Duality of hypergraphs

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Abstract

In this paper, some notions and basic concepts related to hypergraph are presented. Next, hypergraph modeling examples are described. Finally, several results obtained from the matching, covering, independent and transversal of hypergraphs such as the Konig property and the dual Konig property are added.

Keyword: k-regular, rank, incidence matrix, subhypergraph, a stable, a transversal, a matching, a covering.

Basic Definitions

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite st, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ be a family of subsets of X. The pair $H = (X, \mathcal{D})$ is called a **hypergraph** with **vertex set** X also denoted by V(H) and with **edge set** $\mathcal{D}$ also denoted by $\mathcal{D}(H)$.

$|X| = n$ is called the **order** of the hypergraph, written also as n, or n(H). The elements x_1, x_2, …, x_n are called the vertices and the sets D_1, D_2, … , D_m are called the **edges (hyperedges)**. The number of edges is usually denoted by m or m(H).

A hypergraph which contains no vertices and no edges is called the **empty set.** A hypergraph is called **simple** if it contains no included edges,hence simple hypergraphs do not have empty and multiple edges. Simple hypergraphs are also known as **sperner families.** In a hypergraph, two vertices are said to be **adjacent** if there is an edge $D \in \mathcal{D}$ that contains both vertices. The adjacent vertices are sometimes called **neighbour** to each other, and all the neighbours for a given vertex x are called the **neighbourhood** of x in a hypergraph. The neighbourhood of x is denoted by N(x). Two edges are said to be **adjacent** if their intersection is not empty. If a vertex $x_i \in X$ belongs to an edge $D_j \in \mathcal{D}$, then we say that they are **incident** to each other.

$\mathcal{D}(x), x \in X$, will denote **all the edges containing the vertex** x. The number $|\mathcal{D}(x)|$ is called the **degree of the vertex** x, the number $|D_i|$ is called the **degree (size, cardinality) of the edge** D_i. The **maximum degree** of the hypergraph H is denoted by $\Delta(H) = \max_{x \in X} |\mathcal{D}(x)|$.

A hypergraph in which all vertices have the same degree $k \geq 0$ is called **k-regular**. A hypergraph in which all edges have the same degree $r \geq 0$ is called **r-uniform**.

The **rank** of a hypergraph H is $r(H) = \max_{D \in \mathcal{D}} |D|$. Two simple hypergraphs H_1 and H_2 are called **isomorphic** if there exists a one-to-one correspondence between their vertex sets such that any subset of vertices form an edge in H_1if and only if the corresponding subset of vertices forms an edge in H_2.

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Hypergraph modeling examples.

Hypergraphs can model concepts in different sciences in a much more general settings than graphs do. In addition, they help to find optimal solutions for many few optimization problems. While vertices represent elements of a set, the hyperedges represent properties of different subsets, or, even more generally, arbitrary statements about arbitrary subsets. Let us mention just a few examples.

Mathematics:

- the vertices are a finite set of points on the real line; the edges are some subsets of the points which form intervals in the ordering of the points, one hyperedge for one such subset;
- the vertices are vertices of a 3-dimensional polyhedron; each face of the polyhedron forms a hyperedge.

Computer science:

- the vertices are all possible inputs for a chip; the edges are the subsets of inputs which determine some internal defects, one subset for each defect;
- the vertices are files in a data base; the edges are files needed to open for a query, one subset of every query.

Physics/Chemistry:

- the vertices are the atoms in a molecule; hyperedges of degree 2 correspond to simple covalent bonds, and hyperedges of degree greater than 2 correspond to polycentric bonds;
- the vertices are chemical compounds produced by a chemical factory; the edges are the subsets of compounds that might explode.

Geographical maps:

- the vertices are cities; the edges are the cities which are on the same highway;
- the vertices are street crossings in the city map; the edges are the subsets corresponding to the bus routes, one edge for each bus route.

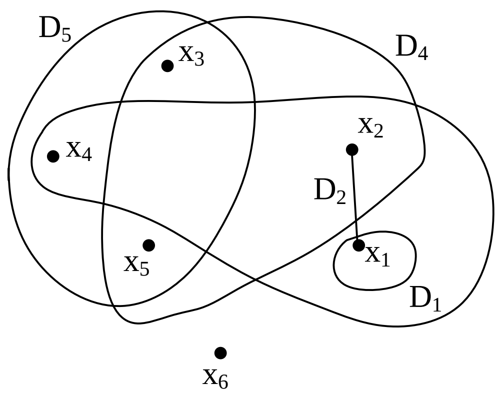
Incidence and Duality

The **incidence matrix** of a hypergraph $H = (X, D)$ is a matrix $I(H)$ with n rows that represent the vertices and m columns that represent the edges of H such that

$$(i, j) - \text{entry} = \begin{cases} 1 & \text{if} \quad x_i \in D_j, \\ 0 & \text{if} \quad x_i \notin D_j. \end{cases}$$

Let $H = (X, D)$ be a hypergraph with $X = \{x_1, x_2, \ldots, x_n\}$ and $D = \{D_1, D_2, \ldots, D_m\}$. The **dual** of the hypergraph H is a hypergraph $H^* = (Y, Z)$ whose vertex set is $Y = \{d_1, d_2, \ldots, d_m\}$, and the edge set is defined as follows: $Z = \{X_1, X_2, \ldots, X_n\}$, $X_i = \{d_j \mid X_i \in D_j \text{ in } H\}$.

For a hypergraph $H=(X,\mathcal{D})$, the **bipartite representation** of H to be the bipartite graph $B(H)=(W,E)$ with the vertex set $W=X\cup\mathcal{D}$, where X is the left part, $\mathcal{D}$ is the right part, and E is the edge set; vertex $x\in X$ is adjacent to vertex $D\in\mathcal{D}$ in $B(H)$ if and only if vertex $x\in X$ is incident to edge $D\in\mathcal{D}$ in H..



	D_1	D_2	D_3	D_4	D_5
x_1	1	1	1	0	0
x_2	0	1	1	1	0
x_3	0	0	0	1	1
x_4	0	0	1	0	1
x_5	0	0	0	1	1
x_6	0	0	0	0	0

I(H)

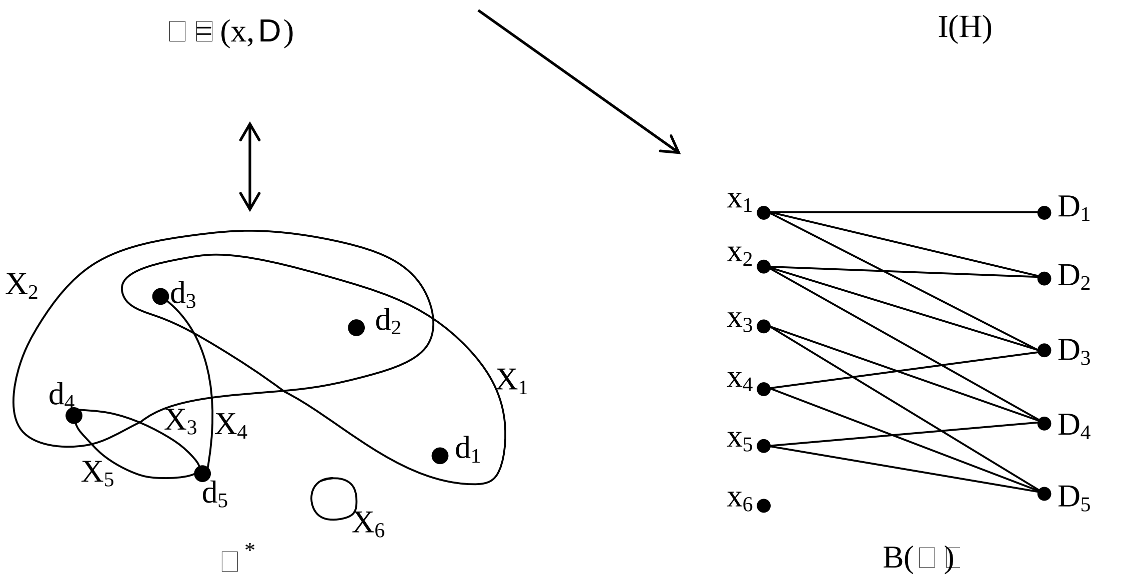


Figure 1

Figure 1 shows example of a hypergraph H, its incidence matrix $I(H)$, dual H^* and bipartite representation $B(H)$.

Proposition. 1 (Degree equality) For a hypergraph $H=(X,\mathcal{D})$, the sum of all vertex degrees equals the sum of all edge cardinalities, i.e., $\sum_{i=1}^{n}|\mathcal{D}(x_i)|=\sum_{j=1}^{m}|D_j|$.

Proof. Consider the bipartite representation of H, i.e. the bipartite graph $B(H)=(W,E)$. If we sum the degrees of vertices in the first part, we obtain the left side of the equality; if we sum the degrees of vertices in the second part, we obtain the right side of the equality. Evidently, they coincide because both are equal to the number of edges in $B(H)$.

Konig Property and Dual Konig Property

Let H = (x, $\mathcal{D}$) be a hypergraph. Any hypergraph H′ = (X′, $\mathcal{D}'$) where $X' \subseteq X$, and $\mathcal{D}' \subset \mathcal{D}$ is called a **subhypergraph** of H.

A hypergraph H′ = (X′, $\mathcal{D}'$) is called an **induced subhypergraph** of a hypergraph H = (X, $\mathcal{D}$) if $X' \subseteq X$ and all edges of H completely contain in X′ form the family $\mathcal{D}'$.

For a hypergraph H = (X, $\mathcal{D}$), any subhypergraph $H' \subseteq H$ such that H′ = (X, $\mathcal{D}'$) is called a **partial subhypergraph.**

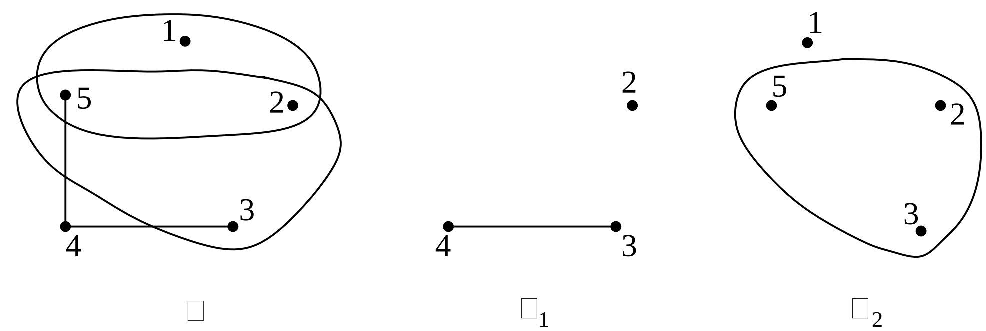


Figure 2

Figure 2 shows hypergraph H, induced subhypergraphH_1, and subhypergraphH_2.

Let H = (X, $\mathcal{D}$) be a hypergraph. A subset of vertices which contains no edge of H is called the **stable set**, or the **independent set.**

The largest size of a stable set over all maximal by inclusion stable sets is called the **stability (independence) number,** denoted by α (H).

For a hypergraph H = (X, $\mathcal{D}$), a subset of vertices $S \subseteq X$ is called a **strongly independent (stable) set** if $|S \cap D| \leq 1$ for every hyperedge $D \in \mathcal{D}$.

The cardinality of a maximum strongly independent set is denoted by $\bar{\alpha}(H)$.

A set $T \subseteq X$ is called a **transversal** of a hypergraph H = (X, $\mathcal{D}$) if $|T \cap D| \geq 1$ for every edge $D \in \mathcal{D}$. The cardinality of a minimum transversal is denoted by $\tau(H)$.

In a hypergraph H, a set of edges pairwise have no vertices in common is called a **matching.** A **perfect matching** is a matching which contains every vertex of a hypergraph.

The maximum size of a matching (over all matchings) is denoted by v (H).

For a hypergraph H = (X, $\mathcal{D}$), a subset of edges $\mathcal{D}'$ is called a **covering** if the union of all edges from $\mathcal{D}'$ coincides with X. The minimum number of edges in a covering is denoted by $\rho(H)$.

The hypergraph H = (X, $\mathcal{D}$) has the **Konig property** if $\tau(H) = v(H)$ and the **dual Konig property** if $\rho(H) = \bar{\alpha}(H)$.

For hypergraph H in Figure 2,

$\tau(H) = 2, v(H) = 2, \rho(H) = 2$ and $\bar{\alpha}(H) = 2.$

So this hypergraph has the Konig property and the dual Konigproperty.

Conclusion

The hypergraphs gives us a more compact notation for describing various problems in traditional graph theory. It is used outside the area of graph theory as well, having applications in, for example, in combinations and linear algebra.

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References

C. Berge, *Hypergraphs*, Vol. 45, North-Holland Mathematical Library, North-Holland, 1989.

A. Bretto, *Hypergraph Theory*: *An Introduction*, Springer, 2012.

V. I. Voloshin, *Introduction to Graph and Hypergraph Theory*, Nova Science Publishers, New York, 2009.