MATHEMATICAL MODELLING AND SOLUTION PROCEDURE FOR THE BEHAVIOR OF SUCKER-ROD PUMPING SYSTEM

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Abstract

This paper presents a simple and efficient approach for predicting the dynamic behavior of the sucker rod string. The dynamic behavior of the sucker rod string is an important factor for predicting the performance of a pumping system. Prediction of sucker-rod system behavior involves the solution of a boundary value problem. Such a problem includes a differential equation and a set of boundary conditions. For the sucker-rod problem, the wave equation is used, together with boundary conditions which describe the initial stress and velocity of the sucker rods, the motion of the polished rod and the operation of the downhole pump. Of these items, the wave equation, the polished rod motion condition and the down-hole pump conditions are of primary importance. Discussion of the mathematical model centers about these factors. In this study, the model consists of two analysis that are used to calculate the displacement and loads of the operation conditions of the sucker-rod pump. Practical usefulness of the proposed approach is shown by simple examples and tests. This study shows that the proposed approach is very efficient for predicting the performance of a sucker rod pumping system.

Keywords: Mathematical modelling, sucker rod pump, kinematic analysis, polished rod load.

1. Introduction

Sucker rod pumping systems are the most popular artificial lift method in the oil industry. Accurate prediction of the performance of the system can improve efficiency of a system. The most important parts to model are the subsurface elements such as the sucker rod string and downhole pump. The sucker-rod string's most important feature is its elasticity, which is responsible for the difficulty in calculating the downhole conditions from surface data.[1] The mathematical models for the sucker rod pumping system for the digital computer started in early 1960's. Gibbs was the first who successfully modeled a sucker rod pumping system, and one of his most important contribution is simulation of the subsurface elements. The finite difference solution of the wave equation and its initial and boundary conditions, which together can be used to predict the performance of the sucker rod string: [2] For the sucker rod string, other models were developed by using different solution techniques for the partial differential equation. These models all ignore the effect of the fluid inertia and assume that the fluid surrounding the rod is incompressible. There are models that take into consideration the dynamic effect of fluid. In the later models, the dynamic effects of fluid were modeled by a system of partial differential equations, which were solved by the method of characteristics. [3]

The one dimensional wave equation with viscous damping is used in the sucker-rod boundary value problem to simulate the behavior of the rod string. This equation describes the longitudinal vibration in a long selendar rod and, here, is ideal for the sucker-rod application. This paper presents two analysis to find the displacement of the sucker rod string instead of solving the analytical methods that are difficult to solve these models.

2. The Components of a Sucker-rod Pumping System

The components of a sucker-rod pumping system [4] is shown in the figure1.

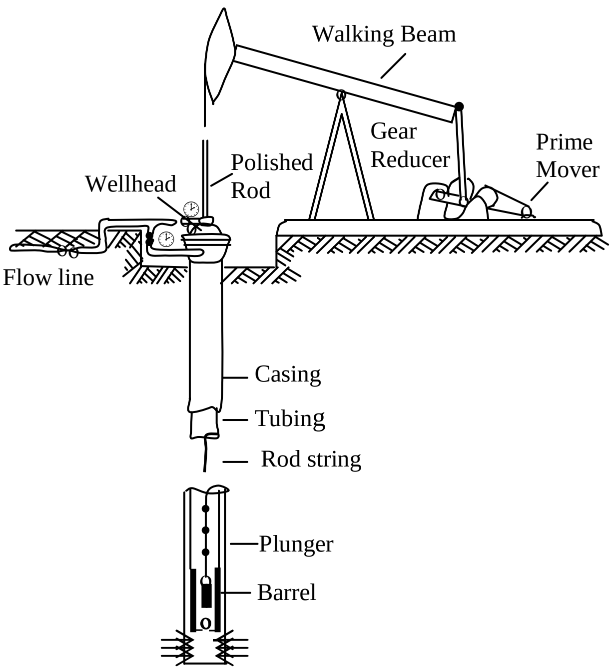

Figure 1. The Components of a Sucker-Rod Pumping System

2.1 Polished-Rod Loads

The components of the polished-rod load, in general, are the following:

(i) The weight of the rod string,

(ii) A buoyant force that decreases the rod weight,

(iii) Mechanical and fluid friction forces along the rod string,

(iv) Dynamic forces occurring in the string, and

(v) The fluid load on the pump plunger.

The sum of the rod string weight and the buoyant force is usually expressed by the 'wet' weight of the rods, which is quite simple to calculate. The effects of the friction forces are not included in most calculation because they are difficult or impossible to predict. The dynamic forces stem from the inertia of the moving masses: the rod string and the fluid column. They are additive to the static loads during the up-stroke and must be subtracted from the static rod weight on the down-stroke. The inertial forces are calculated by multiplying the mass being moved with the acceleration at the polished-rod. From Mill's 'acceleration factor' formula,

$$\delta = \pm \frac{SN^2}{70{,}500}$$

where

δ = acceleration factor,

S = polished -rod stroke length, in.,

and

N = pumping speed, SPM.

Next, fluid load on the plunger is found from:

F_0 = 0.433 H A_p (SpGr)

where

H = depth of the dynamic fluid level, ft.,

A_p = plunger area, sq-in.,

and SpGr = specific gravity of the produced fluid.

Then an expression to approximate the peak polished-rod load can now be written as the sum of the fluid load on the plunger and the static plus dynamic loads.

$$PPRL = F_o + W_r (1 + \delta)$$

where

PPRL = peak polished- rod load, lb.,

F_o = fluid load on the plunger, lb.,

W_r= total rod-string weight in air, lb., and

δ = acceleration factor.

During the down stroke, the buoyant weight of the rod-string must be decreased by the dynamic force to find the minimum polished-rod load because they act in opposite direction.

$$MPRL = W_{rf} - W_r \delta$$

The buoyant rod weight (W_{rf}) can be expressed as

$$W_{rf} = W_r [1- (0.128) (SpGr)]$$

Then $MPRL = W_r [1 - (0.128) (SpGr)- \delta]$

where

MPRL = minimum polished -rod load, lb.,

W_r = total rod-string weight in air, lb,

SpGr =specific gravity of the produced fluid

3. Down -hole Pump Simulation

The most important boundary condition in the sucker-rod problem is the one which describes the operation of the down-hole pump .Undoubtedly, the mathematical description of the down-hole pump has been the greatest difficulty in analytical treatment of the sucker-rod system .In order to get around this impasse, it is convenient to write the pump condition as

$$\alpha\, u(L,t) + \beta \frac{\partial u(L,t)}{\partial x} = p(t) \qquad (1)$$

where the parameters α, β and $p(t)$ depend upon the type of pump operation to be simulated . With the pump condition written in this manner, the flexibility needed to simulate widely varying pumping conditions can be achieved.

For example, the choice $\alpha = 0,\ \beta = 1,\ p(t) = 0$ gives (1) the form

$$\frac{\partial u(L,t)}{\partial x} = 0,$$

which implies that the down-hole pump is free and unloaded. This situation occurs or is approached in a real sucker-rod installation when the pump is descending with the traveling valve open.

As a further example, take $\alpha = 1, \beta = 0, P(t) = u_c$.

In this case, the pump condition becomes

$$u(L,t) = u_c,$$

which implies that the pump is stationary at some position u_c. This situation is approached in a high-pump-efficiency installation while the fluid load is being transferred from the rods to the tubing or from the tubing to the rods.

As a final example, take $\alpha=0, \beta=1, P(t)=\frac{W_1}{EA}$. The pump boundary condition then reduces to

$$EA\frac{\partial u(L,t)}{\partial x}=W_{1,}$$

where

W_1 = steady load of the rod string, lb.,

E = Young's modulus of elasticity for the rod's material, psi, and

A = cross-sectional area of the rod, sq- in.

which means that a steady load W_1 is being applied at the pump .This condition exists while fluid is being lifted to the surface.

4.Determination of the Damping Coefficient

The damping term in the wave equation stands for the irreversible energy losses that occur along the rod string during its movement. The available model for damping coefficient determination is discussed as follows:

The method involves an energy balance written for the two ends of the rod string and can be solved for the damping coefficient and the following formula is obtained:

$$c = \frac{(550)(144)g_c\left(\text{PRHP - P}_{\text{hydr}}\right)T^2}{\sqrt{2}\,\pi\,W\,S^2} \qquad (2)$$

where

c = damping coefficient, $\frac{1}{s}$,

g_c = 32.2, gravitational constant,

PRHP = polished-rod power, HP,

P_{hydr} = hydraulic power used for fluid lifting, HP,

T = period of the pumping cycle , s ,

W = total weight of the rod string, lb.,

S = polished-rod stroke length, in.

and the above equation (2) is valid for any rod material.

5. Rod-String Simulation with the Wave Equation

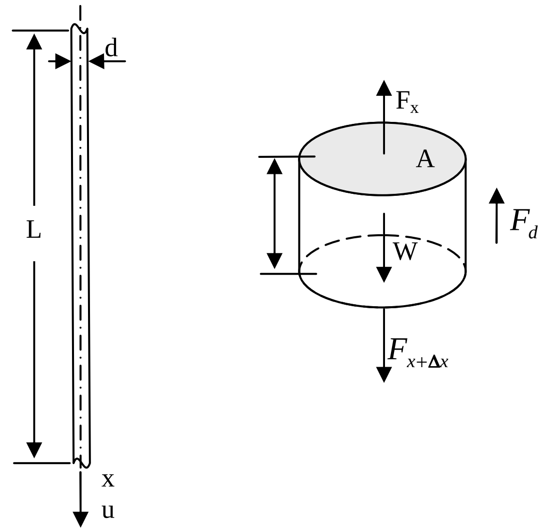

Figure 2. Illustration of the Forces Acting on an Incremental Element of the Rod String

Figure 2 shows a rod string section with a uniform cross-sectional area, A, and length, L .The coordinate axes x and u are directed downwards and represent axial distance and displacement of the rod along the string, respectively. As shown in the figure, the following forces act on the rod element:

W = the buoyant weight of the rod element,

F_x = a tension force that represents the pull from above on the rod element,

$F_{x+\Delta x}$ = another tension force representing the downward pull on the rod element, and

F_d = a damping force opposing the movement of the rod element, which is the result of fluid and mechanical friction on the rod elements surface.

Using Newton's Second Law of motion, the sum of the forces acting on the element should equal the mass of the element times its acceleration:

$$-F_x + F_{x+\Delta x} + W - F_d = m\frac{\partial^2 u}{\partial t^2} \qquad (3)$$

The weight of the rod element, W, is a static force that is constant during the pumping cycle. Therefore it can be dropped from the equation .The tension forces F_x and $F_{x+\Delta x}$ can be expressed with the mechanical stresses present in the rod sections at the axial distance x and $x+\Delta x$:

$$F_x = S_x A \text{ , and } \qquad F_{x+\Delta x} = S_{x+\Delta x}\, A$$

where

S_x and $S_{x+\Delta x}$ = rod stresses at sections x and $x+\Delta x$, psi, and A = cross-sectional area of the rod, sq- in. Then equation (3) becomes,

$$\left(S_{x+\Delta x} - S_x\right)A - F_d = m\frac{\partial^2 u}{\partial t^2} \qquad (4)$$

Since sucker rods in normal operation undergo an elastic deformation, Hooke's Law can be applied[8], which states that the stress at any cross section is proportional to the deformation of the actual rod element.

$$S = E\frac{\partial u}{\partial x} \qquad (5)$$

where

E = Young's modulus of elasticity for the rod material, psi, and $\frac{\partial u}{\partial x}$ = rod strain, i.e. the change of rod displacement over rod length[9]. Using equation (5) and define for rod stress, and substituting the appropriate terms in equation (4), we get

$$EA\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] - F_d = m\frac{\partial^2 u}{\partial t^2}. \qquad (6)$$

The multiplier of the term EA on the left-hand side can be expressed with second derivative of displacement, u, with respect to distance, x . Introducing this and expressing the mass, m, with volume and density of the rod element, we arrive at the following equation:

$$E\,A\,\Delta x\frac{\partial^2 u}{\partial x^2} - F_d = \frac{\Delta x A\rho}{144 g_c}\frac{\partial^2 u}{\partial t^2} \qquad (7)$$

where

ρ = density of rod material, lb/ft^3. ,

and g_c = 32.2, gravitational constant.

In order to develop the final form of the wave equation, the damping force, F_d, remains to be determined. During the pumping cycle, energy is continuously lost along the rod string because the well fluids impart a viscous force at the outer surface of the rods. This viscous damping force is proportional to the relative velocity, i.e. the shear velocity, $\frac{\partial u}{\partial t}$, between the fluid and the rods. Then the damping force to be proportional to rod mass and gave the following semi empirical formula:[2]

$$F_d \propto \frac{\Delta x \rho A}{144 g_c}\frac{\partial u}{\partial t}$$

or

$$F_d = c\frac{\Delta x \rho A}{144 g_c}\frac{\partial u}{\partial t} \qquad (8)$$

where

c = damping coefficient, $\frac{1}{s}$.

From (7) and (8),

$$EA\frac{\partial^2 u}{\partial x^2} - c\frac{A\rho}{144 g_c}\frac{\partial u}{\partial t} = \frac{\rho A}{144 g_c}\frac{\partial^2 u}{\partial t^2}. \qquad (9)$$

Equation (9) is the final form of the one-dimensional wave equation describing the propagation of force waves in the sucker rod string. In this form, it is valid for variable rod diameters, i.e. for tapered rod strings. From it, the more familiar equation for a uniform rod section is found by a simple mathematical operation:

$$v_s^2\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} \qquad (10)$$

where

$v_s = \sqrt{\frac{144 g_c E}{\rho}}$ = sound velocity in the rod material, ft/s.

This is the most widely used form of the wave equation and is a linear, second-order hyperbolic partial differential equation.

6. Solution of the Wave Equation

The damped wave equation allows the calculation of the displacement of the rods, u, at any axial distance, x, and time, t. Thus the movement of any rod element is a function of both the vertical distance and the time. Depending on the boundary conditions used, there are basically two ways to find displacements and forces along the rod string: either starting at the surface and proceeding downwards or vice versa. The two main uses of the wave equation can be classified according to this criterion and are:

1. The diagnostic analysis, which involves calculating the down hole displacements and forces based on surface measurements, i.e. the surface dynamometer card.
2. The predictive analysis, which predicts the surface conditions based on the description of the sucker rod pump's operation.

6.1. The Analytical Solution

The analytical solution of the wave equation with the polished-rod load and polished-rod displacement vs. time functions are described by Fourier series approximations. The relevant harmonic functions are as follows:

$$D(t) = F(t) - W_{rf} = \frac{\sigma_0}{2} \sum_{n=1}^{N} \left[\sigma_n \cos(n\omega t) + \tau_n \sin(n\omega t)\right] \tag{11}$$

$$u(t) = \frac{v_0}{2} + \sum_{n=1}^{M} \left[v_n \cos(n\omega t) + \delta_n \sin(n\omega t)\right] \tag{12}$$

where

$D(t)$ = dynamic load at the polished rod vs. time,

$F(t)$ = polished-rod load vs. time,

W_{rf} = buoyant weight of the rod string,

σ_n, τ_n = Fourier coefficients of the dynamic load function,

v_n, δ_n =Fourier coefficient of the displacement function,

ω = angular frequency of pumping,

N = number of dynamic load coefficients, and

M = number of displacement coefficients.

The number of the coefficients in (11) and (12) are evaluated with the classical integration formula:

$$\sigma_n = \frac{\omega}{\pi} \int_{t=0}^{2\pi} D(t) \cos(n\omega t) dt$$

where n = 0,1,…,N.

In practice, however, these integrals cannot be evaluated analytically because the load and displacement functions are known at some discrete points only. Therefore, a numerical integration procedure, such as trapezoidal rule, can be used. The final formulae for the rod displacement and the dynamic force in the functions of the time, t, and the distance from the surface, x is

$$u(x,t) = \frac{\sigma_0}{2EA} + \frac{v_0}{2} + \sum_{i=1}^{M} \left[0_n \cos(n\omega t) + P_n \sin(n\omega t)\right], \text{and}$$

$$D(x,t) = EA\left\{\frac{\sigma_0}{2EA} + \sum_{i=1}^{N} \left[0'_n \cos(n\omega t) + P'_n \sin(n\omega t)\right]\right.$$

With the knowledge of these coefficients, the displacements and forces for different assumed times are determined easily at the bottom of a uniform rod section, enabling a down hole dynamometer card (dynagraph) to be potted.

6.2 Numerical Solution

As in the case with every differential equation, the damped wave equation also can be recast in a difference form[1]. The resultant finite difference equation for damp wave equation is

$$v_s^{\,2}\left[\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2}\right]-c\left[\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}\right] = \frac{u(x,t+\Delta t)-2u(x,t)+u(x,t-\Delta t)}{(\Delta t)^2}$$

$$\frac{v_s^2(\Delta t)^2}{(\Delta x)^2}\left[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)\right]-c\Delta t\left[u(x,t+\Delta t)-u(x,t)\right] = u(x,t+\Delta t)-2u(x,t)+u(x,t-\Delta t)$$

$$u(x,t+\Delta t)+c\Delta t\left[u(x,t+\Delta t)-u(x,t)\right] = \frac{v_s^2(\Delta t)^2}{(\Delta x)^2}\left[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)\right] + 2u(x,t)-u(x,t-\Delta t)$$

$$u(x,t+\Delta t) = \frac{\frac{v_s^2(\Delta t)^2}{(\Delta x)^2}\left[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)\right]+2u(x,t)-u(x,t-\Delta t)+c\Delta t u(x,t)}{1+c\Delta t}$$

and it can be solved numerically.

There are two possibilities: such equations can be solved for $u(x, t+\Delta t)$ (i.e. for the rod displacement at the same place but ahead in time), or they can be solved for $u(x+\Delta x, t)$, (i.e. for the displacement at the same time but at the next distance). These two approaches conform to the diagnostic and the predictive analysis methods of simulating the behavior of the sucker-rod string.

Likewise, the boundary conditions are recast into partial difference form. In particular, the down-hole pump condition becomes

$$u(x,t) = \frac{\Delta x p(t) + 2\beta u(x-\Delta x,t) - \frac{1}{2}\beta u(x-2\Delta x,t)}{\alpha\Delta x + \frac{3}{2}\beta} \tag{13}$$

in which $u(x,t)$ denotes the displacement of the pump. Equation (13) is obtained directly from equation (1) by replacing the derivative by a difference quotient. A timely choice of α, β and $p(t)$ is needed, and these choices depend on valve operation. The times of valve opening and closing are “ sensed” by the computer with the following test:

Test for t_1 - while

$$\frac{3}{2}u(x,t)-2u(x-\Delta x,t)+\frac{1}{2}u(x-2\Delta x,t)=0$$

(no load on pump), the computer senses when $u(x,t)-u(x,t-\Delta t)$ changes from positive to negative. This indicates that the pump has reached its lowest position, at which time the traveling valve closes. This is the computer's signal to make the appropriate choices α, β and $p(t)$ to simulate the desired pump condition.

Test for t_2 - while

$$\frac{3}{2}u(x,t)-2u(x-\Delta x,t)+\frac{1}{2}u(x-2\Delta x,t)\rangle 0$$

(tension at the pump), the computer makes tests to determine when

$$\frac{EA}{\Delta x}\left[\frac{3}{2}u(x,t)-2u(x-\Delta x,t)+\frac{1}{2}u(x-2\Delta x,t)\right]=W_1$$

At this time the fluid load is completely borne by the rods and the standing valve opens.

Test for t_3 - while

$$\frac{EA}{\Delta x}\left[\frac{3}{2}u(x,t)-2u(x-\Delta x,t)+\frac{1}{2}u(x-2\Delta x,t)\right]=W_1$$

(fluid load imposed on the pump), the computer senses when $u(x,t)-u(x,t-\Delta t)$ changes from negative to positive. At this time the pump has reached its highest position, and the standing valve closes.

Test for t_4 - while

$$\frac{3}{2}u(x,t)-2u(x-\Delta x,t)+\frac{1}{2}u(x-2\Delta x,t)\rangle 0$$

(tension at the pump) , the computer determines when

$$\frac{3}{2}u(x,t)-2u(x-\Delta x,t)+\frac{1}{2}u(x-2\Delta x,t)=0\,.$$

At this time the fluid load is completely borne by the tubing , and the traveling valve opens.

In this manner the computer continually senses the forces and movements which affect valve action and make the proper choices in the pump boundary condition to simulate the desired down-hole dynagraph card.

6.2.1 Diagnostic Analysis

The diagnostic analysis involves the calculation of displacements, u, along the length of the rod string, x, for the same values of the time, t. The boundary conditions that must be used are provided by the surface dynamometer card, which gives the time history of the dynamic force and polished-rod movement at place x = 0, i.e. the functions D (0 , t) and u (0 , t).

Application of the finite difference method involves dividing the rod string into a number of Δx segments to facilitate a stepwise solution. A time increment, Δt , has also been assumed, which is usually defined by the number of points read from the surface dynamometer diagram. The values of the two increments are interrelated and the following stability criterion applies:

$$\Delta x \le \Delta t\, v_s$$

where v_s = sound velocity in the rod material.

The main calculation steps of the diagnostic model using finite differences can be summarized as follows:

(i) The polished-rod load and displacement vs. time functions are determined, and their values are found at given Δt time intervals. The rod string length increment, Δx ,is established based on the stability criterion, and the rod string is divided into the appropriate number of segments.

(ii) The initial displacements of the rod string at the surface, u (0, t), are set to the polished–rod positions at every time step.

(iii) The displacement at the next lower segment, $u(x+\Delta x,t)$, is calculated with the finite difference formula. This is repeated for all time steps involved to cover the whole pumping cycle.

(iv) Step 3 is repeated for the next consecutive rod string elements until a junction of the different taper sections is reached. At such points, rod displacements are corrected for the static rod stretch and dynamic forces are calculated with Hooke's Law.

(v) At the bottom of the string, after correction for buoyant rod weight, the calculated displacements and loads define the operating conditions of the sucker-rod pump.

6.2.2 Predictive Analysis

In contrast to the diagnostic model, the predictive analysis model also considers time and uses a calculation formula that gives the rod displacements ahead in time, $u(x,t+\Delta t)$. Just as in the diagnostic case, the rod string is divided into a number of segments, and the segment length is determined in a similar fashion as in the diagnostic case.

The stability of the solution requires that the time increment, Δt , satisfy the condition

$$\Delta t \leq \frac{\Delta x}{v_s}.$$

The greatest difficulty in the predictive analysis is the simulation of the down -hole pump's performance. The use of this technique allows the simulation of different down -hole conditions encountered in practice.

In summary, the predictive analysis consists of the following main calculation steps:

(i) The rod length increment, Δx ,is defined and the time increment, Δt , that satisfies the stability criterion is found.

(ii) The initial conditions, i.e. the values of the polished-rod displacement vs. time function, u (0,t), are determined from the pumping unit's kinematic evaluation.

(iii) The displacement at the next lower rod segment is determined using the finite difference formula. This procedure is repeated, taking into account the changes in rod size, until the bottom of the string is reached.

(iv) The action of the sucker-rod pump is taken into account as detailed above.

(v) Step 3 and 4 are repeated for all time steps.

(vi) The whole procedure is repeated for several pumping cycles to reach a steady-state solution without any transient effects.

(vii)The final rod displacements and loads valid at the end of the calculations represent the conditions at the polished-rod and at the subsurface pump. Then the surface and down-hole dynamometer diagrams (dynagraphs) can be plotted to analyze the operation of the pumping system.

7. Conclusion

Our concentration is centered on the displacement of the sucker-rod string and the important role of damping coefficient, and finite difference equations are used to find the displacement of the sucker-rod string. Most of the problems in Engineering subjects may come across to solve the corresponding partial differential equation models. In practice, analytical methods are difficult to solve these models. So the finite difference methods are appropriable to use instead of analytical methods.

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