Sequential Bouligand Tangent Cone and Clarke Tangent Cone in a Real Normed Space

Saw Win[1]

Abstract

In this paper, we discuss several basic properties of cone, convex, tangent cone, and bring to a focus on the sequential Bouligand tangent cone which is also called the contingent cone. For this tangent cone we prove several basic properties.

Keywords: Cone, Convex, Sequential Bouligand tangent cone, Clarke tangent cone.

Definition Let X be a given set. Assume that an addition on X, i.e., a mapping from $X \times X$ to X, and a scalar multiplication on X, i.e., a mapping from $\mathbb{R} \times X$ to X, is defined. The set X is called a **real linear space**, if the following axioms are satisfied (for arbitrary $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$):

(a) $(x+y)+z = x+(y+z)$,

(b) $x+y = y+x$,

(c) there is an element $0_X \in X$ with $x + 0_X = x$ for all $x \in X$,

(d) for every $x \in X$ there is a $y \in X$ with $x + y = 0_X$,

(e) $\lambda(x+y) = \lambda x + \lambda y$,

(f) $(\lambda + \mu)x = \lambda x + \mu x$,

(g) $\lambda(\mu x) = (\lambda \mu) x$,

(h) $1x = x$.

The element 0_X given under (c) is called the **zero element** of X.

Definition Let C be a nonempty subset of a real linear space X.

(a) The set C is called a **cone**, if

$$x \in C, \lambda \geq 0 \Rightarrow \lambda x \in C.$$

[1] Tutor, Department of Mathematics, Yadanabon University

(b) A cone C is called **pointed**, if

$$x \in C, -x \in C \Rightarrow x = 0_X.$$

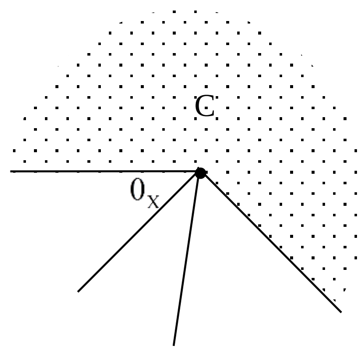


Figure 1: Cone.

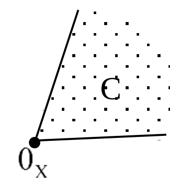


Figure 2: Pointed cone.

Example (a) The set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i \in \{1,\dots,n\}\}$ is a pointed cone.

(b) The set $C = \{x \in C[0,1] \mid x(t) \geq 0 \text{ for all } t \in [0,1]\}$ is a pointed cone.

In order theory and optimization theory convex cones are of special interest. Such cones may be characterized as follows:

Theorem* A cone C in a real linear space is convex if and only if for all $x, y \in C$

$$x + y \in C. \tag{1}$$

Proof. (a) Let C be a convex cone. Then it follows for all $x, y \in C$

$$\frac{1}{2}(x+y) = \frac{1}{2}x + \frac{1}{2}y \in C$$

which implies $x + y \in C$.

(b) For arbitrary $x, y \in C$ and $\lambda \in [0,1]$ we have $\lambda x \in C$ and $(1-\lambda)y \in C$.

Then we get with the condition (1) $\lambda x + (1-\lambda)\, y \in C$.

Consequently, the cone C is convex. □

Definition Let S be a subset of a real linear space X.

(a) Let some $\bar{x} \in S$ be given. The set S is called **starshaped** at $\bar{x}$, if for every $x \in S$

$$\lambda\, x + (1-\lambda)\,\bar{x} \in S \quad \text{for all } \lambda \in [0,1].$$

(b) The set S is called **convex,** if for every x, y $\in$ S

$$\lambda x + (1-\lambda)y \in S \quad \text{for all } \lambda \in [0, 1].$$

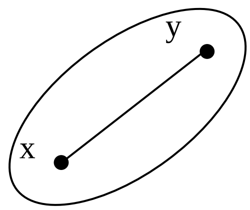


Figure 3: Convex set.

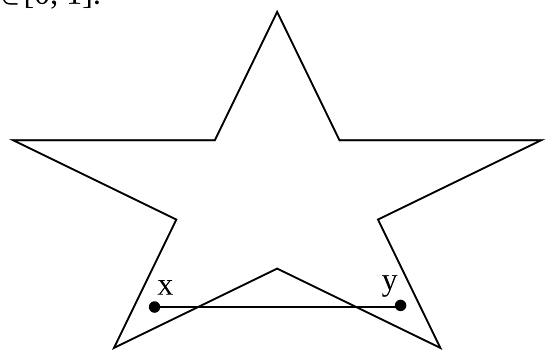


Figure 4: Non-convex set.

Obviously, the empty set is convex and a set which is starshaped at every point is convex as well.

Lemma Let S and T be convex subsets of a real linear space X.

(a) The intersection of arbitrarily many convex sets is convex.

(b) If S and T are nonempty, then the algebraic sum $\alpha S + \beta T$ is convex for all $\alpha, \beta \in \mathbb{R}$. Consequently, for every $\bar{x} \in X$ the translated set $S + \{\bar{x}\}$ is convex as well.

Proof. (a) Let $S_1, S_2, ..., S_n$ be convex sets.

Take any $x, y \in \bigcap_{i=1}^{n} S_i$ and $\lambda \in [0,1]$.

So, $x, y \in S_i$ for all $i = 1, 2, ..., n$.

Since S_i is convex, $\lambda x + (1-\lambda) y \in S_i$ for all $i = 1, 2, ..., n$.

Hence, $\lambda x + (1-\lambda) y \in \bigcap_{i=1}^{n} S_i$ and $\bigcap_{i=1}^{n} S_i$ is convex.

Therefore, the intersection of arbitrarily many convex sets is convex.

(b) Take any $x, y \in \alpha S + \beta T$ and $\lambda \in [0,1]$.

Then there exist $s_1, s_2 \in S$ and $t_1, t_2 \in T$ such that $x = \alpha s_1 + \beta t_1$ and $y = \alpha s_2 + \beta t_2$.

$$\begin{aligned} \lambda x + (1-\lambda) y &= \lambda (\alpha s_1 + \beta t_1) + (1-\lambda)(\alpha s_2 + \beta t_2) \\ &= \alpha (\lambda s_1 + (1-\lambda) s_2) + \beta (\lambda t_1 + (1-\lambda) t_2) \end{aligned}$$

Since $\lambda s_1 + (1-\lambda) s_2 \in S$ and $\lambda t_1 + (1-\lambda) t_2 \in T$, $\lambda x + (1-\lambda) y \in \alpha S + \beta T$.

Therefore, $\alpha S+\beta T$ is convex.

For every $\bar{x} \in X$.

Take any $x, y \in S+\{\bar{x}\}$ and $\lambda \in [0,1]$.

Then there exist $s_1, s_2 \in S$ such that $x = s_1+\bar{x}$ and $y = s_2+\bar{x}$.

$$\begin{aligned}\lambda x+(1-\lambda) y &= \lambda (s_1+\bar{x})+(1-\lambda)(s_2+\bar{x}) \\ &= \lambda s_1+(1-\lambda) s_2+\bar{x}\end{aligned}$$

Since $\lambda s_1+(1-\lambda) s_2 \in S$, $\lambda x+(1-\lambda) y \in S+\{\bar{x}\}$.

Therefore, $S+\{\bar{x}\}$ is convex. □

In the sequel, we also define cones generated by sets.

Definition Let S be a nonempty subset of a real linear space. The set

$$\text{cone}(S) = \{\lambda s \mid \lambda \geq 0 \text{ and } s \in S\}$$

is called the **cone generated** by S.

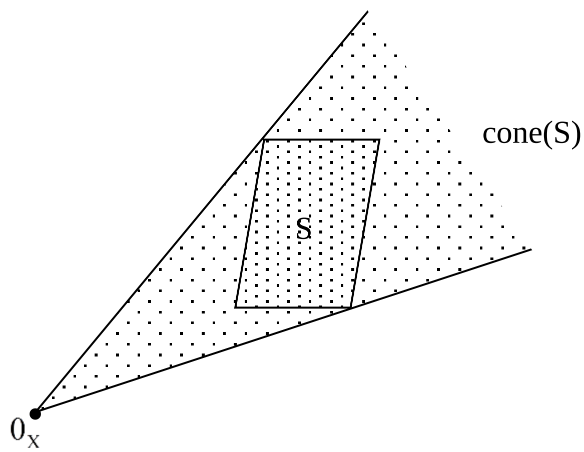


Figure 5: Cone generated by S.

Example (a) Let $B(0_X,1)$ denote the closed unit ball in a real normed space $(X,\|.\|)$.

Then the cone generated by $B(0_X,1)$ equals the linear space X.

(b) Let S denote the graph of the function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{Bmatrix} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{Bmatrix}.$$

Then the cone generated by S is given as $\text{cone}(S) = \{(x, y) \in \mathbb{R}^2 \mid |y| \leq |x|\}$.

Now we turn our attention to tangent cones.

Definition Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$.

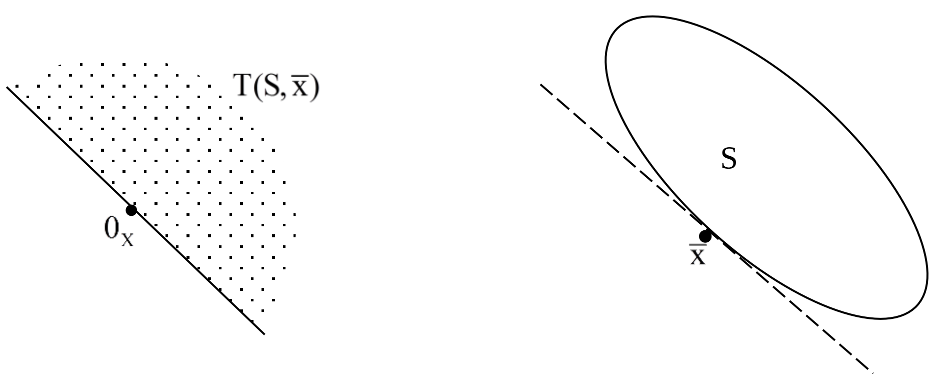
(a) Let $\bar{x} \in \text{cl}(S)$ be a given element. A vector $h \in X$ is called a **tangent vector** to S at $\bar{x}$, if there are a sequence $(x_n)_{n \in N}$ of elements in S and a sequence $(\lambda_n)_{n \in N}$ of positive real numbers with

$$\bar{x} = \lim_{n \to \infty} x_n$$

and

$$h = \lim_{n \to \infty} \lambda_n (x_n - \bar{x}).$$

(b) The set $T(S, \bar{x})$ of all tangent vectors to S at $\bar{x}$ is called **sequential Bouligand tangent cone** to S at $\bar{x}$ or **contingent cone** to S at $\bar{x}$.



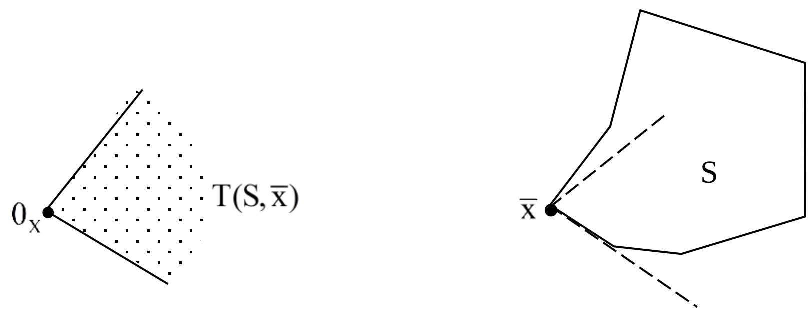


Figure 6: Two examples of contingent cones.

Notice that $\bar{x}$ needs only to belong to the closure of the set S in the definition of $T(S,\bar{x})$. But later we will assume that $\bar{x} \in S$.

By the definition of tangent vectors it follows immediately that the contingent cone is in fact a cone.

Before investigating the contingent cone we briefly present the definition of the Clarke tangent cone which is not used any further in this paper.

Remark Let $\bar{x}$ be an element of the closure of a nonempty subset S of a real normed space $\left(X,\|\cdot\|\right)$.

(a) The set

$$\begin{aligned}T_{Cl}(S,\bar{x}) = \{h \in X \,|\, &\text{for every sequence } (x_n)_{n\in N} \text{ of elements of S with } \bar{x} = \lim_{n\to\infty} x_n \\ &\text{and for every sequence } (\lambda_n)_{n\in N} \text{ of positive real numbers converging} \\ &\text{to 0 there is a sequence } (h_n)_{n\in N} \text{ with } h = \lim_{n\to\infty} h_n \text{ and } x_n + \lambda_n h_n \in S \\ &\text{for all } n \in \mathbb{N}\}\end{aligned}$$

is called (**sequential**) **Clarke tangent cone** to S at $\bar{x}$.

(b) It is evident that the Clarke tangent cone $T_{Cl}(S,\bar{x})$ is always a cone.

(c) If $\bar{x} \in S$, then the Clarke tangent cone $T_{Cl}(S,\bar{x})$ is contained in the contingent cone $T(S,\bar{x})$.

For the proof of this assertion let some $h \in T_{Cl}(S,\bar{x})$ be given arbitrarily. Then we choose the special sequence $(\bar{x})_{n\in N}$ and an arbitrary sequence $(\lambda_n)_{n\in N}$ of positive real numbers converging to 0. Consequently, there is a sequence $(h_n)_{n\in N}$ with $h = \lim_{n\to\infty} h_n$ and $\bar{x} + \lambda_n h_n \in S$ for all $n \in \mathbb{N}$. Now we set

$$y_n = \bar{x} + \lambda_n h_n \text{ for all } n \in \mathbb{N}$$

and

$$t_n = \frac{1}{\lambda_n} \text{ for all } n \in \mathbb{N}.$$

Then it follows

$$y_n \in S \text{ for all } n \in \mathbb{N},$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(\bar{x} + \lambda_n h_n) = \bar{x}$$

and

$$\lim_{n\to\infty} t_n(y_n - \bar{x}) = \lim_{n\to\infty} \frac{1}{\lambda_n}(\bar{x} + \lambda_n h_n - \bar{x}) = \lim_{n\to\infty} h_n = h.$$

Consequently, h is a tangent vector.

(d) The Clarke tangent cone $T_{Cl}(S,\bar{x})$ is always a closed convex cone. We mention this result without proof. Notice that this assertion is true without any assumption on the set S.

Next, we come back to the contingent cone and we investigate the relationship between the contingent cone $T(S,\bar{x})$ and the cone generated by $S - \{\bar{x}\}$.

Theorem Let S be a nonempty subset of a real normed space. If S is starshaped with respect to some $\bar{x} \in S$, then it follows $\text{cone}\left(S - \{\bar{x}\}\right) \subset T(S,\bar{x})$.

Proof. Let the set S be starshaped with respect to some $\bar{x} \in S$, and let an arbitrary element $x \in S$ be given. Then we define a sequence $(x_n)_{n\in N}$ with

$$x_n = \bar{x} + \frac{1}{n}(x - \bar{x}) = \frac{1}{n}x + (1 - \frac{1}{n})\bar{x} \in S \text{ for all } n \in \mathbb{N}.$$

For this sequence we have

$$\lim_{n\to\infty} x_n = \bar{x}$$

and

$$\lim_{n\to\infty} n(x_n - \bar{x}) = x - \bar{x}.$$

Consequently, $x - \bar{x}$ is a tangent vector, and we obtain

$$S - \{\bar{x}\} \subset T(S, \bar{x}).$$

Since $T(S, \bar{x})$ is a cone, we conclude

$$\operatorname{cone}\left(S - \{\bar{x}\}\right) \subset \operatorname{cone}(T(S, \bar{x})) = T(S, \bar{x}). \qquad \square$$

Theorem Let S be a nonempty subset of a real normed space. For every $\bar{x} \in S$ it follows $T(S, \bar{x}) \subset \operatorname{cl}\left(\operatorname{cone}\left(S - \{\bar{x}\}\right)\right)$.

Proof. We fix an arbitrary $\bar{x} \in S$ and we choose any $h \in T(S, \bar{x})$.

Then there are a sequence $(x_n)_{n\in N}$ of elements in S and a sequence $(\lambda_n)_{n\in N}$ of positive real numbers with $\bar{x} = \lim_{n\to\infty} x_n$ and $h = \lim_{n\to\infty} \lambda_n (x_n - \bar{x})$. The last equation implies

$$h \in \operatorname{cl}\left(\operatorname{cone}\left(S - \{\bar{x}\}\right)\right)$$

which has to be shown. $\square$

By the two preceding theorems we obtain the following inclusion chain for a set S which is starshaped with respect to some $\bar{x} \in S$:

$$\operatorname{cone}\left(S - \{\bar{x}\}\right) \subset T(S, \bar{x}) \subset \operatorname{cl}\left(\operatorname{cone}\left(S - \{\bar{x}\}\right)\right). \tag{2}$$

The next theorem says that the contingent cone is always closed.

Theorem** Let S be a nonempty subset of a real normed space $\left(X, \|.\|\right)$. For every $\bar{x} \in S$ the contingent cone $T(S, \bar{x})$ is closed.

Proof. Let $\bar{x} \in S$ be arbitrarily chosen, and let $(h_n)_{n \in N}$ be an arbitrary sequence of tangent vectors to S at $\bar{x}$ with $\lim\limits_{n \to \infty} h_n = h \in X.$

For every tangent vector h_n there are a sequence $(x_{n_i})_{i \in N}$ of elements in S and a sequence $(\lambda_{n_i})_{i \in N}$ of positive real numbers with $\bar{x} = \lim\limits_{i \to \infty} x_{n_i}$ and $h_n = \lim\limits_{i \to \infty} \lambda_{n_i}(x_{n_i} - \bar{x}).$

Consequently, for every $n \in \mathbb{N}$ there is a number $i(n) \in \mathbb{N}$ with

$$\|x_{n_i} - \bar{x}\| \le \frac{1}{n} \text{ for all } i \in \mathbb{N} \text{ with } i \ge i(n)$$

and

$$\|\lambda_{n_i}(x_{n_i} - \bar{x}) - h_n\| \le \frac{1}{n} \text{ for all } i \in \mathbb{N} \text{ with } i \ge i(n).$$

If we define the sequences $(y_n)_{n \in N}$ and $(t_n)_{n \in N}$ by

$$y_n = x_{n_{i(n)}} \in S \text{ for all } n \in \mathbb{N}$$

and

$$t_n = \lambda_{n_{i(n)}} > 0 \text{ for all } n \in \mathbb{N},$$

then we obtain $\lim\limits_{n \to \infty} y_n = \bar{x}$ and

$$\begin{aligned} \|t_n(y_n - \bar{x}) - h\| &= \|\lambda_{n_{i(n)}}(x_{n_{i(n)}} - \bar{x}) - h_n + h_n - h\| \\ &\le \frac{1}{n} + \|h_n - h\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence we have

$$h = \lim_{n \to \infty} t_n(y_n - \bar{x})$$

and h is a tangent vector to S at $\bar{x}$. $\square$

Since the inclusion chain (2) is also valid for the corresponding closed sets, it follows immediately with the aid of Theorem**:

Corollary* Let S be a nonempty subset of a real normed space. If the set S is starshaped with respect to some $\bar{x} \in S$, then it is $T(S, \bar{x}) = \mathrm{cl}\left(\mathrm{cone}\left(S - \{\bar{x}\}\right)\right).$

If the set S is starshaped with respect to some $\bar{x} \in S$, then Corollary* says essentially that for the determination of the contingent cone to S at $\bar{x}$ we have to consider only rays emanating from $\bar{x}$ and passing through S.

Finally, we show that the contingent cone to a nonempty convex set is also convex.

Theorem If S is a nonempty convex subset of a real normed space $(X, \|.\|)$, then the contingent cone $T(S, \bar{x})$ is convex for all $\bar{x} \in S$.

Proof. We choose an arbitrary $\bar{x} \in S$ and we fix two arbitrary tangent vectors $h_1, h_2 \in T(S, \bar{x})$ with $h_1, h_2 \neq 0_X$.

Then there are sequences $(x_n)_{n\in N}, (y_n)_{n\in N}$ of elements in S and sequences $(\lambda_n)_{n\in N}$, $(\mu_n)_{n\in N}$ of positive real numbers with

$$\bar{x} = \lim_{n\to\infty} x_n,\ h_1 = \lim_{n\to\infty} \lambda_n (x_n - \bar{x})$$

and

$$\bar{x} = \lim_{n\to\infty} y_n,\ h_2 = \lim_{n\to\infty} \mu_n (y_n - \bar{x}).$$

Next, we define additional sequences $(\nu_n)_{n\in N}$ and $(z_n)_{n\in N}$ with

$$\nu_n = \lambda_n + \mu_n \text{ for all } n \in \mathbb{N}$$

and

$$z_n = \frac{1}{\nu_n}(\lambda_n x_n + \mu_n y_n) \text{ for all } n \in \mathbb{N}.$$

Because of the convexity of S we have

$$z_n = \frac{\lambda_n}{\lambda_n + \mu_n} x_n + \frac{\mu_n}{\lambda_n + \mu_n} y_n \in S \text{ for all } n \in \mathbb{N},$$

and we conclude

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{1}{\nu_n}(\lambda_n x_n + \mu_n y_n)$$

$$= \lim_{n\to\infty} \frac{1}{\nu_n}(\lambda_n x_n - \lambda_n \bar{x} + \mu_n y_n - \mu_n \bar{x} + \lambda_n \bar{x} + \mu_n \bar{x})$$

$$= \lim_{n\to\infty}\left(\frac{\lambda_n}{\nu_n}(x_n - \bar{x}) + \frac{\mu_n}{\nu_n}(y_n - \bar{x}) + \bar{x}\right)$$

$$= \bar{x}$$

and

$$\lim_{n\to\infty} \nu_n (z_n - \bar{x}) = \lim_{n\to\infty}(\lambda_n x_n + \mu_n y_n - \nu_n \bar{x})$$

$$= \lim_{n\to\infty}(\lambda_n (x_n - \bar{x}) + \mu_n (y_n - \bar{x}))$$

$$= h_1 + h_2.$$

Hence it follows $h_1 + h_2 \in T(S,\bar{x})$. Since $T(S,\bar{x})$ is a cone, Theorem* leads to the assertion. □

Notice that the Clarke tangent cone to an arbitrary nonempty set S is already a convex cone, while we have shown the convexity of the contingent cone only under the assumption of the convexity of S.

Acknowledgements

I would like to express my heartfelt thanks to Rector Dr Khin Maung Oo and also thanks to Pro-Rector Dr Aung Aung Min, Yadanabon University, for their kind permission to make this research. I am also grateful to Dr Ye` Shwe, Professor and Head, Department of Mathematics, Yadanabon University for his suggestions to conduct this research. Finally, my thanks go to my colleagues for their valuable advices.

References

Jahn, J. (1985). “Scalarization in multiobjective optimization in Mathematics of Multiobjectuve Optimization”, P. Serafini (ed.), Springer-Verlag, Berlin, Germany.

Jahn, J. (1986). “Mathematical Vector Optimization in Partially Ordered Linear Spaces”, Verlag Pete Lang, Frankfurt.

Jahn, J. (1994). “Introduction to the Theory of Nonlinear Optimization”, Springer-Verlag, Berlin, Germany.

Jahn, J. (2011). “Vector Optimization, Theory, Applications, and Extensions”, Second Edition, Springer -Verlag, Berlin, Germany.

Rockafellar, R.T. (1970). “Convex Analysis”, Princeton University Press,Princeton.