Method of Characteristics for Traffic Flow Problem

Pa Pa Lwin*

Abstract

In this paper a powerful method of characteristics is introduced to solve the partial differential equations. Moreover, traffic flow models are also discussed and solved by using method of characteristics.

1. Introduction

A partial differential equation (PDE) is an equation that must be solved for an unknown function of at least two independent variables when the equation contains partial derivatives of the unknown function. The order of a PDE is the highest-order partial derivative contained therein. The method of characteristics to solve linear and quasilinear equations is used.

In the method of characteristics, integration of the initial-value problem associated with a first-order PDE is reduced to integration of the initial value problem for a system of ordinary differential equations. For simplicity in calculations and for geometric visualizations, problems in two independent variables are considered, but extensions to higher numbers of independent variables are algebraically straightforward. The quasilinear PDE is

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \qquad (1)$$

for u as a function of x and y. It is assumed that coefficient functions $a(x,y,u)$, $b(x,y,u)$, and $c(x,y,u)$ are continuous in some region of xyu-space, and solutions of the PDE that have continuous first partial derivatives u_x and u_y are sought. If $c(x,y,u)$ is of the form $c(x,y)u + d(x,y)$, and $a(x,y,u)$ and $b(x,y,u)$ are independent of u, the equation is said to be linear. Quasilinear equations are linear in u_x and u_y, but not in u itself. Theory for linear equations is the same as that given here for more general quasilinear equations, but simplifications in calculations occur in linear case.

If $u(x,y)$ is a solution of PDE (1), a normal to the surface $u = u(x,y)$ is $\nabla[u(x,y) - u] = \langle u_x, u_y, -1\rangle$. Since the PDE can be expressed in the form

$$0 = a\,u_x + b\,u_y - c = \langle u_x, u_y, -1\rangle \cdot \langle a, b, c\rangle,$$

the PDE demands that at each point on a solution surface, the vector $\langle a,b,c\rangle$ must be normal to the vector $\langle u_x, u_y, -1\rangle$, and hence lies in the tangent plane to the surface at that point. Thus, the PDE defines a direction field $\langle a(x,y,u), b(x,y,u), c(x,y,u)\rangle$, called the characteristic directions, such that $u = u(x,y)$ is a solution surface if and only if, at each point $(x, y, u(x,y))$ on the surface, the tangent plane to the surface contains the characteristic direction (Figure 1). In Figure 2, a number of these tangent vectors at various points on the solution surface were shown. A curve that begins at a point on the surface, lies in the surface, and remains tangent to the characteristic direction at every point is called a characteristic curve (C-curve, for short). The solution surface can be thought of as being comprised of C-curves; in fact a one-parameter family of C-curves.

* Dr, Lecturer, Department of Mathematics, Yadanabon University

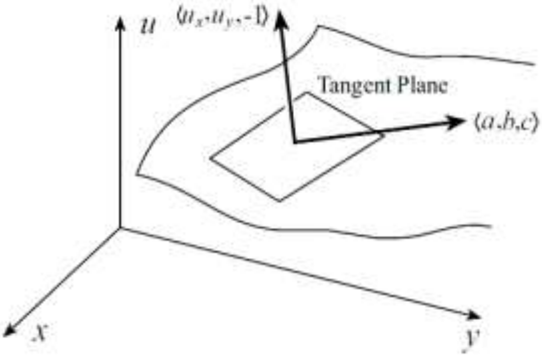

Figure 1

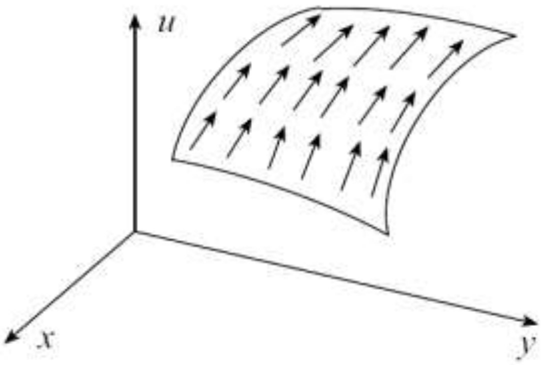

Figure 2

This suggests that solution surfaces to (1) can be obtained by finding all $C-$curves and extracting form them one-parameter families. C-curves are defined by the ordinary differential equations

$$\frac{dx}{a}=\frac{dy}{b}=\frac{du}{c}, \tag{2}$$

called the characteristic equations (C-equations). Solving these equations gives a two-parameter family of C-curves that can be expressed in the form

$$F(x,y,u,\alpha,\beta)=0,\ G(x,y,u,\alpha,\beta)=0. \tag{3}$$

Through each point (x,y,u) in space there is a unique C-curve and a tangent vector to a C-curve at every such point is $\langle a,b,c\rangle$ (Figure 3). Any smooth surface composed of C-curves is a solution of PDE (1). Such surfaces can be found analytically by specifying β as a function of α, $\beta=\beta(\alpha)$. This creates a one-parameter family of C-curves, a surface,

$$F\left[x,y,u,\alpha,\beta(\alpha)\right]=0, G\left[x,y,u,\alpha,\beta(\alpha)\right]=0. \tag{4}$$

The equation of the surface is found implicitly or explicitly by eliminating α between these equations.

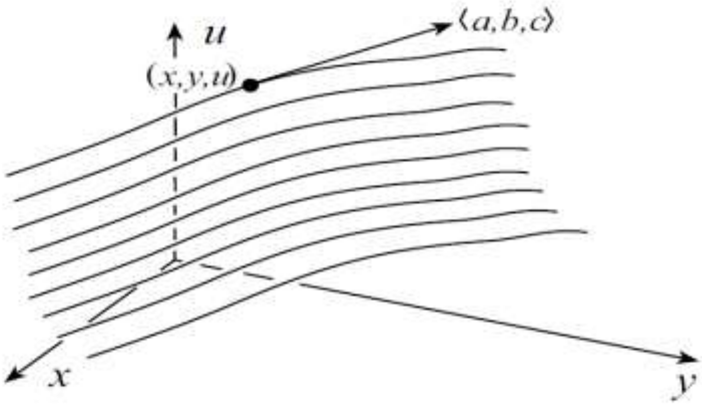

Figure 3

2. Method of Characteristics

2.1 Theorem

Every one-parameter family of characteristic curves generates a solution surface to PDE(1). Conversely, every solution surface may be considered as a one-parameter family of characteristic curves.

2.2 Example

We will find the solution of the quasilinear PDE $u_x + u\,u_y = 0$, (for $x > 0$) that takes on the values $u(0, y) = \begin{cases} k_1, & y \le 0 \\ k_2, & y > 0 \end{cases}$.

Characteristic equations for the PDE are

$$dx = \frac{dy}{u},\ du = 0.$$

If they follow that $u = \beta$ so that u is constant along C-curves, and $y = \beta x + \alpha$. A one-parameter family of C-curves is

$$y = \beta(\alpha)x + \alpha,\ \ u = \beta(\alpha).$$

For C-curves to take on the values $u = k_1$ when $x = 0$ and $y \le 0$, we set

$$y = \alpha,\ k_1 = \beta(\alpha).$$

Thus, C-curves through this part of the initial curve $(y \le 0)$ are

$$y = k_1 x + \alpha,\ \ u = k_1.$$

Similarly, C-curves passing through the other half of the initial curve $(y > 0)$ are

$$y = k_2 x + \alpha,\ \ u = k_2.$$

The solution always has value $u = k_1$ or $u = k_2$. To determine regions of the xy-plane where these values of u are taken on, we draw base C-curves. Base C-curves along which $u = k_1$ are straight lines $y = k_1 x + \alpha$ $(\alpha \le 0)$ defining region R_1 in Figure 4 a. Base C-curves along which $u = k_2$ are straight lines $y = k_2 x + \alpha$ $(\alpha > 0)$ defining region R_2 in Figure 4 a. Base C-curves have been drawn only for $x > 0$ as specified in the original problem. This leaves the solution undefined in the wedge R_3 between the lines $y = k_1 x$ and $y = k_2 x$. To obtain a solution in R_3, imagine a fan of straight lines $y = mx$ eliminating from $(0,0)$ into R_3. Since u takes on constant values along the C-curves in R_1 and R_2, supposing $u = m$ along $y = mx$. In other words, let equations for the lines in the fan be $y = ux$, $k_1 < u < k_2$, and let the value of the solution $u(x, y)$ along each line in the fan be the slope u of the line. It is straightforward to show that when this equation is inverted, the function $u = \dfrac{y}{x}$ satisfies the PDE in R_3. It also joins the planes $u = k_1$ and $u = k_2$ in R_1 and R_2 to create a continuous solution (Figure 4 b). The straight lines of the fan in R_3 satisfy the C-equations, and are called fanlike base C-curves.

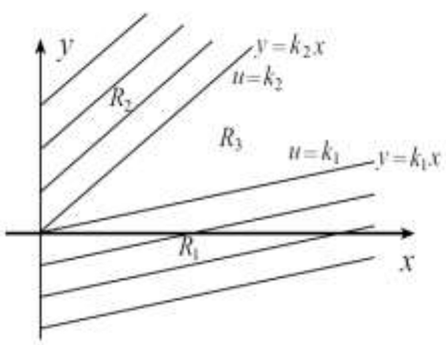

Figure 4 a

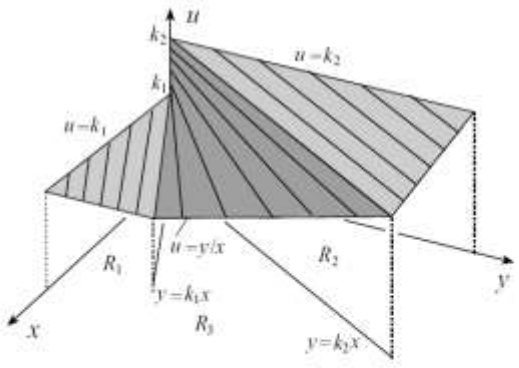

Figure 4 b

3. Traffic Flow

First order PDEs can be used to model traffic flow on a single-lane highway represented by the x-axis with flow to the right. Two important quantities in the analysis are density $\rho(x,t)$ of vehicles on the highway (number of vehicles per unit length) at position x and time t, and speed $v(x,t)$ of vehicles. The integral

$$\int_a^x \rho(\zeta,t)d\zeta$$

represents the number of vehicles on that part of the highway between a fixed point $x=a$ and any other point $x>a$. Its time-derivative is the rate of change of the number of vehicles on this part of the highway. It must be equal to the rate at which vehicles enter this part of the highway at $x=a$ less the rate at which they leave at x. Since ρv represents the number of vehicles passing point x on the highway per unit time, it may be written like this

$$\frac{\partial}{\partial t}\int_a^x \rho(\zeta,t)d\zeta = \rho(a,t)\,v(a,t) - \rho(x,t)\,v(x,t)\,.$$

When we differentiate this equation with respect to x and interchange order of operations on the left,

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v) \Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0\,. \qquad (5)$$

This is the fundamental equation of traffic flow; it relates the density and speed of the flow. It is necessary to specify a functional relationship between ρ and v in order to solve this PDE for both quantities. It is assumed that v is a function of ρ only, $v = v(\rho)$ which logically should be a decreasing function $\frac{dv}{d\rho} < 0$, velocity decreasing as density increases. Equation (5) then becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho v(\rho)\right] = 0 \Rightarrow \frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial x} + \rho\frac{dv}{d\rho}\frac{\partial\rho}{\partial x} = 0\,.$$

Replacing (5) is

$$\frac{\partial\rho}{\partial t} + \left(v + \rho\frac{dv}{d\rho}\right)\frac{\partial\rho}{\partial x} = 0\,. \qquad (6)$$

Once $v(\rho)$ is specified, this is a quasilinear first-order PDE for $\rho(x,t)$.

The simplest conceivable relationship would be $v(\rho) = v_m(1-\frac{\rho}{\rho_m})$, a linear function with maximum $v = v_m$ when $\rho = 0$, and minimum $v = 0$ when $\rho = \rho_m$ is a maximum. In this case,

$$v + \rho\frac{dv}{d\rho} = v_m\left(1-\frac{\rho}{\rho_m}\right) + \rho\left(-\frac{V_m}{\rho_m}\right) = v_m\left(1-\frac{2\rho}{\rho_m}\right),$$

and (6) becomes

$$\frac{\partial\rho}{\partial t} + v_m\left(1-\frac{2\rho}{\rho_m}\right)\frac{\partial\rho}{\partial x} = 0\,. \qquad (7)$$

C-equations for this first-order quasilinear PDE are

$$\frac{dx}{v_m\left(1-\frac{2\rho}{\rho_m}\right)} = dt,\;\; d\rho = 0\,.$$

C-curves are therefore

$$x = v_m\left(1 - \frac{2\beta}{\rho_m}\right)t + \alpha, \quad \rho = \beta .$$

They are straight lines along which density is constant, but the value of ρ varies from C-curve to C-curve.

To proceed further, we must specify the initial density of vehicles on the road. We do so in the following example.

3.1 Example

We will find the density and velocity of traffic when a traffic light at $x = 0$ turns green at time $t = 0$. Assume that initially there is no traffic to the right of the light and traffic to the left is stationary at maximum density ρ_m.

The initial data is

$$\rho(x,0) = \begin{cases} \rho_m \ , & x < 0 \\ 0 \ , & x > 0. \end{cases}$$

For C-curves to pass through this curve when $x < 0$, we set $\beta = \beta(\alpha)$ and

$$x = \alpha,\ \rho_m = \beta(\alpha).$$

C-curves for $x < 0$ are therefore

$$x = v_m\left(1 - \frac{2\rho_m}{\rho_m}\right)t + \alpha = -v_m t + \alpha,\ \rho = \rho_m .$$

For C-curves to pass through the initial curve when $x > 0$, we again set $\beta = \beta(\alpha)$, and

$$x = \alpha,\ 0 = \beta(\alpha).$$

C-curves for $x > 0$ are therefore

$$x = v_m t + \alpha,\ \rho = 0 .$$

Base C-curves (Figure 5 a) are two sets of parallel lines. For any point x and time t in region R_1, the density of traffic flow is a maximum, meaning that motion has not yet commenced. On the other hand, for any point and time in region R_2, density is a minimum and traffic would flow at maximum velocity except for the fact that no cars have reached this value for x at this time.

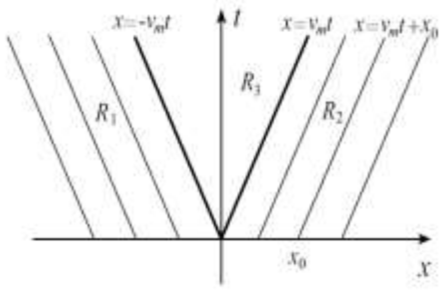

Figure 5 a

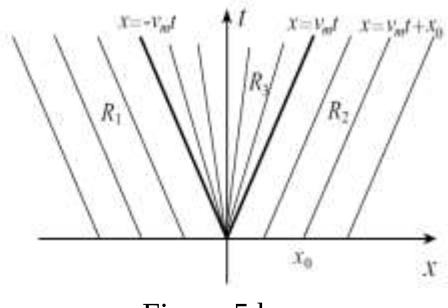

Figure 5 b

What is missing is the transition from zero velocity to maximum velocity, and this corresponds to the fact that the solution is as yet undefined in region R_3. To remedy this, we introduced fanlike base C-curves in R_3. They are straight lines through the origin, and in order for the definition of $\rho(x,t)$ in this region to satisfy (7), equations for these lines are specified in the form

$$x = v_m\left(1-\frac{2\rho}{\rho_m}\right)t, \quad -v_m < v_m\left(1-\frac{2\rho}{\rho_m}\right) < v_m .$$

When we solve this equation for ρ, the result is

$$\rho(x,t) = \frac{\rho_m}{2}\left(1-\frac{x}{v_m t}\right), -v_m t < x < v_m t .$$

This is the transition traffic density in region R_3, and it does indeed satisfy (7). Substitution into $v = v_m\left(1-\frac{\rho}{\rho_m}\right)$ gives the transition flow velocity,

$$v(x,t) = \frac{1}{2}\left(v_m + \frac{x}{t}\right), \quad -v_m t < x < v_m t .$$

Complete specifications of $\rho(x,t)$ and $v(x,t)$ are

$$\rho(x,t) = \begin{cases} \rho_m, & x < -v_m t \\ \frac{\rho_m}{2}\left(1-\frac{x}{v_m t}\right), & -v_m t < x < v_m t \\ 0, & x > v_m t \end{cases}$$

$$v(x,t) = \begin{cases} 0, & x < -v_m t \\ \frac{1}{2}\left(v_m + \frac{x}{t}\right), & -v_m t < x < v_m t \\ v_m, & x > v_m t. \end{cases}$$

It is interesting, informative, and surprising to plot these as functions of x for fixed t. Plots are shown in Figures 6.

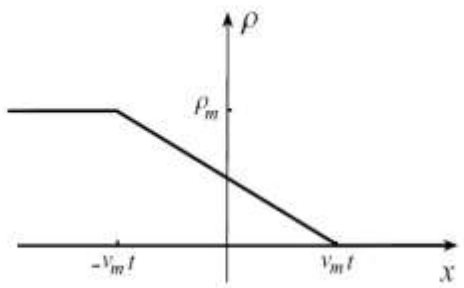

Figure 6 a

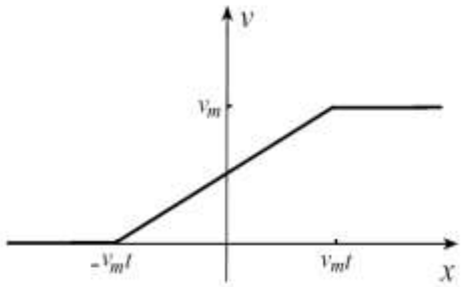

Figure 6 b

As time increases, the slanted portions of each of these graphs increase in length and become more horizontal. The point $x = -v_m t$ on the x-axis of the velocity graph moves to the left with velocity $-v_m$. This indicates that when the light turns green, all cars do not begin to move; there is a time delay for all but the first car. For a car at position x on the negative x-axis, the time delay is $t = -\frac{x}{v_m}$. Cars further back in line experience longer delays. It is if a signal to move propagates back through the line of stationary cars at velocity v_m.

In Figures 7 a, b, we have plotted velocity as a function of time for fixed $x < 0$ and $x > 0$, respectively. Figure 7 a confirms what we saw in Figure 6 b.

The car at position x in the line of stationary cars experiences a time delay $t = -\frac{x}{v_m}$ before it begins to move. Thereafter, velocities of cars at this position gradually increase, ultimately approaching $\frac{v_m}{2}$. On the other hand, for $x > 0$, Figure 6 b indicates velocity v_m at position x until $t = \frac{x}{v_m}$. It is not that cars move at this velocity at this position for these times because until time $\frac{x}{v_m}$ no cars will have reached position x. The leading car in line travels with velocity v_m, reaching position x at time $\frac{x}{v_m}$. Thereafter, velocities of cars at this position gradually decrease, approaching $v_m / 2$.

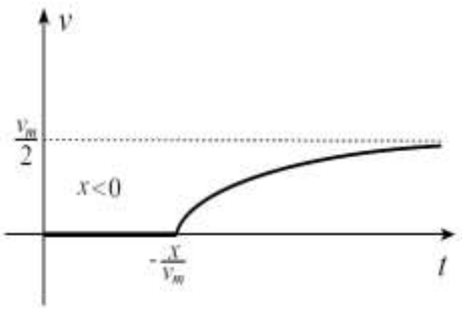

Figure 7 a

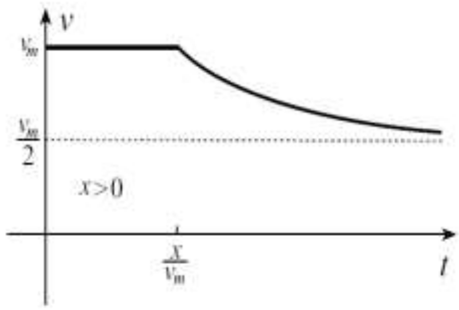

Figure 7 b

What may seem strange at first is that for very large t, all cars move with velocity $\frac{v_m}{2}$. This is a result of the fact that the number of cars on the highway must be preserved; the initial number of the cars to the left of the traffic light with density ρ_m eventually spreads out over the entire highway at density $\frac{\rho_m}{2}$.

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