems and corollaries which are concerning constant c(f) of magic labeling of odd cycles are stated. By using these theorems and corollaries, it is summarized that every odd cycle has four values of c(f). Finally, it is shown that deleting of an edge with label 1 of a magic graph is also a magic graph.
Keywords: Magic Graph, Endpoints, Edge-magic labeling.

Introduction

A **graph** is a finite set of vertices and edges where every edge connects two vertices. If G= G(V,E) is a graph, then V(G) is a finite non-empty set of elements called **vertices** and E(G) is a set (possibly empty) of unordered pairs of vertices called **edges**. (Marr and Waills, 2013)

The cardinality of the vertex set V(G) is called the **order** of G, commonly denoted by |V(G)| = p. The cardinality of the edge set E(G) is the **size** of G denoted by |E(G)| = q. (Wallis, 2001)

Example

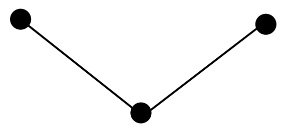

Figure 1 Graph G

In the above graph, |V(G)| = p = 3 and |E(G)| = q = 2.

A **labeling** for a graph is a map that takes graph elements such as vertices and edges to alphabets or numbers (usually positive or non-negative integers). [Wallis, 2001]

Figure 2 Graph elements labeled (a) by numbers (b) by alphabets

Preliminaries

A (p,q) graph G = (V, E) is said to be magic if there exists a bijection $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ such that for all edges uv of G, $f(u)+f(v)+f(uv)$ is constant, c(f). Such a bijection is called a **magic labeling** of G. This is also called by **edge magic labeling** of G. If vertex magics, same property holds for vertices. A graph G with magic labeling is called a **magic graph**. [Marr and Waills, 2013, Wallis, 2001]

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Example

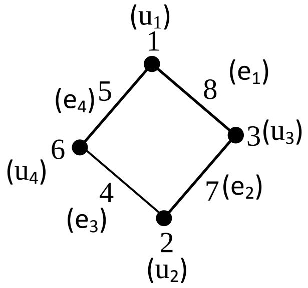

Figure 3 A graph G

$f:\{u_1,u_2,u_3,u_4,e_1,e_2,e_3,e_4\}\to\{1,2,3,4,5,6,7,8\}.$

$f(u_1)=1,\qquad f(e_1)=8.$

$f(u_2)=2,\qquad f(e_2)=7.$

$f(u_3)=3,\qquad f(e_3)=4.$

$f(u_4)=6,\qquad f(e_4)=5.$

For the edge e_1, $f(u_1)+f(u_3)+f(u_1u_3)=1+3+8=12.$
For the edge e_2, $f(u_2)+f(u_3)+f(u_2u_3)=2+3+7=12.$
For the edge e_3, $f(u_2)+f(u_4)+f(u_2u_4)=2+6+4=12.$
For the edge e_4, $f(u_1)+f(u_4)+f(u_1u_4)=1+6+5=12.$
For any magic labeling of G, there is a constant c(f) such that all the edges uv of G, $f(u)+f(v)+f(uv)=c(f).$ In Figure 3, c(f) = 12.

Example [Gallian, 2019]

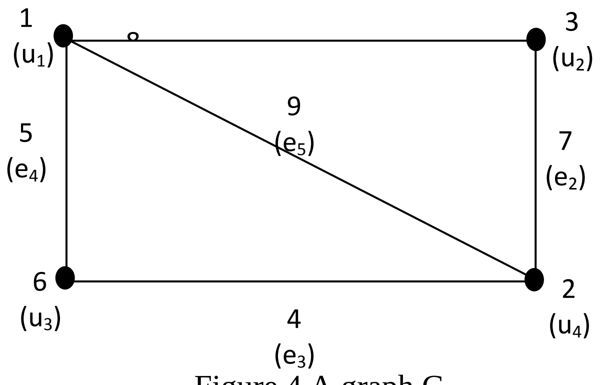

Figure 4 A graph G

$f:\{u_1,u_2,u_3,u_4,e_1,e_2,e_3,e_4,e_5\}\to\{1,2,3,4,\ldots,9\}.$

$f(u_1)=1,\qquad f(e_1)=8.$

$f(u_2)=3,\qquad f(e_2)=7.$

$f(u_3)=6,\qquad f(e_3)=4.$

$f(u_4)=2,\qquad f(e_4)=5.$

$f(e_5)=9.$

In the above figure, $c(f)=12.$

If there is a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, p+q\}$ such that for all edges $xy, f(x)+f(y)+f(xy)$ are all distinct, then G is called **antimagic**.

Example

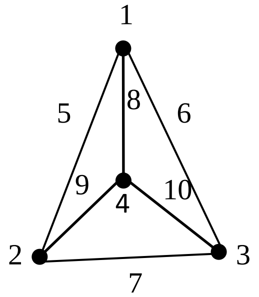

Figure 5 Anantimagic graph

A **walk** of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_1.The walk joins v_0 and v_n, and it is called a $v_0 - v_n$ **walk.** (Marr and Waills, 2013, Wallis, 2001)

A $v_0 - v_n$ walk is called **closed** if $v_0 = v_n$. A closed walk $v_0, v_1, v_2, \ldots, v_n, v_0$ in which $n > 3$ and $v_0, v_1, v_2, \ldots, v_{n-1}$ are distinct is called a **cycle** of length n. A cycle on n vertices is denoted by C_n.. [Marr and Waills, 2013, Wallis, 2001]

Example

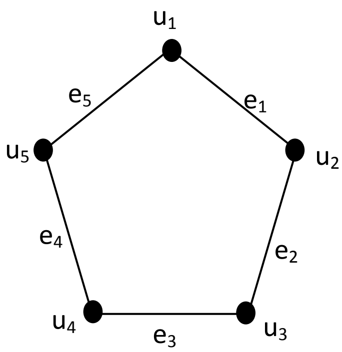

Figure 6 A cycle C_5.

Discussion on Some Magic Graph

Theorem 1

Every odd cycle has a magic labeling with $c(f)=\frac{1}{2}(5p + 3)$.

Proof: See [Wallis, 2000]. ■

Corollary 1

Every odd cycle has a magic labeling with $c(f)=\frac{1}{2}(7p + 3)$.

Proof: See [Wallis, 2000]. ■

Theorem 2

Every odd cycle has a magic labeling with c(f)=3p + 1 .

Proof: See [Wallis, 2000]. ■

Corollary 2

Every odd cycle has a magic labeling with c(f) = 3p + 2 .

Proof: See [Wallis, 2000]. ■

According to the above theorems and corollaries, it is summarized that every odd cycle has four values of c(f). Four values of c(f) of cycles C_3, C_5,C_7 are shown in Table 1.

Table 1. Four values of c(f) of some cycles.

	C(f) of C_3	C(f) of C_5	C(f) of C_7
By Theorem 1	9	14	19
By Corollary 1	12	19	26
By Theorem 2	10	16	22
By Corollary 2	11	17	23

Example

According to Table 1, cycle C_3 has four values of c(f). Four magic labelings of C_3 are illustrated in Figure 7.

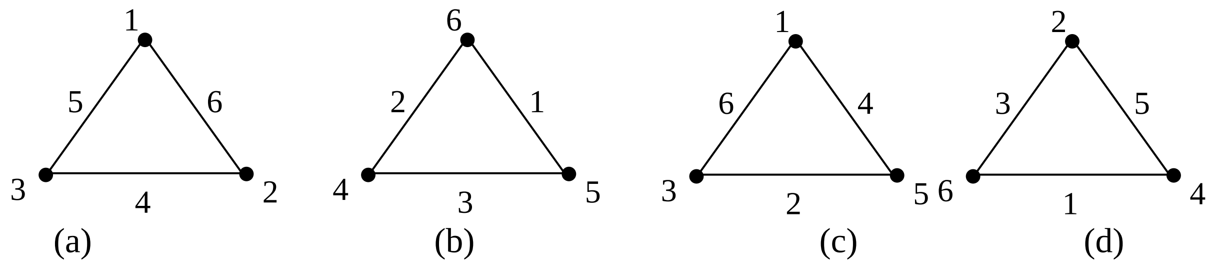

Figure 7 (a) Cycle C_3 with c(f) = 9 (b) Cycle C_3 with c(f) = 12 (c) Cycle C_3 with c(f) = 10 .

(d) Cycle C_3 with c(f) = 11 .

Example

According to Table 1, cycle C_5 has four values of c(f). Four magic labelings of C_5 are illustrated in Figure 8.

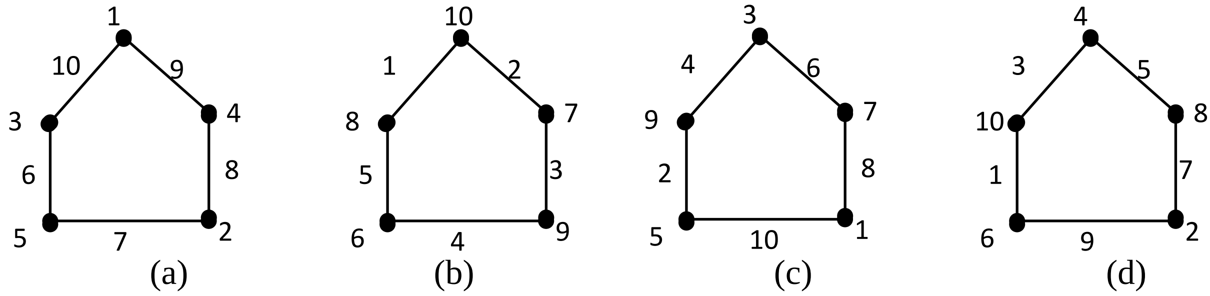

Figure 8 (a) Cycle C_5 with c(f) = 14 (b) Cycle C_5 with c(f) = 19 (c) Cycle C_5 with c(f) = 16 (d) Cycle C_5 with c(f) = 17 .

Example

According to Table 1, cycle C_7has four values of c(f). Four magic labelings of C_7 are illustrated in Figure 9.

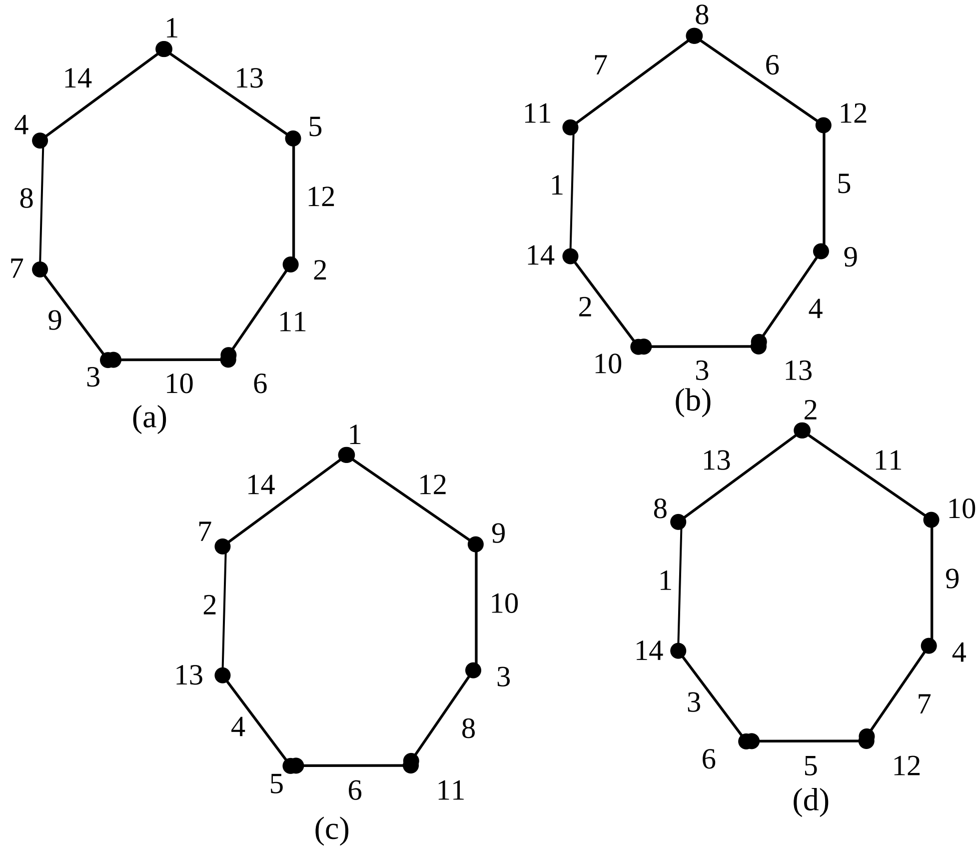

Figure 9 (a) Cycle C_7 with c(f) = 19 (b) Cycle C_7 with c(f) = 26 (c) Cycle C_7 with c(f) = 22 .

(d) Cycle C_7 with c(f) = 23

Theorem 3 [Gallian, 2019]

If G is a magic graph and f is a magic labeling of G for which there exists $e \in E(G)$ such that $f(e) = 1$. Then G – e is a magic.

Proof:

Let G (p, q) be magic graph. And f is a magic labeling.

Let $f : V \bigcup E \to \{1, 2, 3, \ldots, p+q\}$.

Let $g : V(G) \bigcup E(G) - \{e\} \to \{1, 2, 3, \ldots, p+q-1\}$.

$g(x) = f(x) - 1, \forall x \in (V(G) \bigcup E(G) - \{e\})$.

G is also magic labeling. Therefore, G – e is magic.

The proof is complete. ■

Example

We consider a cycle $C_3$and magic labeling edge is shown in Figure 10. Then C_3 is a magic graph.

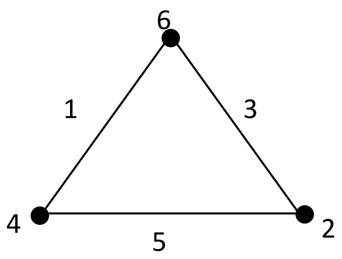

Figure 10 A cycle C_3

Let e be the edge with label 1. Then C_3 – e is shown in Figure 11.C_3 – e has c(f) = 11.

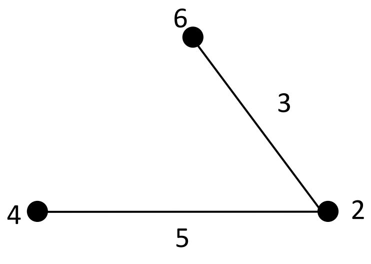

Figure 11 C_3 – e with c(f) = 11

We consider the another magic labeling g such that g(x) = f(x) – 1, $\forall x \in \left[V(C_3) \bigcup E(C_3) - \{e\} \right]$. Then C_3 – e is also magic graph with c(g) = 8.

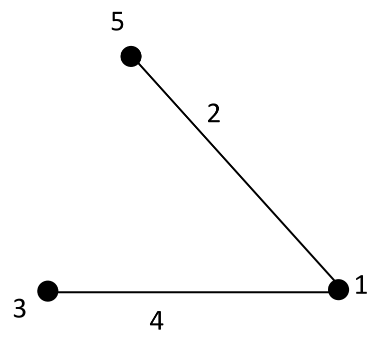

Figure 12 C_3 – e with c(g) = 8.

Conclusion

In this research work, some magic graphs are analyzed by using the survey of graph labeling and edge-magic labeling of graphs that lead to successful approaches to other label problems.

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