Some Properties of Convex Set and Maps on Linear Spaces

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Abstract

This paper starts with definitions of convex, cone, pointed and some related properties in a linear space. Then we consider a partial ordering in such a linear setting and we investigate some special partially ordered linear spaces and list various known properties. Finally, we consider convex maps and their generalizations and also several types of differentials.

Keywords: Cone, Convex map, Concave map, Epigraph of map.

1. Linear Spaces and Convex Sets

1.1 Definition Let X be a given set. Assume that an addition on X, i.e., a map from $X \times X$ to X, and a scalar multiplication on X, i.e., a map from $\mathbb{R} \times X$ to X, is defined. The set X is called a **real linear space**, if the following axioms are satisfied (for arbitrary $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$):

(a) $(x+y)+z = x+(y+z)$,

(b) $x+y = y+x$,

(c) there is an element $0_X \in X$ with $x + 0_X = x$ for all $x \in X$,

(d) for every $x \in X$ there is a $y \in X$ with $x + y = 0_X$,

(e) $\lambda(x+y) = \lambda x + \lambda y$,

(f) $(\lambda + \mu)x = \lambda x + \mu x$,

(g) $\lambda(\mu x) = (\lambda\mu)x$,

(h) $1x = x$.

The element 0_X given under (c) is called the **zero element** of X.

1.2 Definition Let S and T be nonempty subsets of a real linear space X. Then we define the **algebraic sum** of S and T as

$$S + T := \{x + y \mid x \in S \text{ and } y \in T\}$$

and the **algebraic difference** of S and T as

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$$S - T := \{x - y \mid x \in S \text{ and } y \in T\}.$$

For an arbitrary $\lambda \in \mathbb{R}$ the notation λS will be used as

$$\lambda S := \{\lambda x \mid x \in S\}.$$

1.3 Definition Let X be a real linear space. The set X' is defined to be the set of all linear mappings from X to $\mathbb{R}$. If we define for all $\phi, \psi \in X'$ and all $\lambda \in \mathbb{R}$

$$(\phi + \psi)(x) = \phi(x) + \psi(x) \text{ for all } x \in X$$

and $$(\lambda\phi)(x) = \lambda\phi(x) \text{ for all } x \in X,$$

then X' is a real linear space itself and it is called the **algebraic dual space** of X. The algebraic dual space of X' is denoted by X'' and it is called the **second algebraic dual space** of X.

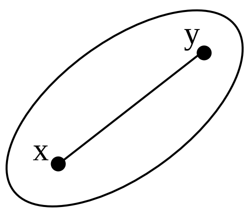
1.4 Definition Let S be a subset of a real linear space X.

(a) Let some $\bar{x} \in S$ be given. The set S is called **starshaped** at $\bar{x}$, if for every $x \in S$

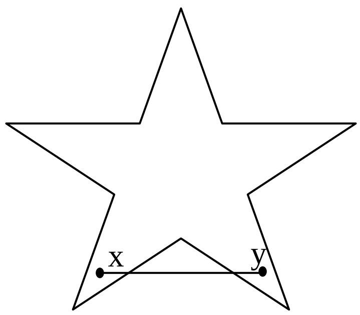
$$\lambda x + (1 - \lambda)\bar{x} \in S \quad \text{for all } \lambda \in [0,1].$$

(b) The set S is called **convex,** if for every x, y $\in$ S

$$\lambda x + (1 - \lambda) y \in S \quad \text{for all } \lambda \in [0, 1].$$



Convex set.



Non-convex set.

(c) The set S is called **balanced**, if it is nonempty and $\alpha S \subset S$ for all $\alpha \in [-1, 1]$.

(d) The set S is called **absolutely convex**, if it is convex and balanced.

Obviously, the empty set is convex and a set which is starshaped at every point is convex as well.

1.5 Remark

(a) The intersection of arbitrarily many convex sets of a real linear space is convex.

(b) If S and T are nonempty convex subsets of a real linear space X, then the algebraic sum $\alpha S + \beta T$ is convex for all $\alpha, \beta \in \mathbb{R}$. Consequently, for every $\bar{x} \in X$ the translated set $S + \{\bar{x}\}$ is convex as well.

1.6 Definition Let S be a nonempty subset of a real linear space X. The intersection of all convex subsets of X that contain S is called the **convex hull** of S and is denoted $\text{co}(S)$.

1.7 Remark For two nonempty subsets S and T of a real linear space we obtain for all $\alpha, \beta \in \mathbb{R}$

$$\text{co}(\alpha S + \beta T) = \alpha\, \text{co}(S) + \beta\, \text{co}(T).$$

1.8 Definition Let S be a nonempty subset of a real linear space X.

(a) The set

$$\text{cor}(S) := \{\bar{x} \in S \mid \text{for every } x \in X \text{ there is a } \bar{\lambda} > 0 \text{ with } \bar{x} + \lambda x \in S \text{ for all } \lambda \in [0, \bar{\lambda}]\}$$

is called the **algebraic interior** of S (or the **core** of S).

(b) The set S with $S = \text{cor}(S)$ is called **algebraically open**.

(c) The set of all elements of X which do not belong to $\text{cor}(S)$ and $\text{cor}(X \setminus S)$ is called the **algebraic boundary** of S.

(d) An element $\bar{x} \in X$ is called **linearly accessible** from S, if there is an $x \in S, x \neq \bar{x}$, with the property $\lambda x + (1 - \lambda)\bar{x} \in S$ for all $\lambda \in (0, 1]$.

The union of S and the set of all linearly accessible elements from S is called **the algebraic closure of S** and it is denoted by

$$\text{lin}(S) := S \cup \{x \in X \mid x \text{ is linearly accessible from } S\}.$$

In the case of $S = \text{lin}(S)$ the set S is called **algebraically closed.**

(e) The set S is called **algebraically bounded**, if for every $\bar{x} \in S$ and every $x \in X$ there is a $\bar{\lambda} > 0$ such that $\bar{x} + \lambda x \notin S$ for all $\lambda \geq \bar{\lambda}$.

These algebraic notions have a special geometric meaning. Take the intersections of the set S with each straight line in the real linear space X and consider these intersections as subsets of the real line $\mathbb{R}$. Then the set S is algebraically open, if these subsets are open; S is algebraically closed, if these subsets are closed; and S is algebraically bounded, if these subsets are bounded.

1.9 Lemma For a nonempty convex subset S of a real linear space we have:

(a) $\bar{x} \in \mathrm{cor}(S), \tilde{x} \in \mathrm{lin}(S) \Rightarrow \{\lambda \tilde{x} + (1-\lambda)\bar{x} \mid \lambda \in [0,1)\} \subset \mathrm{cor}(S)$,

(b) $\mathrm{cor}(\mathrm{cor}(S)) = \mathrm{cor}(S)$,

(c) $\mathrm{cor}(S)$ and $\mathrm{lin}(S)$ are convex,

(d) $\mathrm{cor}(S) \neq \emptyset \Rightarrow \mathrm{lin}(\mathrm{cor}(S)) = \mathrm{lin}(S)$ and $\mathrm{cor}(\mathrm{lin}(S)) = \mathrm{cor}(S)$.

Proof. See [3].

1.10 Definition Let C be a nonempty subset of a real linear space X.

(a) The set C is called a **cone**, if

$$x \in C, \lambda \geq 0 \Rightarrow \lambda x \in C.$$

(b) A cone C is called **pointed**, if

$$C \cap (-C) = \{0_X\}.$$

(c) A cone C is called **reproducing**, if

$$C - C = X.$$

(d) A nonempty convex subset B of a convex cone $C \neq \{0_X\}$ is called a **base** for C, if each $x \in C \setminus \{0_X\}$ has a unique representation of the form

$$x = \lambda b \text{ for some } \lambda > 0 \text{ and some } b \in B.$$

Sometimes a cone is also called a **wedge** and a pointed wedge is called a **cone**.

By definition each cone contains the zero element of the real linear space. The simplest cones in a real linear space X are $\{0_X\}$ and X itself. $\{0_X\}$ is also called the **trivial cone**. From a geometric point of view a nontrivial cone is a set of rays emanating from the origin. Consequently, each cone is starshaped at 0_X.

1.11 Lemma A cone D in a real linear space is convex if and only if

$$D + D \subset D.$$

Proof. Assume that D is a convex cone. Then for every x, y$\in$ D we have

$$\lambda x + (1-\lambda)\, y \in D \text{ for all } \lambda \in [0,1].$$

Choose $\lambda = \frac{1}{2}$. Therefore, $\frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}(x+y) \in D.$

Since D is a cone, we obtain $x + y \in D$ and hence D + D $\subset$ D.

For arbitrary x, y $\in$ D and $\lambda \in [0,1]$, we obtain

$$\lambda\, x \in D \text{ and } (1-\lambda)\, y \in D.$$

With the inclusion D + D $\subset$ D we then get

$$\lambda x + (1-\lambda)\, y \in D,$$

i.e., the cone D is convex.

1.12 Lemma Let C be a convex cone in a real linear space X with a nonempty algebraic interior. Then:

(a) $\text{cor}(C) \cup \{0_X\}$ is a convex cone,

(b) $\text{cor}(C) = C + \text{cor}(C)$.

Proof. (a) Take arbitrary $\bar{x} \in \text{cor}(C)$ and $\mu > 0$.

For every $x \in X$ there is a $\bar{\lambda} > 0$ with $\bar{x} + \frac{\lambda}{\mu} x \in C$ for all $\lambda \in [0, \bar{\lambda}]$.

Since C is a cone, we get $\mu\,(\bar{x} + \frac{\lambda}{\mu} x) = \mu\,\bar{x} + \lambda\, x \in C$ for all $\lambda \in [0, \bar{\lambda}]$.

So, we obtain $\mu\,\bar{x} \in \text{cor}(C)$ and with Lemma 1.9(c) the assertion is obvious.

(b) The inclusion $\text{cor}(C) = \{0_X\} + \text{cor}(C) \subset C + \text{cor}(C)$ is clear.

For the proof of the converse inclusion we take arbitrary $\tilde{x} \in C, \bar{x} \in \text{cor}(C)$ and $x \in X$.

Then there is a $\bar{\lambda} > 0$ with $\bar{x} + \lambda\, x \in C$ for all $\lambda \in [0, \bar{\lambda}]$.

Since C is assumed to be convex, we conclude with Lemma 1.11

$$\tilde{x} + \bar{x} + \lambda\, x \in C \text{ for all } \lambda \in [0, \bar{\lambda}]$$

implying $\tilde{x} + \bar{x} \in \text{cor}(C)$. So, we conclude $C + \text{cor}(C) \subset \text{cor}(C)$.

1.13 Lemma A cone C in a real linear space X is reproducing, if $\mathrm{cor}(C) \neq \emptyset$.

Proof. If $\mathrm{cor}(C)$ is nonempty, take some $\bar{x} \in \mathrm{cor}(C)$ and any $x \in X$.

Then there is a $\bar{\lambda} > 0$ with $\bar{x} + \bar{\lambda} x \in C$ implying

$$x \in \frac{1}{\bar{\lambda}} C - \left\{ \frac{1}{\bar{\lambda}} \bar{x} \right\} \subset C - C.$$

So, we get $X \subset C - C$ and together with the trivial inclusion $C - C \subset X$ we obtain the assertion.

1.14 Lemma Each nontrivial convex cone with a base in a real linear space is pointed.

Proof. Let C be a nontrivial convex cone with base B.

Take any $x \in C \cap (-C)$ and assume that $x \neq 0_X$.

Then there are $b_1, b_2 \in B$ and $\lambda_1, \lambda_2 > 0$ with $x = \lambda_1 b_1 = -\lambda_2 b_2$ implying

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} b_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} b_2 = 0_X \in B.$$

But this is a contradiction to the afore-mentioned remark.

1.15 Definition Let S be a nonempty subset of a real linear space. The cone

$$\mathrm{cone}(S) := \{x \in X \mid x = \lambda s \text{ for some } \lambda \geq 0 \text{ and some } s \in S\}$$

is called the **cone generated by S**.

2. Partially Ordered Linear Spaces

2.1 Definition Let X be a real linear space.

(a) Each nonempty subset R of the product space $X \times X$ is called a **binary relation** R on X (we write xRy for $(x, y) \in R$).

(b) Every binary relation $\leq$ on X is called a **partial ordering** on X, if the following axioms are satisfied (for arbitrary $w, x, y, z \in X$):

(i) $x \leq x$;

(ii) $x \leq y,\ y \leq z \Rightarrow x \leq z$;

(iii) $x \leq y,\ w \leq z \Rightarrow x + w \leq y + z$;

(iv) $x \leq y,\ \alpha \in \mathbb{R}_+ \Rightarrow \alpha x \leq \alpha y$.

(c) A partial ordering $\leq$ on X is called **antisymmetric** , if the following implication holds for arbitrary x, y $\in$ X:

$$x \leq y,\ y \leq x \Rightarrow x = y.$$

In Definition 2.1, (b) with axiom (i) the partial ordering is **reflexive** and with (ii) it is **transitive**. The axioms (iii) and (iv) guarantee the compatibility of the partial ordering with the **linear structure** of the space.

2.2 Definition A real linear space equipped with a partial ordering is called a **partially ordered linear space.**

2.3 Theorem Let X be a real linear space.

(a) If $\leq$ is a partial ordering on X, then the set

$$D := \{x \in X \mid 0_X \leq x\}$$

is a convex cone. If, in addition, $\leq$ is antisymmetric, then D is pointed.

(b) If D is a convex cone in X, then the binary relation

$$\leq_D := \{(x, y) \in X \times X \mid y - x \in D\}$$

is a partial ordering on X. If, in addition, D is pointed, then $\leq_D$ is antisymmetric.

Proof. (a) Suppose $\leq$ is a partial ordering on X.

Take any $x \in D$ and $\lambda \geq 0$. So, $0_X \leq x$.

Since $\leq$ is a partial ordering on X, $0_X \leq \lambda x$. Therefore, D is a cone.

Then take any $x, y \in D$ and $\lambda \in [0,1]$.

Since D is a cone, $\lambda x \in D$ and $(1-\lambda) y \in D$. So, $0_X \leq \lambda x$ and $0_X \leq (1-\lambda) y$.

Since $\leq$ is a partial ordering on X, $0_X \leq \lambda x + (1-\lambda) y$. Therefore, D is a convex.

Suppose $\leq$ is antisymmetric.

Take any $x \in D \cap (-D)$. So, $x \in D$ and $-x \in D, 0_X \leq x$ and $0_X \leq -x$. Then $x \leq 0_X$.

Since $\leq$ is antisymmetric, $x = 0$.

Therefore, D is pointed.

(b) Suppose D is a convex cone in X and x, y, z $\in$ X.

Let $\leq_D := \{(x, y) \in X \times X \mid y - x \in D\}$ be a binary relation.

Since $x - x = 0 \in D$, we get $x \leq_D x$. Therefore, the relation is reflexive.

Suppose $x \leq_D y$ and $y \leq_D z$. Therefore, $y - x \in D$ and $z - y \in D$.

Then there exist $d_1 \in D$ such that $y - x = d_1$ and $d_2 \in D$ such that $z - y = d_2$.

Since D is convex, by Lemma 1.11, we obtain $x \leq_D z$ and hence the relation is transitive.

Suppose $x \leq_D y$ and $w \leq_D z$. Therefore, $y - x \in D$ and $z - w \in D$.

Then there exist $d_1 \in D$ such that $y - x = d_1$ and $d_2 \in D$ such that $z - w = d_2$.

Since D is convex, by Lemma 1.11, we obtain $x + w \leq_D y + z$.

Suppose $x \leq_D y$ and $\alpha \in \mathbb{R}_+$. Therefore, $y - x \in D$.

Since D is a cone, we get $\alpha(y - x) \in D$ which implies $\alpha x \leq_D \alpha y$.

Therefore, $\leq_D$ is a partial ordering on X.

In addition, suppose D is pointed. Take any $x \leq_D y$ and $y \leq_D x$.

Therefore, $y - x \in D$ and $x - y \in D$. This implies $y - x \in D$ and $y - x \in (-D)$ and hence $y - x \in D \cap (-D)$. Since D is pointed, $y - x = 0_X$ and we obtain x = y.

Therefore, $\leq_D$ is antisymmetric.

2.4 Definition A convex cone characterizing a partial ordering in a real linear space is called an **ordering cone.**

Several authors also call an ordering cone a **positive cone**. We denote $\leq_C$ as a partial ordering induced by a convex cone C.

2.5 Definition Let X be a partially ordered linear space. For arbitrary elements $x, y \in X$ with $x \leq y$ the set

$$[x, y] := \{z \in X \mid x \leq z \leq y\}$$

is called the **order interval between x and y**.

If C is the ordering cone in a partially ordered linear space, then the order interval between x and y can be written as

$$[x, y] = (\{x\} + C) \cap (\{y\} - C).$$

2.6 Lemma Let X be a partially ordered linear space with the ordering cone C. Let $x, y \in X$ with $x \in \{y\} - C$ (i.e., $x \leq_C y$) be arbitrarily given. Then we have for $z := \frac{1}{2}(x + y)$:

(a) The order interval $[x - z, y - z]$ is absolutely convex.

(b) If $\mathrm{cor}(C) \neq \emptyset$ and $x \in \{y\} - \mathrm{cor}(C)$, then $z \in \mathrm{cor}([x, y])$.

(c) If C is algebraically closed, then [x, y] is algebraically closed.

(d) If C is algebraically closed and pointed, then [x, y] is algebraically bounded.

Proof. (a) With the equality

$$[x - z, y - z] = \left[-\frac{1}{2}(y - x), \frac{1}{2}(y - x)\right]$$

the assertion is obvious.

(b) Since

$$z = x + \frac{1}{2}(y - x) \in \{x\} + \mathrm{cor}(C)$$

and
$$z = y - \frac{1}{2}(y - x) \in \{y\} - \mathrm{cor}(C),$$

we conclude $z \in \mathrm{cor}([x, y])$.

(c) Because of the equality $[x, y] = (\{x\} + C) \cap (\{y\} - C)$ this assertion is evident.

(d) First, if the pointed convex cone C is algebraically closed, then the complement set $X \setminus C$ is algebraically open.

For if we assume that $X \setminus C$ is not algebraically open, then there is an $\bar{x} \in X \setminus C$ and an $h \in X$ so that for all $\bar{\lambda} > 0$

$$\bar{x} + \lambda h \in C \text{ for some } \lambda \in (0, \bar{\lambda}].$$

Since C is convex, we conclude for some $x := \bar{x} + \lambda h \in C$

$$\mu x + (1 - \mu)\bar{x} \in C \text{ for all } \mu \in (0,1]$$

which implies $\bar{x} \in \mathrm{lin}(C) = C$. But this contradicts the assumption $\bar{x} \notin C$.

So, the complement set $X \setminus C$ is algebraically open.

In order to prove that [x,y] is algebraically bounded we take any $v \in [x, y]$ and any $w \in X \setminus \{0_X\}$.

Then we consider the two cases $w \notin C$ and $w \in C$.

Assume that $w \notin C$. Since $X \setminus C$ is algebraically open, there is a $\bar{\lambda} > 0$ with

$$w + \lambda(v - x) \in X \setminus C \text{ for all } \lambda \in [0, \bar{\lambda}].$$

The set $(X \setminus C) \cup \{0_X\}$ is a cone and, therefore, we obtain

$$\frac{1}{\lambda}(w + \lambda(v - x)) \in X \setminus C \text{ for all } \lambda \in (0, \bar{\lambda}]$$

or alternatively

$$\lambda(w + \frac{1}{\lambda}(v - x)) \in X \setminus C \text{ for all } \lambda \in [\frac{1}{\bar{\lambda}}, \infty).$$

But then we have

$$v - x + \lambda w \in X \setminus C \text{ for all } \lambda \in [\frac{1}{\bar{\lambda}}, \infty)$$

and $$v + \lambda w \notin \{x\} + C \text{ for all } \lambda \in [\frac{1}{\bar{\lambda}}, \infty)$$

which implies $v + \lambda w \notin [x, y]$ for all $\lambda \in [\frac{1}{\bar{\lambda}}, \infty)$.

Next, assume that $w \in C$. Since the ordering cone C is assumed to be pointed and $w \neq 0_X$, we conclude $w \notin -C$.

With the same arguments as before there is a $\bar{\bar{\lambda}} > 0$ with

$$v + \lambda w \notin [x, y] \text{ for all } \lambda \in [\frac{1}{\bar{\bar{\lambda}}}, \infty).$$

Hence, the order interval [x, y] is algebraically bounded.

2.7 Definition Let X be a real linear space with a convex cone C_X.

(a) The cone $C_{X'} := \{x' \in X' \mid x'(x) \geq 0 \text{ for all } x \in C_X\}$ is called the **dual cone** for C_X. The partial ordering in X' which is induced by $C_{X'}$ is called the **dual partial ordering**.

(b) The set $C^{\#}_{X'} := \{x' \in X' \mid x'(x) > 0 \text{ for all } x \in C_X \setminus \{0_X\}\}$ is called the **quasi-interior** of the dual cone for C_X.

Notice that $C_{X'}$ is a convex cone so that Definition 2.7, (a) makes sense. For $C_X = \{0_X\}$ we obtain $C_{X'} = X'$, and for $C_X = X$ we have $C_{X'} = \{0_{X'}\}$. If the quasi-interior $C^{\#}_{X'}$ of the dual cone for C_X is nonempty, then $C^{\#}_{X'} \cup \{0_{X'}\}$ is a nontrivial convex cone. With the following lemma we list some useful properties of dual cones without proof.

2.8 Lemma Let C_X and D_X be two convex cones in a real linear space X with the dual cone $C_{X'}$ and $D_{X'}$, respectively. Then:

(a) $C_X \subset D_X \Rightarrow D_{X'} \subset C_{X'}$;

(b) $C_{X'} \cap D_{X'}$ is the dual cone for $C_X + D_X$;

(c) $C_X \cup D_X$ and $C_X + D_X$ have the same dual cone;

(d) $C_{X'} + D_{X'}$ is a subset of the dual cone for $C_X \cap D_X$.

In general, the quasi-interior of the dual cone does not coincide with the algebraic interior of the dual cone but the following inclusion holds.

2.9 Lemma If C_X is a convex cone in a real linear space X and X' separates elements in X (i.e., two different elements in X may be separated by an hyperplane), then $\operatorname{cor}(C_{X'}) \subset C^{\#}_{X'}$.

Proof. The assertion is trivial for $C_X = \{0_X\}$ and for $\operatorname{cor}(C_{X'}) = \emptyset$.

If $C_X \neq \{0_X\}$ and $\operatorname{cor}(C_{X'}) = \emptyset$, then take any $\bar{x} \in \operatorname{cor}(C_{X'})$ and assume that $\bar{x} \notin C^{\#}_{X'}$.

Consequently, there is an $x \in C_X \setminus \{0_X\}$ with $\bar{x}(x) = 0$.

Since X' separates elements in X, there is a linear functional $x' \in X'$ with the property $x'(x) < 0$.

Then we conclude $(\lambda x' + (1-\lambda)\bar{x})(x) < 0$ for all $\lambda > 0$ which contradicts the assumption that $\bar{x} \in \operatorname{cor}(C_{X'})$.

2.10 Lemma If C_X is a convex cone in a real linear space X, then

$$\mathrm{cor}(C_X) \subset \{x \in X \mid x'(x) > 0 \text{ for all } x' \in C_{X'} \setminus \{0_{X'}\}\}.$$

Proof. Take any $\bar{x} \in \mathrm{cor}(C_X)$ and any $x' \in C_{X'} \setminus \{0_{X'}\}$.

Consequently, there are an $x \in X$ with $x'(x) < 0$ and a $\bar{\lambda} > 0$ with $\bar{x} + \bar{\lambda} x \in C_X$.

Hence, we obtain $x'(\bar{x} + \bar{\lambda} x) \geq 0$ and $x'(\bar{x}) \geq -\bar{\lambda} x'(x) > 0$ which leads to the assertion.

2.11 Lemma Let C_X be a convex cone in a real linear space X.

(a) If $\mathrm{cor}(C_X)$ is nonempty, then $C_{X'}$ is pointed.

(b) If $C^{\#}_{X'}$ is nonempty, then C_X is pointed.

Proof. (a) For every $x' \in C_{X'} \cap (-C_{X'})$ we have

$$x'(x) = 0 \text{ for all } x \in C_X$$

and especially for some $\bar{x} \in \mathrm{cor}(C_X)$ we get $x'(\bar{x}) = 0$.

With Lemma 2.10 we obtain $x' = 0_{X'}$, and this implies

$$C_{X'} \cap (-C_{X'}) = \{0_{X'}\}.$$

(b) Take any $x \in C_X \cap (-C_X)$. If we assume that $x \neq 0_{X'}$, we obtain for every $x' \in C^{\#}_{X'}$

$$x'(x) > 0 \text{ and } x'(x) < 0$$

which is a contradiction.

2.12 Lemma Let C_X be a nontrivial convex cone in a real linear space X.

(a) For every $x' \in C^{\#}_{X'}$ the set $B := \{x \in C_X \mid x'(x) = 1\}$ is a base for C_X.

(b) In addition, let C_X be reproducing and let C_X have a base. Then there is an

$x' \in C^{\#}_{X'}$ with $B = \{x \in C_X \mid x'(x) = 1\}$.

Proof. (a) Choose any $x' \in C^{\#}_{X'}$.

Then we obtain for every $x \in C_X \setminus \{0_X\}$, $x'(x) > 0$ and, therefore, x can be uniquely represented as

$$x = x'(x)\frac{1}{x'(x)}x \quad \text{for } \frac{1}{x'(x)}x \in B.$$

Hence, the assertion is evident.

(b) We define the functional $x' : C_X \setminus \{0_X\} \to \mathbb{R}_+$ with

$x'(x) = \lambda(x)$ for all $x \in C_X \setminus \{0_X\}$ where $\lambda(x)$ is the positive number in the representation formula for x.

It is obvious that x' is positively homogeneous.

In order to see that it is additive pick some elements $x, y \in C_X \setminus \{0_X\}$.

Then we obtain

$$\frac{1}{x'(x)+x'(y)}(x+y) = \frac{x'(x)}{x'(x)+x'(y)}\frac{1}{x'(x)}x + \frac{x'(y)}{x'(x)+x'(y)}\frac{1}{x'(y)}y \in B$$

because $\frac{1}{x'(x)}x \in B$, $\frac{1}{x'(y)}y \in B$ and B is convex.

Consequently, we get

$$x'(x+y) = x'(x) + x'(y) \text{ for all } x, y \in C_X \setminus \{0_X\}.$$

Hence, x' is a positively homogeneous and additive functional on $C_X \setminus \{0_X\}$.

Next, we define $x'(0_X) := 0$ and we see that this extension is positively homogeneous and additive on C_X as well.

Finally we extend x' to $X = C_X - C_X$ by defining

$$x'(x-y) := x'(x) - x'(y) \text{ for all } x, y \in C_X.$$

It is obvious that x' is positively homogeneous and additive on X, and since

$$x'(x-y) = x'(x) - x'(y) = -x'(y-x) \text{ for all } x, y \in C_X,$$

x' is also linear on X. With

$$x'(x) > 0 \text{ for all } x \in C_X \setminus \{0_X\},$$

we obtain $x' \in C^{\#}_{X'}$. The set equation

$$B = \{ x \in C_X \mid x'(x) = 1\}$$

is evident, if we use the definition of x'.

3. Convex Maps

The importance of convex maps is based on the fact that the image set of such a map has useful properties. One of these properties is also valid for so-called convex-like maps which are investigated in this section as well.

3.1 Definition Let X and Y be real linear spaces. A map $T: X \to Y$ is called **linear**, if for all $x, y \in X$ and all $\lambda, \mu \in \mathbb{R}$

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y).$$

The set of continuous (bounded) linear maps between two real normed spaces $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ is a linear space as well and it is denoted $B(X,Y)$. With the norm $\|\,.\,\|: B(X,Y) \to \mathbb{R}$ given by

$$\|T\| = \sup_{x \neq 0_X} \frac{\|T(x)\|_Y}{\|x\|_X} \text{ for all } T \in B(X,Y)$$

$\left(B(X,Y), \|\,.\,\|\right)$ is even a normed space.

3.2 Definition Let X and Y be real separated locally convex linear spaces, and let $T: X \to Y$ be a linear map. A map $T^*: Y^* \to X^*$ given by

$$T^*(y^*)(x) = y^*(T(x)) \text{ for all } x \in X \text{ and all } y^* \in Y^*$$

is called the **adjoint** (or **conjugate** and **dual**, respectively) of T.

3.3 Theorem Let X and Y be real separated locally convex linear spaces, and let the elements $x \in X, x^* \in X^*, y \in Y$ and $y^* \in Y^*$ be given.

(a) If there is a linear map $T: X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$, then $y^*(y) = x^*(x)$.

(b) If $x \neq 0_X, y^* \neq 0_{Y^*}$ and $y^*(y) = x^*(x)$, then there is a continuous linear map $T: X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$.

Proof. (a) let a linear map $T: X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$ be given. Then we get

$$y^*(y) = y^*(T(x)) = T^*(y^*)(x) = x^*(x)$$

which completes the proof.

(b) Assume that for $x \neq 0_X$ and $y^* \neq 0_{Y^*}$ the functional equation

$$y^*(y) = x^*(x). \tag{1}$$

is satisfied. In the following we consider the two cases $x^*(x) \neq 0$ and $x^*(x) = 0$.

(i) First assume that $x^*(x) \neq 0$. Then we define a map $T: X \to Y$ by

$$T(z) = \frac{x^*(z)}{x^*(x)} y \text{ for all } z \in X. \tag{2}$$

Evidently, T is linear and continuous. From (1) and (2) we conclude $T(x) = y$ and

$$y^*(T(z)) = \frac{x^*(z)}{x^*(x)} y^*(y) = x^*(z) \text{ for all } z \in X$$

which means $x^* = T^*(y^*)$.

(ii) Now assume that $x^*(x) = 0$. Because of $y^* \neq 0_{Y^*}$ there is a $\tilde{y} \neq 0_Y$ with $y^*(\tilde{y}) = 1$.

Since in a separated locally convex space X^* separates elements of X, $x \neq 0_X$ implies the existence of some $\tilde{x}^* \in X^*$ with $\tilde{x}^*(x) = 1$.

Then we define the map $T: X \to Y$ as follows

$$T(z) = x^*(z)\tilde{y} + \tilde{x}^*(z)y \text{ for all } z \in X. \tag{3}$$

It is obvious that T is a continuous linear map. With (3) we conclude

$$T(x) = x^*(x)\tilde{y} + \tilde{x}^*(x)y = y.$$

Furthermore, we obtain with (3) and (1)

$$y^*(T(z)) = x^*(z)y^*(\tilde{y}) + \tilde{x}^*(z)y^*(y) = x^*(z) \text{ for all } z \in X$$

which implies $x^* = T^*(y^*)$.

3.4 Definition Let X and Y be real linear spaces, C_Y be a convex cone in Y, and let S be a nonempty convex subset of X. A map $f: S \to Y$ is called **convex** (or C_Y**-convex**), if for all $x, y \in S$ and all $\lambda \in [0,1]$

$$\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) \in C_Y. \quad (4)$$

A map $f: S \to Y$ is called **concave** (or C_Y**-concave**), if $-f$ is convex.

If $\leq_{C_Y}$ is the partial ordering in Y induced by C_Y, then the condition (4) can also be written as

$$f(\lambda x + (1-\lambda) y) \leq_{C_Y} \lambda f(x) + (1-\lambda) f(y).$$

If f is a linear map, then f and $-f$ are convex maps.

3.5 Definition Let X and Y be real linear spaces, let C_Y be a convex cone in Y, let S be a nonempty subset of X, and let $f: S \to Y$ be a given map. The set

$$\operatorname{epi}(f) = \{(x,y) \mid x \in S, y \in \{(f(x)\} + C_Y\} \quad (5)$$

is called the **epigraph** of f.

Notice that the epigraph in (5) can also be written as

$$\operatorname{epi}(f) = \{(x,y) \mid x \in S, f(x) \leq_{C_Y} y\}.$$

It turns out that a convex map can be characterized by its epigraph.

3.6 Theorem Let X and Y be real linear spaces, let C_Y be a convex cone in Y, let S be a nonempty subset of X and let $f: S \to Y$ be a given map. Then f is convex if and only if $\operatorname{epi}(f)$ is a convex set.

Proof. (a) Let f be a convex map (then S is a convex set).

For arbitrary $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \operatorname{epi}(f)$ and $\lambda \in [0,1]$ we obtain

$\lambda x_1 + (1-\lambda) x_2 \in S$ and

$$\begin{aligned}\lambda y_1 + (1-\lambda) y_2 &\in \lambda\left(\{f(x_1)\} + C_Y\right) + (1-\lambda)\left(\{f(x_2)\} + C_Y\right) \\ &= \{\lambda f(x_1) + (1-\lambda) f(x_2)\} + C_Y \\ &\subset \{f(\lambda x_1 + (1-\lambda) x_2)\} + C_Y.\end{aligned}$$

Consequently, we have $\lambda z_1 + (1-\lambda) z_2 \in \operatorname{epi}(f)$. Thus, $\operatorname{epi}(f)$ is a convex set.

(b) If $epi(f)$ is a convex set, then S is convex as well.

For arbitrary $x_1, x_2 \in S$ and $\lambda \in [0,1]$ we obtain

$$\lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) \in epi(f)$$

and $$f(\lambda x_1 + (1-\lambda)x_2) \leq_{C_Y} \lambda f(x_1) + (1-\lambda) f(x_2).$$

Hence, f is a convex map.

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