Existence and Uniqueness of Solutions of The Quantum Hydrodynamic Model for Semiconductors

Dr. Aye Theingi, Department of Computational Mathematics, University of Computer Studies, Yangon, Myanmar
ayetheingi@gmail.com

Abstract

We consider the quantum hydro - dynamic model which describes tunneling of electrons through potential barriers in semiconductor devices. The existence and uniqueness of solutions of the thermal equilibrium states of the quantum hydro-dynamic model for semiconductors is studied in one space dimension.

1. Introduction

Nowadays, semiconductor materials are contained in almost all electronic devices, Depending on the device structure, the main transport phenomena of the electrons may be different. We can divide semiconductor models in three classes: quantum models, kinetic models and fluiddynamical (macroscopic) models.

We study the existence result of the thermal equilibrium states of the quantum hydro-dynamic model (QHD) for semi-conductors. The QHD describes electron transport in a semiconductor as a quantum field. It is an extension of the classical hydrodynamic model including quantum corrections. These quantum terms allow a description of quantum effects, for example, tunneling of electrons through potential barriers in semiconductor devices like a resonant tunneling diode.

The equations of the QHD in one space dimension read

(1) $$n_t + (nv)_x = 0$$

(2) $$(nv)_t + (nv^2)_x + \frac{k}{m}(nT(n))_x - \frac{\hbar^2}{4m^2}(n(\ln n)_{xx})_x - \frac{q}{m}nV_x = -\frac{nv}{\tau_p}$$

where we have assumed that the isentropic case , i.e., the temperature T is given as a function of the charge density n .The electric potential V is given by

(3) $$\varepsilon V_{xx} = q(n - C)$$

Where ε = dielectric constant

q = the electric charge

C = the velocity profile

k = the Boltzmann constant

τ_p = the momentum relaxation time

m = the effective electron mass

v = the velocity of the electron

$\hbar$ = the reduced Planck's constant

h = Planck's constant

$(6.626068 \times 10^{-34} \mathrm{m^2 kg/s})$

The electric field is given by $E = -qV_x$. In the isentropic case, the charge density dependence of the temperature is given by

(4) $$T(n) = const.n^{\gamma-1} \quad 1 \le \gamma \le 3$$

The equations describing thermal equilibrium can be obtained from stationary case of (2) by setting the current density $J = nv$ to zero. That is

(5) $$\frac{k}{m}(nT(n))_x - \frac{\hbar^2}{4m^2}(n(\ln n)_{xx})_x - \frac{q}{m}nV_x = 0$$

At the boundary of the domain, we prescribe the charge density. As additional boundary condition, we take a vanishing first derivative of the charge density at the boundary. After an appropriate scaling, we obtain for $x \in (0,1) = \Omega$ (semiconductor interval after scaling)

(6) $-\delta^2 (n(\ln n)_{xx})_x + n_x + nE = 0,$

(7) $\lambda^2 E_x = C - n$

with boundary conditions for the charge density n

$n(0) = n_0, n(1) = n_1, n_x(0) = n_x(1) = 0$

where δ = the scaled Planck's constant

λ = the scaled Debye length

Dividing (6) by n, taking derivative , using (7) and Setting u=*ln n*, we obtain

$$-\delta^2 (u_{xx}^2 + u_{xxxx} + u_x u_{xxx}) + u_{xx} - \frac{1}{\lambda^2}(e^u - C) = 0$$

$$(8) -\delta^2 \left[u_{xx} + \frac{u_x^2}{2} \right]_{xx} + u_{xx} - \frac{1}{\lambda^2}(e^u - C) = 0$$

With boundary conditions for u

(9) $u(0) = u_0, u(1) = u_1, u_x(0) = u_x(1) = 0$

From (6), we obtain

$$nE = \delta^2 (n(\ln n)_{xx})_x - n_x$$

$$E = \frac{\delta^2}{n}(n(\ln n)_{xx})_x - \frac{n_x}{n}$$

$$-qV_x = \frac{\delta^2}{n}\left(\frac{1}{\sqrt{n}}(\sqrt{n})_{xx}\right)_x 2n - (\ln n)_x$$

Using $(n(\ln n)_{xx})_x = 2n\left(\frac{1}{\sqrt{n}}(\sqrt{n})_{xx}\right)_x$

$$V_x = -\frac{2\delta^2}{q}\left(\frac{1}{\sqrt{n}}(\sqrt{n})_{xx}\right)_x + \frac{(\ln n)_x}{q}$$

$$(10)\, V = \frac{\ln n}{q} - \frac{2\delta^2}{q}\left(\frac{1}{\sqrt{n}}(\sqrt{n})_{xx}\right) + V_0$$

The nonlinear fourth order equation (8) with boundary condition (9) is the problem to be analyzed in this paper, which is organized in the following way.

2. Modeling of The Quantum Hydro-dynamic Models

The nonlinear Schrödinger equation (NLS) is

$$(1)\, i\varepsilon\partial_t \Psi + \frac{\varepsilon^2}{2}\Psi_{xx} - V'(|\Psi|^2)\Psi = 0 \quad t > 0, x \in \Re^n$$

$$\Psi(x,0) = \Psi_0(x)$$

is a fundamental equation in many physical problems that involve the evolution of the envelope of nonlinear wave packet. Here Ψ denotes the wave function, V a smooth real function and ε is the scaled Planck constant. A physical example is the focusing of a laser beam, which is modeled by the NLS with the coordinate along the beam playing the role of time. If we introduce the complex-valued wave function

$$(2) \qquad \Psi = A\exp\left(i\frac{S}{\varepsilon}\right)$$

where *A*, *S* are the real valued functions, *A* is called the amplitude and *S* the classical action. Easy calculation yields

$$\Psi_t = A_t e^{\frac{iS}{\varepsilon}} + \frac{Ai}{\varepsilon} e^{\frac{iS}{\varepsilon}} S_t$$

$$\Psi_x = A_x e^{\frac{iS}{\varepsilon}} + \frac{Ai}{\varepsilon} e^{\frac{iS}{\varepsilon}} S_x$$

$$\Psi_{xx} = A_{xx} e^{\frac{iS}{\varepsilon}} + \frac{2i}{\varepsilon} A_x S_x e^{\frac{iS}{\varepsilon}} + i\frac{A}{\varepsilon}\left[e^{\frac{iS}{\varepsilon}} S_{xx} + \frac{i}{\varepsilon} e^{\frac{iS}{\varepsilon}} S_x^2 \right]$$

$$= e^{\frac{iS}{\varepsilon}} \left[A_{xx} + \frac{2i}{\varepsilon} A_x S_x + i\frac{A}{\varepsilon} S_{xx} - \frac{A}{\varepsilon^2} S_x^2 \right]$$

(1) becomes

$$e^{\frac{iS}{\varepsilon}} \left[i\varepsilon A_t - AS_t + \frac{\varepsilon^2}{2} A_{xx} + i\varepsilon A_x S_x + i\frac{A\varepsilon}{2} S_{xx} - \frac{A}{2} S_x^2 - V'A \right] = 0$$

By separating real and imaginary parts,

$\varepsilon A_t + \varepsilon A_x S_x + \frac{\varepsilon}{2} AS_{xx} = 0$, this implies

(3) $A_t + A_x S_x + \frac{A}{2} S_{xx} = 0$ and

$-AS_t + \frac{\varepsilon^2}{2} A_{xx} - \frac{A}{2} S_x^2 - V'A = 0$ and so

(4) $S_t - \frac{\varepsilon^2}{2}\frac{A_{xx}}{A} + \frac{1}{2} S_x^2 + V' = 0$

Let (5) $n = A^2 = |\Psi|^2$, $v = \nabla S = S_x$

Differentiating (4) with respect to x, we obtain

(6) $S_{tx} + S_x S_{xx} + V''(A^2)2AA_x - \frac{\varepsilon^2}{2}\left(\frac{A_{xx}}{A}\right) = 0$

Multiplying (3) by *2A*, we obtain

$$2AA_t + 2AA_x S_x + A^2 S_{xx} = 0$$

i.e $\quad n_t + n_x S_x + nS_{xx} = 0$

(7) $\quad n_t + (nv)_x = 0$

Let $\quad J = n\,v, \quad S_x = v$

(8) $\quad n_t + J_x = 0$

which is analogous to the equation of continuity.

(6) can be written as

$$v_t + vv_x + V''(A^2)n_x - \frac{\varepsilon^2}{2}\left(\frac{\left(\sqrt{n}\right)_{xx}}{\sqrt{n}}\right)_x = 0$$

Multiplying by n, we obtain

$$nv_t + nvv_x + nV''(A^2).n_x - \frac{\varepsilon^2}{2} n\left(\frac{\left(\sqrt{n}\right)_x}{\sqrt{n}}\right)_x = 0$$

And this implies

(9) $nv_t + v(nv_x) + P'(n).n_x - \frac{\varepsilon^2}{2} n\left(\frac{\left(\sqrt{n}\right)_x}{\sqrt{n}}\right)_x = 0$

The density is defined by $n = A^2$ and $nV''(A^2) = P'(n)$

But $\quad nv_t + v(nv_x) = nv_t + v(-(n_t + n_x v))$

$$= nv_t + vn_t - 2vn_t - v^2 n_x$$

$$= (nv)_t + (nv^2)_x$$

$$= J_t + \left(\frac{J^2}{n}\right)_x \quad \text{and}$$

$$\frac{\varepsilon^2}{2} n\left(\frac{\left(\sqrt{n}\right)_{xx}}{\sqrt{n}}\right)_x = \frac{\varepsilon^2}{4}\left(n_{xxx} - \frac{2n_x n_{xx}}{n} + \frac{n_x^3}{n^2}\right)$$

$$= \frac{\varepsilon^2}{4}\left(n(\log n)_{xx}\right)_x$$

S0 (9) becomes

(10) $J_t + \left(\frac{J^2}{n}\right)_x + (P(n))_x = \frac{\varepsilon^2}{4}\left(n(\log n)_{xx}\right)_x$

Multiplying (8) by $V'(n)$,

(11) $V'(n)n_t + V'(n)J_x = 0$

Multiplying (8) by $-\frac{J^2}{2n^2}$, we obtain

(12) $-\frac{J^2}{2n^2} n_t - \frac{J^2}{2n^2} J_x = 0$

Multiplying (10) by $\frac{J}{n}$, we obtain

$$\frac{J}{n} J_t + \frac{J}{n}\left(\frac{J^2}{n}\right)_x + J(V''(n)n_x) = \frac{\varepsilon^2}{4}\frac{J}{n}\left(n(\log n)_{xx}\right)_x$$

Let $\quad E = \frac{1}{2}\frac{J^2}{n} + \frac{\varepsilon^2}{8}\frac{n_x^2}{n} + V(n)$

$$E_t = \frac{1}{2}\frac{2nJJ_t - J^2 n_t}{n^2} + \frac{\varepsilon^2}{8}\frac{2nn_x n_{xt} - n_x^2 n_t}{n^2} + V'(n)n_t$$

$$= \frac{JJ_t}{n} - \frac{J^2 n_t}{2n^2} - \frac{\varepsilon^2 n_x^2 n_t}{8n^2} + \frac{\varepsilon^2 n_x n_{xt}}{4n} + V'(n)n_t$$

$$J_t = -\left(\frac{J^2}{n}\right)_x - (P(n))_x + \frac{\varepsilon^2}{4}\left(n(\log n)_{xx}\right)_x$$

$$= -\frac{2nJJ_x - J^2 n_x}{n^2} - P_x + \frac{\varepsilon^2}{4}\left(n_{xxx} - \frac{2n_x n_{xx}}{n} + \frac{n_x^3}{n^2}\right)$$

$$\frac{J}{n} J_t = \frac{J}{n}\left(-\frac{2nJJ_x - J^2 n_x}{n^2}\right) - \frac{J}{n} P_x + \frac{J\varepsilon^2}{4n}\left(n_{xxx} - \frac{2n_x n_{xx}}{n} + \frac{n_x^3}{n^2}\right)$$

and we obtain

(13) $\frac{J}{n} J_t = -\frac{2J^2 J_x}{n^2} + \frac{J^3}{n^3} n_x - \frac{J}{n}\nabla P + \frac{J\varepsilon^2}{4n}\left(n_{xxx} - \frac{2n_x n_{xx}}{n} + \frac{n_x^3}{n^2}\right)$

From(8), we obtain

(14) $\quad \frac{\varepsilon^2 n_x}{4n} n_{tx} = -J_{xx}$

(15) $\quad -\frac{\varepsilon^2 n_x^2}{8n^2} n_t = \frac{\varepsilon^2 n_x^2}{8n^2} J_x$

By (11), (12), (13), (14), (15), we obtain

(16) $E_t = -\left(\frac{3J^2}{2n^2} + V'(n)J_x + \frac{J^3}{n^3} n_x - \frac{J}{n} P_x + \frac{\varepsilon^2}{4n^2}(n_{xxx} nJ - 2Jn_x n_{xx} + \frac{Jn_x^3}{n}\right.$

$$-nn_x J_{xx} + \frac{n_x^2}{2} J_x \Bigg)$$

But $\quad E + P(n) = \frac{1}{2}\frac{J^2}{n} + \frac{\varepsilon^2}{8}\frac{n_x^2}{n} + V(n) + P(n)$

(17) $\left(\frac{J}{n}(E + P(n))\right)_x = \frac{1}{2}\frac{n^2 3J^2 J_x - J^3 2nn_x}{n^4} +$

$$\frac{\varepsilon^2}{8}\frac{n^2(J_x n_x^2 + 2Jn_x n_{xx}) - Jn_x^2 2nn_x}{n^4} +$$

$$\frac{n\big(J_x(V(n) + P(n))\big) + J(V'n_x + P'n_x) - J(V + P)n_x}{n^2}$$

$$= \frac{3J^2}{2n^2}J_x - \frac{J^3}{n^3}n_x + \frac{J_x(V + P) + n_x J(V' + P')}{n}$$

$$-\frac{J(V + P)n_x}{n^2} + \frac{\varepsilon^2}{8n^2}(J_x n_x^2 + 2Jn_x n_{xx}) - \frac{\varepsilon^2}{4n^3}Jn_x^3$$

(18) $\frac{\varepsilon^2}{4}\left(\frac{Jn_x}{n} - \frac{J_x n_x}{n}\right)_x = \frac{\varepsilon^2}{4}\frac{nJn_{xxx} - nJ_{xx}n_x - Jn_x n_{xx} + J_x n_x^2}{n^2}$

Adding (16) and (17)

$$E_t + \left(J/_n (E + P(n))\right)_x = \frac{\varepsilon^2}{4}\left(\frac{Jn_{xx} - J_x n_x}{n}\right)_x$$

Which is the equation of conservation of energy.

3. Existence of Solutions

3.1 Lemma Suppose $C \in L^\infty(0,1)$. Then the estimate $\|u\|_{H^2(0,1)} \le c$ holds. In the sequel, we denote by c generic, positive, not necessarily equal constants.

Proof We define a function $u_D \in C^2[0,1]$ satisfying the boundary conditions

(1) $u_D(0) = u_0, u_D(1) = u_1, u_{D_x}(0) = u_{D_x}(1) = 0$ and with piecewise linear second derivative $(0 \le \varepsilon \le \frac{1}{2})$.

(2) $$u_{D_{xx}}(x) = \begin{cases} \frac{4\alpha}{\varepsilon^2(1-\varepsilon)}x & x \in \left[0, \frac{\varepsilon}{2}\right) \\ -\frac{4\alpha}{\varepsilon^2(1-\varepsilon)}x + \frac{4\alpha}{\varepsilon(1-\varepsilon)} & x \in \left[\frac{\varepsilon}{2}, \varepsilon\right) \\ 0 & x \in \left[\frac{\varepsilon}{2}, \frac{1}{2}\right) \end{cases}$$

where $\alpha = u_1 - u_0$ and $u_{D_{xx}}(1-x) = -u_{D_{xx}}(x)$

$\forall x \in [0,1]$

Multiplying (8) by $(u - u_D)$ and integrating on $\Omega = (0,1)$, we obtain

$$-\delta^2\int_0^1 (u_{xx} + \frac{u_x^2}{2})_{xx}(u - u_D)dx + \int_0^1 u_{xx}(u - u_D)dx$$

$$-\frac{1}{\lambda^2}\int_0^1 (e^u - C)(u - u_D)dx = 0$$

Then

(3) $$-\delta^2\int_0^1 (u_{xx} + \frac{u_x^2}{2})(u_{xx} - u_{D_{xx}})dx + \int_0^1 u_x(u_x - u_{D_x})dx$$

$$-\frac{1}{\lambda^2}\int_0^1 (e^u - e^{u_D})(u - u_D)dx - \frac{1}{\lambda^2}\int_0^1 (e^{u_D} - C)(u - u_D)dx = 0$$

But

$$\left|\int_0^1 u_{xx}u_{D_{xx}}dx\right| \le \int_0^1 \left|u_{xx}u_{D_{xx}}\right|dx \le \int_0^1 |u_{xx}|\left|u_{D_{xx}}\right|dx$$

Using Young's inequality,

$$\left|\int_0^1 u_{xx}u_{D_{xx}}dx\right| \le \frac{1}{2}\|u_{xx}\|_2^2 + \frac{1}{2}\left\|u_{D_{xx}}\right\|_2^2$$

$$\left\|u_{D_{xx}}\right\|_2^2 = \int_0^1 \left|u_{D_{xx}}\right|^2 dx$$

$$= \int_0^{\frac{\varepsilon}{2}} \frac{16\alpha^2 x^2}{\varepsilon^4(1-\varepsilon)^2}dx + \int_{\frac{\varepsilon}{2}}^1 \left[\frac{16\alpha^2}{\varepsilon^2(1-\varepsilon)^2} - \frac{32\alpha^2}{\varepsilon^3(1-\varepsilon)^2} + \frac{16\alpha^2}{\varepsilon^4(1-\varepsilon)^2}\right]dx$$

$$= \frac{\alpha^2}{\varepsilon(1-\varepsilon)^2}\left[\frac{4}{3}\right]$$

Therefore

(4) $$\left|\int_0^1 u_{xx}u_{D_{xx}}dx\right| \le \frac{\|u_{xx}\|^2}{2} + \frac{2\alpha^2}{3\varepsilon(1-\varepsilon)^2} = \frac{\|u_{xx}\|^2}{2} + \frac{c\alpha^2}{\varepsilon}$$

Where $c = \frac{2}{3(1-\varepsilon)^2}$

Let $|u_x(x)| \le \sqrt{x}\|u_{xx}\|_2$ and $|u_x(x)| \le \sqrt{1-x}\|u_{xx}\|_2$

$$\left|\int_0^1 \frac{u_x^2}{2}u_{D_{xx}}dx\right| \le \frac{1}{2}\int_0^1 \left|u_x^2 u_{D_{xx}}\right|dx$$

$$= \frac{1}{2}\int_0^{1/2} |u_x|^2\left|u_{D_{xx}}\right|dx + \frac{1}{2}\int_{1/2}^1 |u_x|^2\left|u_{D_{xx}}\right|dx$$

$$\le \frac{1}{2}\|u_{xx}\|_2^2\int_0^{\frac{1}{2}} x\left|u_{D_{xx}}\right|dx + \frac{1}{2}\|u_{xx}\|_2^2\int_{\frac{1}{2}}^1 (1-x)\left|u_{D_{xx}}\right|dx$$

$$= \frac{1}{2}\|u_{xx}\|_2^2 \left[\int_0^{\frac{\varepsilon}{2}} x^2 \frac{4\alpha}{\varepsilon^2(1-\varepsilon)} dx + \int_{\frac{\varepsilon}{2}}^{\varepsilon} \left(\frac{4\alpha}{\varepsilon(1-\varepsilon)} x - \frac{4\alpha}{\varepsilon^2(1-\varepsilon)} x^2 \right) dx \right]$$

$$= \frac{1}{2}\|u_{xx}\|_2^2 \left[\frac{4\alpha}{\varepsilon^2(1-\varepsilon)} \left[\frac{x^3}{3} \right]_0^{\frac{\varepsilon}{2}} + \frac{4\alpha}{\varepsilon(1-\varepsilon)} \left[\frac{x^2}{2} \right]_{\frac{\varepsilon}{2}}^{\varepsilon} - \frac{4\alpha}{\varepsilon^2(1-\varepsilon)} \left[\frac{x^3}{3} \right]_{\frac{\varepsilon}{2}}^{\varepsilon} \right]$$

$$= \frac{\alpha\varepsilon}{4(1-\varepsilon)} \|u_{xx}\|_2^2 \le c|\alpha|\varepsilon\|u_{xx}\|_2^2$$

(5) $\left| \int_0^1 \frac{u_x^2}{2} u_{D_{xx}} dx \right| \le c|\alpha|\varepsilon\|u_{xx}\|_2^2$ where $c = \frac{1}{4(1-\varepsilon)}$

(6) $\int_0^1 u_x^2 u_{xx} dx = \int_0^1 \left(\frac{u_x^3}{3} \right)_x dx = \frac{1}{3}\int_0^1 \left(u_x^3 \right)_x dx$

$$= \left(\left[u_x^3 \right]_0^1 - \int_0^1 u_x^3 .0 dx \right) \frac{1}{3}$$

$= 0$

If $(u - u_D) \ge 0$, then $(e^u - e^{u_D}) \ge 0$

If $(u - u_D) \le 0$, then $(e^u - e^{u_D}) \le 0$.

Therefore the relation

(7) $(e^u - e^{u_D})(u - u_D) \ge 0$ holds for all values of u and u_D.

Using (4), (5), (6) and (7) in (3) gives

$$-\delta^2 \int_0^1 u_{xx}^2 dx + \delta^2 \int_0^1 u_{xx} u_{D_{xx}} dx - \delta^2 \int_0^1 \frac{u_x^2}{2} u_{xx} dx + \delta^2 \int_0^1 \frac{u_x^2}{2} u_{D_{xx}} dx$$

$$- \int_0^1 u_x^2 dx + \int_0^1 u_x u_{D_x} dx - \frac{1}{\lambda^2} \int_0^1 (e^u - e^{u_D})(u - u_D) dx -$$

$$\frac{1}{\lambda^2} \int_0^1 (e^{u_D} - C)(u - u_D) dx = 0$$

Then we obtain

$$-\delta^2\|u_{xx}\|_2^2 + \frac{\delta^2}{2}\|u_{xx}\|_2^2 + \frac{c\delta^2\alpha^2}{\varepsilon} + \delta^2 c|\alpha|\varepsilon\|u_{xx}\|_2^2 - \|u_x\|_2^2$$

$$+ \frac{\|u_x\|_2^2}{2} + \frac{\|u_{D_x}\|_2^2}{2} + \frac{1}{\lambda^2} \int_0^1 (C - e^{u_D})(u - u_D) dx \ge 0$$

Then we obtain

$$\|u_{xx}\|_2^2 \left(\frac{-\delta^2}{2} + \delta^2 c|\alpha|\varepsilon \right) + \frac{\delta^2 c\alpha^2}{\varepsilon} - \frac{\|u_x\|_2^2}{2} + \frac{\|u_{D_x}\|_2^2}{2}$$

$$+ \frac{1}{\lambda^2} \left(\|C\|_2^2 + \left\|e^{u_D}\right\|_2^2 + \|u\|_2^2 + \|u_D\|_2^2 \right) \ge 0$$

Then we obtain

$$\frac{\delta}{2}\left(1 - 2c|\alpha|\varepsilon\right)\|u_{xx}\|^2 + \frac{\|u_x\|^2}{2}$$

$$\le \frac{c\alpha^2\delta^2}{\varepsilon} + \frac{\|u_D\|^2}{2c^2} + \frac{\|u_D\|^2}{\lambda^2} + \frac{1}{\lambda^2}\left(\|C\|_2^2 + \left\|e^{u_D}\right\|_2^2 + \|u\|_2^2 \right)$$

$$\le \frac{c\alpha^2\delta^2}{\varepsilon} + c_1^2\|u_D\|_2^2 + \frac{\|u_D\|^2}{\lambda^2 k} + \frac{1}{\lambda^2}\left(\|C\|_2^2 + \left\|e^{u_D}\right\|_2^2 + \|u\|_2^2 \right)$$

$$\Rightarrow (8) \quad \frac{\delta}{2}\left(1 - 2c|\alpha|\varepsilon\right)\|u_{xx}\|^2 + \frac{\|u_x\|^2}{2} \le c\left(1 + \frac{\delta^2}{\varepsilon}\right)$$

$\Rightarrow$

$$(9) \quad \frac{\delta}{2}\left(\frac{1}{2c|\alpha|} - \varepsilon \right)\|u_{xx}\|^2 + \frac{\|u_x\|^2}{4c|\alpha|} \le \frac{1}{2c\alpha}\left(1 + \frac{\delta^2}{\varepsilon}\right)$$

Since $0 < \varepsilon < \frac{1}{2c|\alpha|}$, $1 - 2c|\alpha|\varepsilon > 0$

Let $\frac{k}{2} = 1 - 2c|\alpha|\varepsilon$

Therefore

$\frac{k}{2}\delta^2\|u_{xx}\|^2 + \|u_x\|^2 \le 2c\left(1 + \frac{\delta^2}{\varepsilon}\right)$ implies

$k\delta^2\|u_{xx}\|^2 + \|u_x\|^2 + \|u_x\|^2 \le 4c\left(1 + \frac{\delta^2}{\varepsilon}\right)$ implies

$k\delta^2\|u_{xx}\|^2 + \|u_x\|^2 + k^*\|u\|^2 \le 4c\left(1 + \frac{\delta^2}{\varepsilon}\right)$ implies

$$\delta^2\|u_{xx}\|^2 + \|u_x\|^2 + \|u\|^2 \le 4\frac{c}{c^*}\left(1 + \frac{\delta^2}{\varepsilon}\right)$$

Where $c* \le \min(k, k^*, 1)$

$\frac{1}{3}\left(\delta\|u_{xx}\| + \|u_x\| + \|u\|\right)^2 \le 4\frac{c}{c^*}\left(1 + \frac{\delta^2}{\varepsilon}\right)$ implies

$$\left(\delta\|u_{xx}\| + \|u_x\| + \|u\|\right)^2 \le 12c^{**}\left(1 + 2c|\alpha|\delta^2\right)$$

$$= 12c^{**}2c|\alpha|\left(\frac{1}{2c|\alpha|} + \delta^2\right)$$

$$\le 24cc^{**}|\alpha|\left(1 + \delta^2\right)$$

$\le c_1^2\left(1 + \delta^2\right)$, where $c_1^2 = 24cc^{**}$

(10) $\left(\delta\|u_{xx}\| + \|u_x\| + \|u\|\right) \le c(1 + \delta)$ implies

$c^{***}\left(\|u_{xx}\|+\|u_x\|+\|u\|\right)\le c(1+\delta)$ where

$c^{***}=\min(\delta,1)$

$\left(\|u_{xx}\|+\|u_x\|+\|u\|\right)\le c_2$ where $c_2=\frac{c(1+\delta)}{c^{***}}$

$\left(\|u_{xx}\|+\|u_x\|+\|u\|\right)^2\le c_2^2$

$\|u\|^2_{H^2(0,1)}\le c_2^2$ implies

$\|u\|_{H^2(0,1)}\le c_2$

3.2 Theorem: Let $C\in L^{\infty}(0,1)$. Then the problem 1(8) has a solution $u\in H^2(0,1)$.

Proof: Consider the problem

$$(1)\ -\delta^2\left(u_{xx}+\sigma\frac{u_x^2}{2}\right)_{xx}+u_{xx}-\frac{\sigma}{\lambda^2}\left(\frac{e^v-1}{v}u+1-C\right)=0$$

with $\sigma\in[0,1]$, boundary conditions

$(2)\ u(0)=\sigma u_0,\ u(1)=\sigma u_1,\ u_x(0)=u_x(1)=0$ and $v\in C^{0,1}[0,1]$

We multiply (1) by test function ϕ and integrating from 0 to 1, we obtain

$$-\delta^2\int_0^1\left(u_{xx}+\sigma\frac{u_x^2}{2}\right)\phi_{xx}dx-\int_0^1 u_x\phi_x dx-\frac{\sigma}{\lambda^2}\int_0^1\left(\frac{e^v-1}{v}u+1-C\right)\phi dx=0$$

$$\int_0^1\left(\delta^2u_{xx}\phi_{xx}+u_x\phi_x+\frac{\sigma}{\lambda^2}\frac{e^v-1}{v}u\phi\right)dx=-\frac{\delta^2\sigma}{2}\int_0^1\left(u_x^2\phi_{xx}\right)dx+\frac{\sigma}{\lambda^2}\int_0^1(C-1)\phi dx$$

Defining the bilinear form by

$$(3)\ a(w,\phi)=\int_0^1\left[\delta^2w_{xx}\phi_{xx}+w_x\phi_x+\frac{\sigma}{\lambda^2}\frac{e^v-1}{v}w\phi\right]dx$$

We will show that this bilinear form is continuous and coercive.

(i) Continuity

$$|a(w,\phi)|=\left|\int_0^1[\delta^2w_{xx}\phi_{xx}+w_x\phi_x+\frac{\sigma}{\lambda^2}\frac{e^v-1}{v}w\phi]dx\right|$$

$$\le\int_0^1\delta^2|w_{xx}\phi_{xx}|dx+\int_0^1|w_x\phi_x|dx+\frac{\sigma}{\lambda^2}\int_0^1\frac{e^v-1}{v}|w\phi|dx$$

$$\le\delta^2\|w_{xx}\|_2\|\phi_{xx}\|_2+\|w_x\|_2\|\phi_x\|_2+\frac{\sigma}{\lambda^2}\frac{e^v-1}{v}\|w\|_2\|\phi\|_2$$

$$\le\delta^2\|w_{xx}\|_2\|\phi_{xx}\|_2+c_1c_2\|w_{xx}\|_2\|\phi_{xx}\|_2+c_3\|w_{xx}\|_2\|\phi_{xx}\|_2$$

Therefore

$$|a(w,\phi)|\le c_4\left(\|w_{xx}\|^2\|\phi_{xx}\|^2\right)^{1/2}\le c_5\left(\|w\|_2\|\phi_{xx}\|_2\right)$$

$\forall w,\phi\in H^2(\Omega)$ where $c_4=\delta^2+c_1c_2+c_3$

(ii) Coercivity

$$a(w,w)=\int_0^1[\delta^2w_{xx}^2+w_x^2+\frac{\sigma}{\lambda^2}\frac{e^v-1}{v}w^2]dx$$

$$=\int_0^1\delta^2w_{xx}^2dx+\int_0^1w_x^2dx+\frac{\sigma}{\lambda^2}\frac{e^v-1}{v}\int_0^1w^2dx$$

$$\ge\frac{\delta^2}{2}\int_0^1|w_{xx}|^2dx+\frac{1}{2}\int_0^1|w_x|^2dx+\frac{\sigma}{2\lambda^2}\frac{e^v-1}{v}\int_0^1|w|^2dx$$

$$=\frac{\delta^2}{2}\|w_{xx}\|^2+\frac{1}{2}\|w_x\|^2+\frac{c_3^*}{2}\|w\|^2$$

$$\ge\frac{\delta^2}{2c_1^{*2}}\|w\|^2+\frac{1}{2c_2^{*2}}\|w\|^2+\frac{c_3^*}{2}\|w\|^2$$

$$\ge\min\left(\frac{\delta^2}{2c_1^{*2}},\frac{1}{2c_2^{*2}},\frac{c_3^*}{2}\right)\|w\|^2$$

$$=c_6\|w\|^2$$

Therefore

$$a(w,w)\ge c_6\|w\|^2\quad\forall w\in H^2(\Omega)$$

So, by Lax-Miligram theorem, this boundary value problem has a unique weak solution $\forall u\in H^2(0,1)$.

Therefore, we can define the map $T:C^{0,1}[0,1]\times[0,1]\to C^{0,1}[0,1]$ by $T(v,\sigma)=u$. It can be seen easily that the map T is continuous. Due to the compact imbedding of $H^2(0,1)$ in $C^{0,1}[0,1]$, it is compact. Additionally, we have T(v,0)=0 $\forall v\in C^{0,1}[0,1]$ and

$$(4)\ \|u\|_{C^{0,1}[0,1]}\le c\ \ \forall(u,\sigma)\in C^{0,1}[0,1]\times[0,1]$$

with $T(v,\sigma)=u$

Lemma 3.1 settles the case $\sigma = 1$. In the case $\sigma \neq 1$, the same steps as in the proof of lemma 3.1 lead to

$$(5)\quad \frac{\delta^2}{2}(1-2c|\alpha|\varepsilon)\|u_{xx}\|_2^2 + \frac{\|u_x\|^2}{4c|\alpha|} \leq \frac{1}{2|\alpha|}\left(1+\frac{\delta^2}{\varepsilon}+\sigma\right)$$

and

$$\frac{\delta^2}{2}\left(\frac{1}{2c|\alpha|}-\varepsilon\right)\|u_{xx}\|^2 + 4\frac{\|u_x\|^2}{4c|\alpha|} \leq \frac{1}{2|\alpha|}\left(1+\frac{\delta^2}{\varepsilon}\right).$$

follows.

All conditions of the Leray Schauder Fixed Point Theorem with Banach space $C^{0,1}[0,1]$ and compact map T are satisfied. The existence of a fixed point $T(u,1) = u$ follows.

Obviously, with $u \in H^2(0,1)$, the evaluation of the potential V in is straightforward.

4. Uniqueness of the solutions u in $H^2(0,1)$

In order to obtain uniqueness for small values of δ, we prove the following lemma.

4.1 Lemma: Let $u, v \in H^2(0,1)$ be two weak solutions of the problem 1(8).

If $\|u_x + v_x\|_{L^\infty(0,1)} \leq \frac{1}{2\delta}$, then $u = v$ in $H^2(0,1)$ holds.

Proof: Since u is a solution of 1(8), we obtain

$$-\delta^2\left[u_{xx} + \frac{u_x^2}{2}\right]_{xx} + u_{xx} - \frac{1}{\lambda^2}(e^u - C) = 0$$

Multiplying this equation by $(u-v)$ and integrating on $\Omega = (0,1)$ gives

$$(1)\ \delta^2\int_0^1\left(u_{xx}+\frac{u_x^2}{2}\right)(u_{xx}-v_{xx})dx + \int_0^1 u_x(u_x-v_x)dx + \frac{1}{\lambda^2}\int_0^1(e^u - C)(u-v)dx = 0$$

Similarly

$$(2)\ \delta^2\int_0^1\left(v_{xx}+\frac{v_x^2}{2}\right)(u_{xx}-v_{xx})dx + \int_0^1 v_x(u_x-v_x)dx + \frac{1}{\lambda^2}\int_0^1(e^v - C)(u-v)dx = 0$$

Taking the difference of (1) and (2), we obtain

$$0 = \delta^2\int_0^1(u_{xx}-v_{xx})^2dx + \frac{\delta^2}{2}\int_0^1 (u_x^2 - v_x^2)(u_{xx}-v_{xx})dx + \int_0^1(u_x - v_x)^2dx + \frac{1}{\lambda^2}\int_0^1(e^u - e^v)(u-v)dx$$

Since $(e^u - e^v)(u-v) \geq 0$

$$\delta^2\int_0^1(u_{xx}-v_{xx})^2dx + \frac{\delta^2}{2}\int_0^1 (u_x^2 - v_x^2)(u_{xx}-v_{xx})dx + \int_0^1(u_x - v_x)^2dx + \frac{1}{\lambda^2}\int_0^1(e^u - e^v)(u-v)dx \geq$$

$$\delta^2\int_0^1(u_{xx}-v_{xx})^2dx + \frac{\delta^2}{2}\int_0^1 (u_x^2 - v_x^2)(u_{xx}-v_{xx})dx + \int_0^1|u_x - v_x|^2dx \geq$$

$$\int_0^1\left(\delta|u_{xx}-v_{xx}| - |u_x - v_x|\right)^2dx + 2\delta\int_0^1|u_{xx}-v_{xx}||u_x-v_x|dx +$$

$$\frac{\delta^2}{2}\int_0^1|u_{xx}-v_{xx}||u_x-v_x||u_x+v_x|dx$$

$$\geq \int_0^1\left(\delta|u_{xx}-v_{xx}| - |u_x - v_x|\right)^2dx + \frac{7\delta}{4}\int_0^1|u_{xx}-v_{xx}| - |u_x-v_x|dx$$

$$\geq \delta^2\|u_{xx}-v_{xx}\|_2^2 + \|u_x - v_x\|_2^2 - \frac{\delta^2}{2}\|u_{xx}-v_{xx}\|_2^2 - \frac{\|u_x-v_x\|_2^2}{2}$$

≥ 0 which implies

$$\delta^2\|u_{xx}-v_{xx}\|_2^2 + \frac{\|u_x - v_x\|_2^2}{2} + \frac{\|u_x - v_x\|_2^2}{2} = 0$$

By using Poincare's Inequality

$$\delta^2\|u_{xx}-v_{xx}\|_2^2 + \frac{\|u_x - v_x\|_2^2}{2} + \frac{k\|u_x - v_x\|_2^2}{2} = 0$$

this implies

$\|u - v\|_2^2 = 0$ which implies

$u = v$ in $H^2(0,1)$ holds.

This completes the proof.

5. Conclusion

The modern computer and telecommunication industry relies heavily on the use and development of semiconductor devices. A very important fact of the success of semiconductor devices is that the device length is very small compared to previous electronic devices. .We studied the modeling of the quantum hydrodynamic model for semi conductors from nonlinear Schrödinger equation by using Madelung method. Then the existence, uniqueness of weak solutions of the thermal equilibrium states of one-dimensional quantum hydrodynamic model for semi conductors was studied by using modern theory of partial differential equations.

References

[1]ADAMS, R.A: *Sobolev Spaces*, Academic Press, New York, 1975.

[2]Dr. ANSGAR JŰNGEL: *Mathematical Modelling of Semiconductor Devices*, Fachbereich Mathematik und Statistik Universit™t Konstanz, Preliminary Version.

[3]INGENIUN, LIN, C.K and MARKO WICH.: *A Review of Dispersive Limits of Nonlinear Schrödinger-Type Equations*: TAIWANESE Journal of Mathematics, Vol, 4, No,4, pp,501-529

[4]LIN, C.K,: *Remarks on the singularity of the Nonlinear Schrödinger Equation: The Hydrodynamical Approach*, Southeast Asia Bulletin of Mathematics 22, 161-170,1998.

[5] LIN, C.K,: *On the fluid-dynamical analogue of the general of Nonlinear Schrödinger equation* Southeast Asia Bulletin of Mathematics 22,1) 45-46,1998

[6]LIJG HSAIAO and KAIJUN ZAUNG: *Relaxation of the Hydrodynamic model for Semiconductors to the Drift-Diffusion Equations,* Acadamy of Mathematics and systems Sciences, CAS, Beijing 100080, People's Republic of China

[7]RENARDY, M and ROGERS, R.C: *An introduction to Partial Differential Equations,* Springer Verlag, New York, 1993.

[8]MARKOWICH, P.A: *A Singular Perturbation Analysis of the Fundamental Semiconductor Device Equations* Siam J. Appl. Math 44, 896-928,1984

[9]WINGER, E.: *On the quantum correction for thermodyanmic Equilibrium*, Phys, Rev. B44,3199-3210,1932

[10] LIN, C.K,: On the incompressible limit of the compressible Navier-Stokes equations, Communin P. D. E,20, 677-707, 1995.