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Conformal Transformation and it's Applications

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Abstract

This paper investigates flow past a flat plate and two-dimensional irrotational motion of fluid due to singularities between two fixed boundaries. The flow past a flat plate is obtained by using Joukowski transformation. It is also shown by examples that the conformal transformation can make a problem of irrotational flow treatable by converting an awkwardly shaped boundary into one of the simple forms.

Keywords: flat plate, Joukowski transformation, conformal transformation, singularities.

1. Conformal Transformation

Suppose that z and ζ are two complex variables defined by $z = x + iy$ and $\zeta = \xi + i\eta$ where x, y, ξ, η are real variables. Suppose that z describes a certain curve C in the z-plane and ζ is related to z by means of the transformation $\zeta = f(z)$ where f(z) is analytic.

If f(z) is a single-valued function of z, then to each point in the z-plane, we can obtain a corresponding point in the ζ-plane. In this way, the curve C in the z-plane may be mapped into a curve C' in the ζ-plane.

Figure 1

Suppose that the function f (z) is analytic. Let P, Q, R be neighboring points in the z-plane such that OP= z, $OQ = z + \delta z_1$, $OR = z + \delta z_2$.

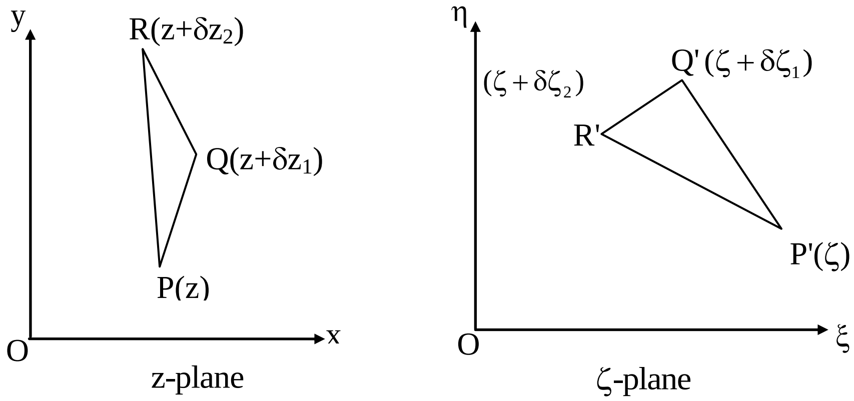

Figure 2

Under the transformation $\zeta = f(z)$, suppose that P,Q, R, map into the points P', Q',R' respectively in the ζ-plane, where OP' = ζ, OQ' = $\zeta + \delta \zeta_1$,OR'=$\zeta + \delta \zeta_2$ .It is assumed that

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$|\delta z_1|$, $|\delta z_2|$, $|\delta \zeta_1|$, $|\delta \zeta_2|$ are small. Since f(z) is analytic, $\frac{d\zeta}{dz}$ is unique at P. Thus, to the first order of smallness, $\frac{\delta\zeta_1}{\delta z_1} = \frac{\delta\zeta_2}{\delta z_2}$ (or) $\frac{\delta\zeta_1}{\delta\zeta_2} = \frac{\delta z_1}{\delta z_2}$. (1)

Therefore, $\left|\frac{\delta\zeta_1}{\delta\zeta_2}\right| = \left|\frac{\delta z_1}{\delta z_2}\right|$, (2) $\arg \delta\zeta_1 - \arg \delta\zeta_2 = \arg \delta z_1 - \arg \delta z_2$, (3)

and $\delta\zeta_1 = \frac{\delta\zeta_1}{\delta z_1}\delta z_1$.

Therefore, $|\delta\zeta_1| = \left|\frac{\delta\zeta_1}{\delta z_1}\right| |\delta z_1|$ and $\arg\delta\zeta_1 = \arg\left(\frac{\delta\zeta_1}{\delta z_1}\right) + \arg\delta z_1$. So in the neighborhood of the point P' distances are multiplied by the value of $\left|\frac{\delta\zeta_1}{\delta z_1}\right|$ at P; this is called the magnification of the transformation.

From (2) and (3), we obtain $\frac{P'Q'}{P'R'} = \frac{PQ}{PR}$ and $\angle R'P'Q' = \angle RPQ$. Thus the triangles $R'P'Q'$ and RPQ are similar. So an infinitesimal triangle in the z-plane maps into a similar infinitesimal triangle in the ζ-plane. Thus the mapping preserves the angles and the similarity of corresponding infinitesimal triangles. Such a transformation which has these properties is said to be conformal.

Example: Transformation of $w = z^2$

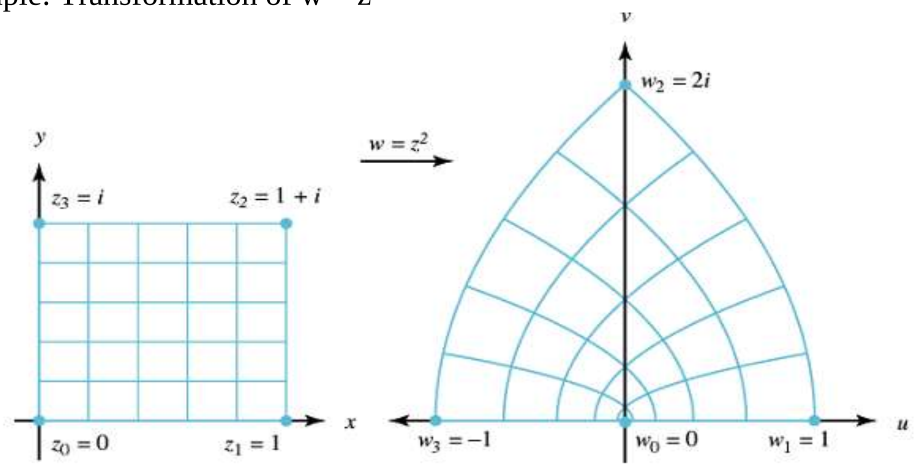

Figure 3

1.1 Applications of Conformal Transformation

Suppose there is a two-dimensional incompressible flow in the z-plane. On applying the conformal transformation $\zeta = g(z)$, the new plane of flow becomes the ζ-plane. Let ρ be the density of the fluid in both cases. Suppose further that C is a rigid boundary in the z-plane which maps into the curve C' in the ζ-plane. Let the complex velocity potential for the z-plane be $w = f(z) = \phi + i\psi$ where the real functions $\phi(x,y)$, $\psi(x,y)$ are the usual velocity potential and stream function respectively. By means of the transformation $\zeta = g(z)$ it can

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express w as a function $\bar{f}(\zeta) = \bar{\phi} + i\bar{\psi}$ where $\bar{\phi} = \bar{\phi}(\xi, \eta)$, $\bar{\psi} = \bar{\psi}(\xi, \eta)$. At the corresponding points t, z, the complex potential w takes the same value so that $\phi = \bar{\phi}$, $\psi = \bar{\psi}$.

Now C is a rigid boundary in the z-plane and so also a streamline for which ψ = constant. Thus along C', $\bar{\psi}$ = constant. Therefore, C' is a streamline and also a rigid boundary. Therefore, under the conformal transformation, points on the streamline through a given point in the z-plane will transform into points on the stream line through the corresponding point in the ζ-plane. In particular, the boundaries of the fluid in z-plane will transform the boundaries in ζ-plane.

1.2 Transformation of source and sink

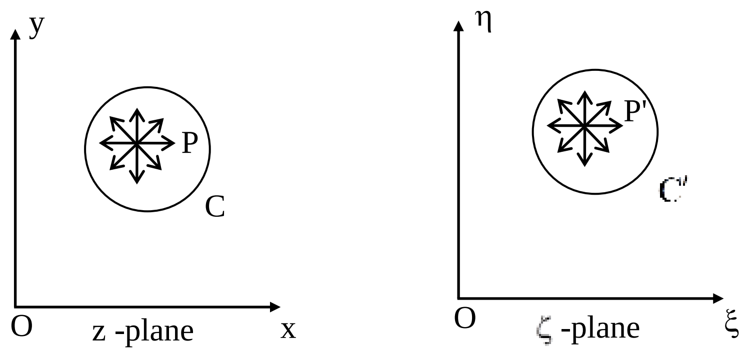

Figure 4

Suppose there is a source of strength m at P in the z-plane surrounded by a small closed curve C. By the definition of source, the flow across C is 2πmρ. Under the conformal transformation the point P transforms into the point P' in the ζ-plane and the small closed curve C surrounding P in the z-plane transforms into a small closed curve C' surrounding P' in the ζ-plane. The flow across C is given in terms of the stream function by $-\rho\int_C d\psi$. Since each point on C' corresponds to one and only one point on C, this is equal to $-\rho\int_{C'} d\psi$ taken in the same sense. So, the flow across C' is 2πmρ and this will be the same for any small closed curve surrounding P'. Therefore a source transforms into an equal source at the corresponding point. Similarly if there is a sink of strength (– m) at P in the z-plane, then it transforms into an equal sink at the corresponding point in the ζ-plane.

In particular, the boundaries of the fluid in z-plane will transform the boundaries in ζ-plane. And a source, sink or vortex at a particular point in the z- plane will transform an equal source, sink or vortex at the corresponding point in the ζ-plane. The kinetic energy of both corresponding regions is equal.

2. Joukowski Transformation

It is the most common conformal transformation which is given by

$$\zeta = f(z) = z + \frac{a^2}{z}, \tag{4}$$

where a is constant. The transformation changes z-plane to ζ-plane where z = x + iy. Then,

$$\zeta = \xi + i\eta = z + \frac{a^2}{z},$$

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$$= x + iy + \frac{a^2(x - iy)}{(x + iy)(x - iy)}.$$

Therefore, $\xi = x\left(1 + \frac{a^2}{x^2 + y^2}\right), \qquad \eta = y\left(1 - \frac{a^2}{x^2 + y^2}\right).$ (5)

If $x^2 + y^2 = r^2$, the circle of radius r is in the z-plane, then

$$\frac{\xi^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{\eta^2}{\left(r - \frac{a^2}{r}\right)^2} = 1, \text{ in the } \zeta\text{-plane.} \qquad (6)$$

Therefore by using Joukowski transformation, a circle on the z-plane of radius r transforms into an ellipse with major axis $A = r + \frac{a^2}{r}$ and minor axis $B = r - \frac{a^2}{r}$ on the ζ-plane.

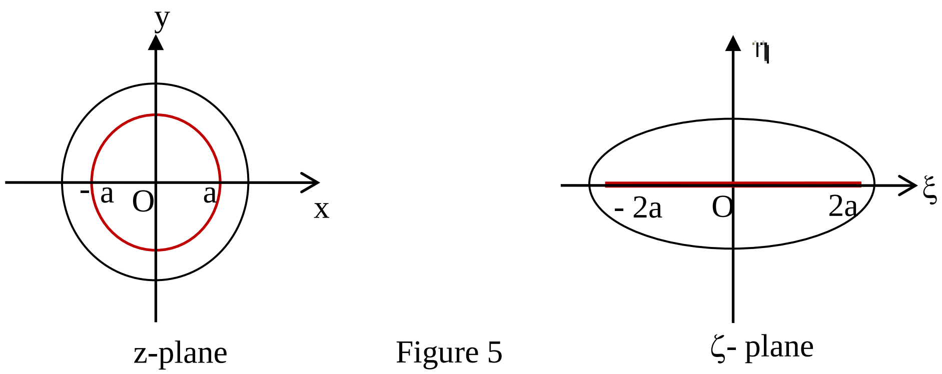

Figure 5

In the special case, when r = a, the ellipse becomes an infinitely thin plate of length 4a in the ζ-plane, since A = 2a and B = 0. So, the Joukowski transformation changes the circle into a flat plate. And then the circle of radius a in the z-plane is called the Joukowski transformation circle.

2.1 Flow Past a Flat Plate with Circulation

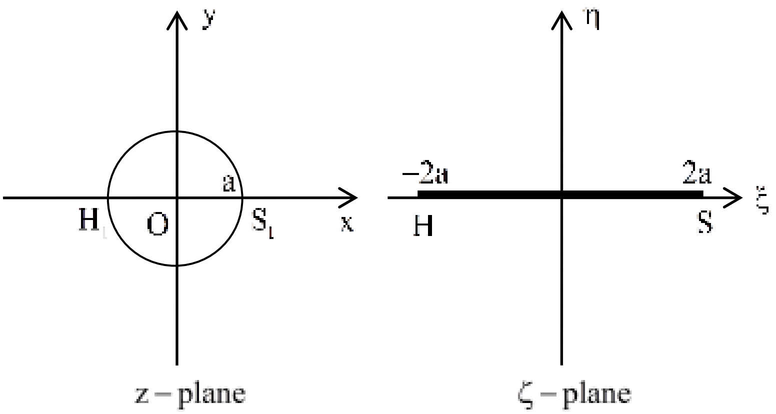

Figure 6

The complex potential for a fixed circular cylinder radius a in a stream whose undisturbed spped U makes an angle α with the X-axis and about which there is a circulation κ is

$$W = U\left(ze^{i\alpha} + \frac{a^2 e^{-i\alpha}}{z}\right) + \frac{i\kappa}{2\pi}\log z. \qquad (7)$$

If the transformation $\zeta = \frac{a^2}{z} + z$ is applied to the whole area outside the circle in the z-plane, it transforms into the whole of the ζ-plane with a rigid barrier between the points $(\pm 2a, O)$. The problem then becomes that a flat plate of width 4a, about which there is circulation, in a stream U inclined at α to the plate. Solving for z in terms of ζ,

$z = \frac{1}{2}\left\{\zeta + \sqrt{\zeta^2 - 4a^2}\right\}$ and $\frac{a^2}{z} = \frac{1}{2}\left\{\zeta - \sqrt{\zeta^2 - 4a^2}\right\}$.

Hence the complex potential (7) becomes

$$W = \frac{1}{2}U\left[\left\{\zeta + \sqrt{\zeta^2 - 4a^2}\right\}e^{i\alpha} + \left\{\zeta - \sqrt{\zeta^2 - 4a^2}\right\}e^{-i\alpha}\right]$$
$$+ \frac{i\kappa}{2\pi}\log\frac{1}{2}\left\{\zeta + \sqrt{\zeta^2 - 4a^2}\right\}.$$
$$= U\left[\zeta\cos\alpha + i\left\{\sqrt{\zeta^2 - 4a^2}\right\}\sin\alpha\right]$$
$$+ \frac{i\kappa}{2\pi}\log\frac{1}{2}\left\{\zeta + \sqrt{\zeta^2 - 4a^2}\right\},$$

neglecting a constant.

The circulation about the plate is given by the decrease in the velocity potential ϕ on describing a circuit round it, this is the same as the decrease in ϕ on describing the corresponding circuit about the cylinder, i.e., there is a circulation κ about the plate.

The velocity at any point can be written

$$-U + iV = \frac{dW}{d\zeta} = \frac{\frac{dW}{dz}}{\left(1 - \frac{a^2}{z^2}\right)}.$$

The denominator vanishes when $z = \pm a$, i.e., $\zeta = \pm 2a$, therefore the velocity is infinite at both edges unless $\frac{dW}{dz}$ has a factor $(z + a)$ or $(z - a)$, when it will be finite at the corresponding edge.

$$\frac{dW}{dz} = U\left(e^{i\alpha} - \frac{a^2}{z^2}e^{-i\alpha}\right) + \frac{i\kappa}{2\pi z},$$

if this is zero when $z = \pm a$, then $\kappa = \pm 4\pi Ua\sin\alpha$. Hence the velocity at the edge $\zeta = 2a$ will be finite.

Example

A flat plate of infinite length and width L is placed in a current of incompressible fluid with its plane at an angle α to the undisturbed stream lines.

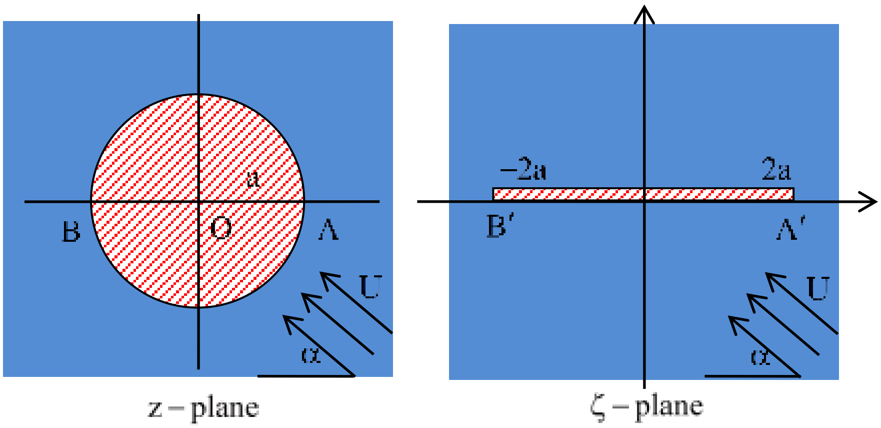

Figure 7

The circle of radius a on BOA as diameter transforms into the flat plate $B'A'$ of length 4a, by means of transformation $\zeta = z + \frac{a^2}{z}$. Taking the centre of the circle as origin, the complex potential in z-plane is given by $W = Uze^{i\alpha} + \frac{Ua^2}{z}e^{-i\alpha} + \frac{i\kappa}{2\pi}\log z$.

$$\frac{dW}{dz} = Ue^{i\alpha} - \frac{Ua^2}{z^2}e^{-i\alpha} + \frac{i\kappa}{2\pi z}.$$

Stagnation points corresponding to $z = \pm a$ are given by $\frac{dW}{dz} = 0$. Therefore $\kappa = 4\pi a U \sin\alpha$.

By using the Blasiu's theorem,

$$X - iY = \frac{1}{2} i\rho \int \left(\frac{dW}{d\zeta}\right)^2 d\zeta.$$

$$= \frac{1}{2} i\rho \int \left\{ \left(1 + \frac{a^2}{z^2} + \ldots\right)\left(Ue^{i\alpha} - \frac{Ua^2e^{-i\alpha}}{z^2} + \frac{i\kappa}{2\pi z}\right)^2 \right\} dz.$$

By using Residue Theorem,

$$X - iY = \frac{1}{2} i\rho \frac{2Ui\kappa e^{i\alpha}}{2\pi} 2\pi i = -i\rho\kappa U e^{i\alpha}$$

$$X = \rho\kappa U \sin\alpha = 4\pi\rho a U^2 \sin^2\alpha$$

$$Y = \rho\kappa U \cos\alpha = 4\pi\rho a U^2 \sin\alpha\cos\alpha.$$

The resultant force R is $4\pi\rho a U^2 \sin\alpha$ and acting at angle, $\tan\theta = \frac{Y}{X}$, $\theta = \frac{\pi}{2} - \alpha$.

$$N = \text{real part of } -\frac{1}{2}\rho \int \left(\frac{dW}{d\zeta}\right)^2 \zeta d\zeta$$

$$= -\frac{1}{2}\rho \int \frac{z + \frac{a^2}{z}}{1 - \frac{a^2}{z^2}} \left(\frac{dW}{dz}\right)^2 dz$$

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$$= \text{real part of } \left[-\frac{1}{2}\rho\left(2U^2a^2e^{2i\alpha} - \frac{\kappa^2}{4\pi^2} - 2U^2a^2 \right)2\pi i \right]$$

$$= 2\pi\rho U^2a^2 \sin 2\alpha = \frac{\pi}{8}\rho L^2U^2 \sin 2\alpha .$$

3. Flow Due to a Source between Two Fixed Boundaries

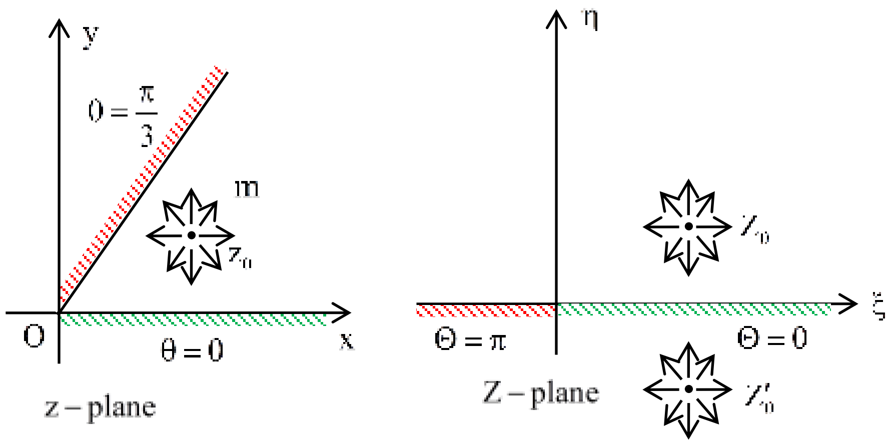

Figure 8

Consider a source m at the point z_0 in the fluid bounded by the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$. The conformal transformation $Z = z^3$ where $z = re^{i\theta}$ from z-plane transform to Z-plane. The boundaries $\theta = 0$ and $\theta = \frac{\pi}{3}$ in z-plane transform to $\Theta = 0$ and $\Theta = \pi$ (real axis) in Z-plane. The point z_0 in z-plane transforms to point Z_0 in Z-plane such that $Z_0 = z_0^3$ and the source m at z_0 transforms to a source m at Z_0. Hence the image system with respect to real axis in Z-plane consists of a source m at Z_0 and a source m at Z_0'. Therefore the complex potential for this motion is

$$W = -m\log(Z - Z_0) - m\log(Z - Z_0')$$
$$= -m\log(z^3 - z_0^3) - m\log(z^3 - z_0'^3)$$
$$\phi + i\psi = -m\log\left\{(z^3 - z_0^3)(z^3 - z_0'^3)\right\}.$$

Example

Suppose that between two fixed boundaries $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, there is two-dimensional liquid motion due to a source of strength m at the point $(a,0)$ and an equal sink at the point $(b,0)$.

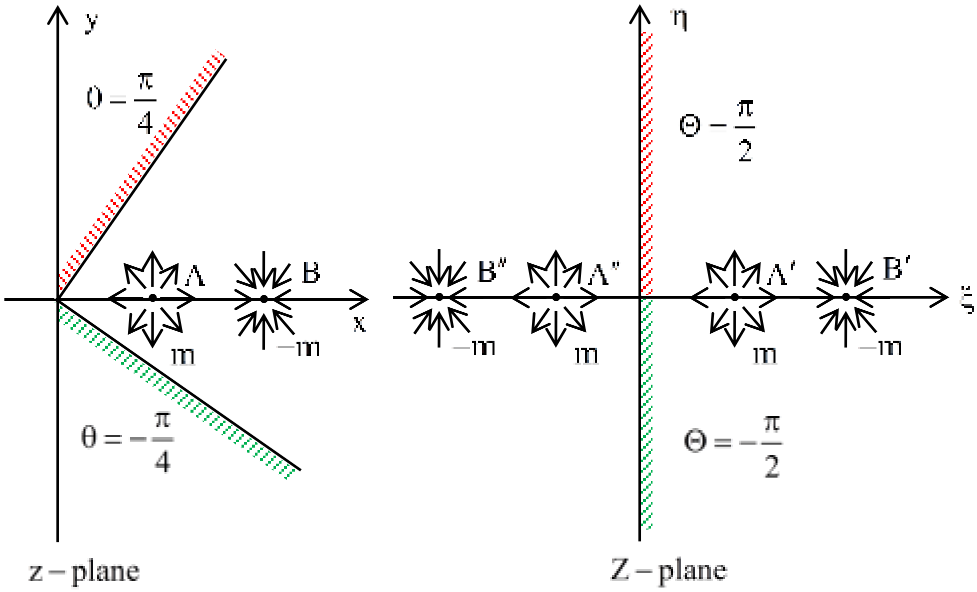

Figure 9

Consider the transformation $Z=z^2$, from xy-plane to $\xi\eta$-plane, where $z=re^{i\theta}$ and $Z=Re^{i\Theta}$.

Hence the boundaries $\theta=\pm\frac{\pi}{4}$ in the z-plane transform to $\Theta=\pm\frac{\pi}{2}$, the imaginary axis of Z-plane. The points $A(a,0)$ and $B(b,0)$ transform to $A'(a^2,0)$ and $B'(b^2,0)$ respectively. Since the source transforms to an equal source at A' and the sink transforms to an equal sink at B', the image system with respective to imaginary axis in Z-plane consists of a source of strength m at $A''(-a^2,0)$ and a sink of strength $-m$ at $B''(-b^2,0)$.

Therefore, the complex potential for this motion is

$$W=-m\log(Z-a^2)+m\log(Z-b^2)-m\log(Z+a^2)+m\log(Z+b^2)$$
$$=-m\log(Z^2-a^4)+m\log(Z^2-b^4).$$

By using the transformation,

$$W=-m\log(z^4-a^4)+m\log(z^4-b^4)$$
$$=-m\log(r^4\cos4\theta-a^4+ir^4\sin4\theta)$$
$$+m\log(r^4\cos4\theta-b^4+ir^4\sin4\theta)$$
$$\psi=-m\left[\tan^{-1}\frac{r^4\sin4\theta}{r^4\cos4\theta-a^4}-\tan^{-1}\frac{r^4\sin4\theta}{r^4\cos4\theta-b^4}\right].$$

Thus the stream function of two-dimensional motion due to a source of strength m at $(a,0)$ and an equal magnitude of sink at $(b,0)$ is

$$-m\tan^{-1}\frac{r^4(a^4-b^4)\sin4\theta}{r^8-r^4(a^4+b^4)\cos4\theta+a^4b^4}.$$

$$\frac{dW}{dz}=-m\frac{4z^3}{z^4-a^4}+m\frac{4z^3}{z^4-b^4}$$

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$$= \frac{-4mr^3\left(\cos 3\theta + i\sin 3\theta\right)\left(a^4 - b^4\right)}{\left(r^4\cos 4\theta - a^4 + ir^4\sin 4\theta\right)\left(r^4\cos 4\theta - b^4 + ir^4\sin 4\theta\right)}.$$

The velocity at the point (r,θ) in two-dimensional liquid motion due to a source and sink is

$$q = \left|\frac{dW}{dz}\right| = \frac{4mr^3\left(a^4 - b^4\right)}{\left(r^8 - 2a^4r^4\cos 4\theta + a^8\right)^{\frac{1}{2}}\left(r^8 - 2b^4r^4\cos 4\theta + b^8\right)^{\frac{1}{2}}}.$$

4. Flow due to a Source and Sink at the Corners of Infinite Rectangle

Consider the infinite rectangle in the z-plane for which $0 \le y \le \pi,\ x \ge 0$. Use the transformation $t = \cosh z$, where $t = \xi + i\eta$ and $z = x + iy$. Therefore, $\xi = \cosh x \cos y$ and $\eta = \sinh x \sin y$, where $0 \le y \le \pi,\ x \ge 0$. If $y = 0$ and $0 < x < \infty$, then $1 < \xi < \infty$. If $x = 0$ and $0 \le y \le \pi$, then $-1 \le \xi \le 1$. If $y = \pi$ and $0 < x < \infty$, then $-\infty < \xi < -1$. Thus, the infinite rectangle in the z-plane for which $0 \le y \le \pi,\ x \ge 0$ into the a half of the t-plane for which η is positive.

Consider the two-dimensional irrotational motion of a liquid due to within the above infinite rectangle with a source and sink are placed at the corners $(0,0)$ and $(0,\pi)$.

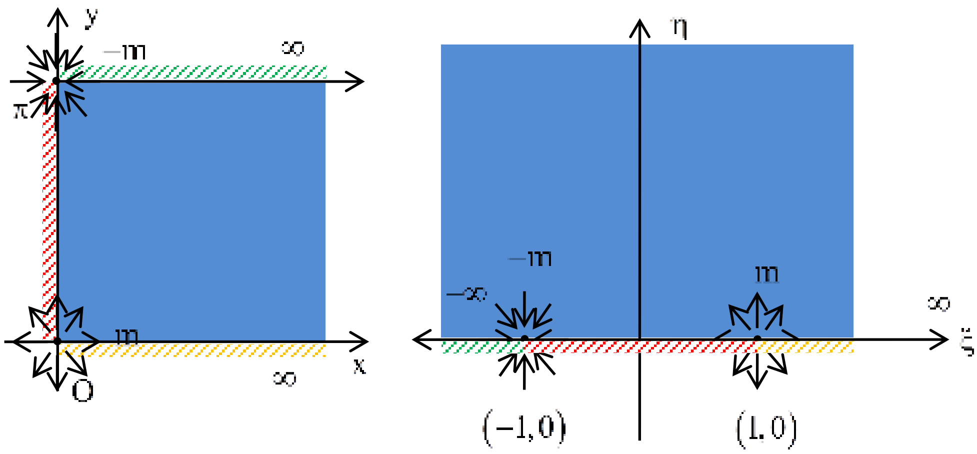

Figure 10

The source transforms into an equal source at $(1,0)$ and the sink transforms into an equal sink at $(-1,0)$. The complex potential for this motion is

$$W = -m\log(t-1) + m\log(t+1)$$

$$= -m\log\frac{\cosh z - 1}{\cosh z + 1} = -2m\log\tanh\frac{z}{2}$$

$$= -\lambda\log\tanh\frac{z}{2},\ \text{where } -2m = -\lambda.$$

$$\frac{dW}{dz} = -\frac{\lambda}{2}\frac{\operatorname{sech}^2\frac{z}{2}}{\tanh\frac{z}{2}} = -\frac{\lambda}{\sinh z}.$$

For the curve of equal pressure in the liquid, the velocity must be constant. Therefore,

$q^2 = \left[-\frac{\lambda}{\sinh z} \right]^2 = C_1^2$, where C_1 is a constant.

$$\sinh z \sinh \overline{z} = C_2^2$$

$$\cosh\left(z+\overline{z}\right) - \cosh\left(z-\overline{z}\right) = C_2^2 .$$

Therefore, the curves of equal pressure in the liquid are given by $\sinh^2 x + \sin^2 y = C^2$.

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