DIFFERENTIABILITY OF NORMS AND ROTUNDITY IN TREE

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Abstract

In this paper, the basic concepts of differentiability and rotundity of norms are first studied. And then the convex function is discussed. After definition Kadec-Klee property and the James tree space JT, the dual space JT^* are discussed. The double dual space is a subspace of a weakly compactly generated and dual space admits an equivalent locally uniformly rotund norm.

1. Gâteaux and Fréchet Derivatives

Throughout this paper, we use S_X for the unit sphere and B_X for the closed unit ball in a Banach space X. Unless stated otherwise, the dual space X^* (i.e., the space of all continuous linear function on X) is considered endowed with the supremum norm, i.e., $\|f\|^* = \sup\{f(x) : x \in B_X\}$, for $f \in X^*$. The w^* or $weak^*$-topology on X^* denotes the pointwise topology on X^* and w-topology denotes the weak topology on Banach space.

1.1 Definition

Let X be a Banach space. Let f be a real-valued function on an open subset U of X. Let $x \in U$. We say that f is **Gâteaux differentiable** at x if there is $F \in X^*$ such that

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t} = F(h),$$

for every $h \in X$. We say that f is **Fréchet differentiable** at x if this limit is uniform for $h \in S_X$. We call F the **Gâteaux** or the **Fréchet derivative** (or differential) of f at x and denote it by $F = f'(x)$. We say that f is **Gâteaux** (resp. **Fréchet**) **differentiable** on U if f is **Gâteaux** (resp. **Fréchet**) at all points of U.

A norm $\|\,.\,\|$ on a Banach space X is called **Fréchet** (resp. **Gâteaux**) differentiable if $\|\,.\,\|$ is Fréchet (resp. Gâteaux) differentiable at every point of $X\setminus\{0\}$.

1.2 Definition

Let U be a convex subset of a vector space V. We say that a function $f : U \to \mathbb{R}$ is **convex** if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y),$$

for all $x, y \in U$ and $\lambda \in [0,1]$.

1.3 Example

(i) Every norm of a normed space X is a convex function on X.
(ii) Every linear functional on a normed space X is a convex function.

1.4 Definition

(i) The norm $\|\,.\,\|$ on a Banach space X is said to be **locally uniformly convex** or **locally uniformly rotund** (LUR) if, whenever x and $x_n (n \in N)$ are elements of X such that $\|x_n\| \to \|x\|$ and $\|x + x_n\| \to 2\|x\|$ as $n \to \infty$, we necessarily have $\|x - x_n\| \to 0$.

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(ii) The norm $\|\,.\,\|$ on a Banach space X is called **rotund** or **strictly convex** (R) if $x = y$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$.

(iii) The norm $\|\,.\,\|$ on a Banach space X is said to have the **Kadec-Klee property** if the relative norm topology and weak topology on the unit ball B_X coincide at each point of the unit sphere S_X.

1.5 Proposition

Every locally uniformly rotund norm on a Banach space has the Kadec-Klee property.

Proof

Let $\|\,.\,\|$ be a locally uniformly rotund norm on X and $x_0 \in S_X$. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that $\|x_0 - x\| < \varepsilon$ whenever $x \in B_X$ and $\left\|\frac{(x_0 - x)}{2}\right\| > 1-\delta$. Assume that $f_0 \in S_{X^*}$ is such that $f_0(x_0) = 1$ and $x \in B_X$ is such that $f_0(x) > 1-\delta$.

Then $\left\|\frac{(x_0 - x)}{2}\right\| \geq \left(\frac{1}{2}\right) f_0(x_0 + x) > 1 - \frac{\delta}{2}$.

Thus $\|x_0 - x\| < \varepsilon$.

Therefore the sets $\{x \in B_X : f_0(x) > 1-\delta\}$ form a neighborhood base in the relative norm topology of B_X at x_0.

1.6 Theorem (Šmulyan Test)

Suppose that $\|\,.\,\|$ is a norm on a Banach space X with dual norm $\|\,.\,\|^*$. Then

(i) The norm $\|\,.\,\|$ is Fréchet differentiable at $x \in S_X$ if and only if whenever $f_n, g_n \in S_{X^*}$, $f_n(x) \to 1$ and $g_n(x) \to 1$, then $\|f_n - g_n\|^* \to 0$.

(ii) The norm $\|\,.\,\|^*$ is Fréchet differentiable at $f \in S_{X^*}$ if and only if whenever $x_n, y_n \in S_X$, $f(x_n) \to 1$ and $f(y_n) \to 1$, then $\|x_n - y_n\| \to 0$.

Proof See [2].

The duality between the rotundity and differentiability of norms is discussed in the following proposition.

1.7 Proposition

If the dual norm $\|\,.\,\|^*$ of $\|\,.\,\|$ is locally uniformly rotund on X^*, then the norm $\|\,.\,\|$ on X is Fréchet differentiable.

Proof

We use Šmulyan's Test. Let $x \in S_X$ and $f_n, g_n \in S_{X^*}$, $n = 1, 2, \ldots$ be such that $f_n(x) \to 1$ and $g_n(x) \to 1$.

We need to show that $\|f_n - g_n\|^* \to 0$.

Choose $f \in S_{X^*}$ such that $f(x) = 1$.

It follows that $2 \geq \|f + f_n\|^* \geq (f + f_n)(x) \to 2$.

From the local uniform rotundity of $\|\,.\,\|^*$ at f, we obtain $\lim_{n\to\infty}\|f_n - f\|^* = 0$.

Similarly we get $\lim_{n\to\infty}\|g_n - f\|^* = 0$. Hence $\lim_{n\to\infty}\|f_n - g_n\|^* = 0$.

Therefore the norm $\|\,.\,\|$ on X is Fréchet differentiable.

1.8 Proposition

(i) If the dual norm $\|\,.\,\|^*$ of $\|\,.\,\|$ is rotund, then the norm $\|\,.\,\|$ on X is Gâteaux differentiable.

(ii) If the dual norm $\|\,.\,\|^*$ of X^* is Gâteaux differentiable, then the norm $\|\,.\,\|$ on X is rotund.

Proof See [2].

1.9 Proposition

Let X be a Banach space, $\|\,.\,\|$ be a norm on X and $x_0 \in S_X$. Then the following are equivalent:

(i) The norm $\|\,.\,\|$ is locally uniformly rotund at x_0.

(ii) If $x_n \in S_X$, $n = 1, 2, \ldots$ and $\lim_{n\to\infty}\|x_0 + x_n\| = 2$, then $\lim_{n\to\infty}\|x_0 - x_n\| = 0$.

(iii) If $\{x_n\} \subset X$ is such that $\lim_{n\to\infty}\left(2\left(\|x_0\|^2 + \|x_n\|^2\right) - \|x_0 + x_n\|^2\right) = 0$, then $\lim_{n\to\infty}\|x_0 - x_n\| = 0$.

Proof

$(ii) \Rightarrow (i)$.

Let $x_n \in X$, $x \in X$ such that $\|x_n\| \to \|x\|$ and $\|x_n + x\| \to 2\|x\|$.

Without loss of generality, assume that $x_n \neq 0$ for all n and $x \neq 0$.

Put $y_n = \dfrac{x_n}{\|x_n\|}$, $y = \dfrac{x}{\|x\|}$.

Then $y_n \in S_X$ and $y \in S_X$, $\|y_n + y\| = \left\|\dfrac{x_n}{\|x_n\|} + \dfrac{x}{\|x\|}\right\|$.

Since $\|x_n\| \to \|x\|$, there exists $B > 0$ such that $\|x_n\| \le B$ for all n.

$$\begin{aligned}
\left\|\frac{x_n}{\|x_n\|} + \frac{x}{\|x\|}\right\| &= \left\|\frac{x_n}{\|x_n\|} - \frac{x_n}{\|x\|} + \frac{x_n}{\|x\|} + \frac{x}{\|x\|}\right\| \\
&\le \left\|\frac{x_n}{\|x_n\|} - \frac{x_n}{\|x\|}\right\| + \left\|\frac{x_n}{\|x\|} + \frac{x}{\|x\|}\right\| \\
&= \left|\frac{1}{\|x_n\|} - \frac{1}{\|x\|}\right|\|x_n\| + \frac{1}{\|x\|}\|x_n + x\| \\
&\le B\left|\frac{1}{\|x_n\|} - \frac{1}{\|x\|}\right| + \frac{1}{\|x\|}\|x_n + x\|.
\end{aligned}$$

By the given condition, $\|x_n + x\| \to 2\|x\|$.

Moreover, $\|x_n\| \to \|x\| \Rightarrow \dfrac{1}{\|x_n\|} \to \dfrac{1}{\|x\|}$.

Therefore $\lim\limits_{n\to\infty}\left\|\frac{x_n}{\|x_n\|}+\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|}\cdot 2\|x\|$

$= 2$,

i.e., $\lim\limits_{n\to\infty}\|y_n + y\| = 2$.

Therefore by (ii) $\lim\limits_{n\to\infty}\|y_n - y\| = 0$,

i.e., $\lim\limits_{n\to\infty}\left\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\right\| = 0$.

Now $\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|} = \frac{\|x\|x_n - \|x_n\|x}{\|x_n\|\|x\|}$

$$= \frac{\|x\|(x_n - x) + \left(\|x\| - \|x_n\|\right)x}{\|x_n\|\|x\|}.$$

Thus $\frac{(x_n - x)}{\|x_n\|} = \left(\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\right) - \frac{\left(\|x\| - \|x_n\|\right)x}{\|x_n\|\|x\|}$,

$$\frac{\|x_n - x\|}{\|x_n\|} \le \left\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\right\| + \frac{\left|\|x\| - \|x_n\|\right|\|x\|}{\|x_n\|\|x\|}.$$

Therefore $\|x_n - x\| \le \|x_n\|\left\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\right\| + \left|\|x\| - \|x_n\|\right|$

$$\le B\left\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\right\| + \left|\|x\| - \|x_n\|\right|.$$

But $\lim\limits_{n\to\infty}\left\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\right\| = 0$

and $\lim\limits_{n\to\infty}\left|\|x\| - \|x_n\|\right| = 0$.

Thus $\lim\limits_{n\to\infty}\|x_n - x\| = 0$.

$(i) \Rightarrow (ii)$.

Suppose that $x_n \in S_n$, $n = 1, 2, \ldots$ such that $\lim\limits_{n\to\infty}\|x_0 + x_n\| = 2$.

Then $\lim\limits_{n\to\infty}\|x_n\| = 1 = \|x_0\|$ since $x_0 \in S_X$.

Moreover, $\lim\limits_{n\to\infty}\|x_0 + x_n\| = 2\cdot 1 = 2\|x_0\|$.

By (i), we have $\lim\limits_{n\to\infty}\|x_0 - x_n\| = 0$.

$(i) \Rightarrow (iii)$.

Let $\{x_n\} \subset X$ such that $\lim\limits_{n\to\infty}\left(2\left(\|x_0\|^2 + \|x_n\|^2\right) - \|x_0 + x_n\|^2\right) = 0$.

Since

$$2\|x_0\|^2 + 2\|x_n\|^2 - \|x_0 + x_n\|^2 \ge 2\|x_0\|^2 + 2\|x_n\|^2 - \left(\|x_0\| + \|x_n\|\right)^2$$

$$= \|x_0\|^2 + \|x_n\|^2 - 2\|x_0\|\|x_n\|$$

$$=\left(\|x_0\|-\|x_n\|\right)^2,$$

it follows that $0\ge\lim_{n\to\infty}\left(\|x_0\|-\|x_n\|\right)^2$.

Therefore $\lim_{n\to\infty}\left(\|x_0\|-\|x_n\|\right)^2=0$. Thus $\lim_{n\to\infty}\|x_n\|=\|x_0\|$.

Since $\lim_{n\to\infty}\left(2\|x_0\|^2+2\|x_n\|^2\right)=\lim_{n\to\infty}\|x_0+x_n\|^2$,

we have

$$2\|x_0\|^2+2\lim_{n\to\infty}\|x_n\|^2=\lim_{n\to\infty}\|x_0+x_n\|^2.$$

Then

$$2\|x_0\|^2+2\|x_0\|^2=\lim_{n\to\infty}\|x_0+x_n\|^2.$$

Hence $\lim_{n\to\infty}\|x_0-x_n\|=2\|x_0\|$.

By the local uniform rotundity, we have $\lim_{n\to\infty}\|x_0-x_n\|=0$.

$(iii)\Rightarrow(i)$.

Take any $x_n, x_0\in X$ such that $\lim_{n\to\infty}\|x_n\|=\|x_0\|$ and $\lim_{n\to\infty}\|x_0+x_n\|=2\|x_0\|$.

Since

$$\lim_{n\to\infty}\left(2\|x_0\|^2+2\|x_n\|^2-\|x_0+x_n\|^2\right)=2\|x_0\|^2+2\|x_0\|^2-4\|x_0\|^2=0,$$

we have $\lim_{n\to\infty}\|x_0-x_n\|=0$, by (iii).

1.10 Proposition

The following conditions on a norm $\|\,.\,\|$ of a Banach space X are equivalent:

(i) The norm $\|\,.\,\|$ is rotund.

(ii) If $x, y\in X$ are such that $2\|x\|^2+2\|y\|^2-\|x+y\|^2=0$, then $x=y$.

(iii) If $x, y\in X$ are such that $\|x+y\|=\|x\|+\|y\|$, $x\neq 0$, and $y\neq 0$, then $x=\lambda y$ for some $\lambda>0$.

Proof

$(i)\Rightarrow(ii)$

Assume that the norm is rotund.

Let $x\in X$, $y\in X$ be such that $2\|x\|^2+2\|y\|^2-\|x+y\|^2=0$.

From the triangle inequality of norm, we have

$$\|x+y\|^2\le\|x\|^2+\|y\|^2+2\|x\|\|y\|.$$

Applying the assumption, we have

$$2\|x\|^2+2\|y\|^2\le\|x\|^2+\|y\|^2+2\|x\|\|y\|,$$

i.e., $\|x\|^2+\|y\|^2-2\|x\|\|y\|\le 0$,

i.e., $\left(\|x\|-\|y\|\right)^2\le 0$ which implies that $\|x\|=\|y\|$.

Let $\|x\|=\|y\|=k$.

Then $\left\|\frac{x}{k}\right\|=\left\|\frac{y}{k}\right\|=1$ and dividing the equation $2\|x\|^2+2\|y\|^2-\|x+y\|^2=0$ by k^2, we have

$$\left\|\frac{x}{k}+\frac{y}{k}\right\|^2 = 2\left\|\frac{x}{k}\right\|^2 + 2\left\|\frac{y}{k}\right\|^2$$
$$= 4 .$$

Thus $\left\|\frac{x}{k}+\frac{y}{k}\right\| = 2$ and $\left\|\frac{x}{k}\right\| = \left\|\frac{y}{k}\right\| = 1$.

Since the norm is rotund, $\frac{x}{k} = \frac{y}{k}$ which shows that $x = y$.

(ii) $\Rightarrow$ (i).

Assume that $\|x\| = \|y\| = 1$ and $\|x+y\| = 2$.

Then

$2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 4 - 4 = 0$.

By (ii) $x = y$.

Hence the norm is rotund.

(i) $\Rightarrow$ (iii).

Assume that $\| \cdot \|$ is rotund and that $x, y \in X$ such that $0 < \|x\| \le \|y\|$ and $\|x+y\| = \|x\| + \|y\|$.

Then we have

$$\begin{aligned}
2 &\ge \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| \\
&= \left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}-\frac{y}{\|x\|}+\frac{y}{\|y\|}\right\| \\
&\ge \left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}\right\| - \left\|\frac{y}{\|x\|}-\frac{y}{\|y\|}\right\| \\
&= \left(\frac{1}{\|x\|}\right)\|x+y\| - \|y\|\left(\frac{1}{\|x\|}-\frac{1}{\|y\|}\right) \\
&= \left(\frac{1}{\|x\|}\right)\left(\|x\|+\|y\|\right) - \|y\|\left(\frac{1}{\|x\|}-\frac{1}{\|y\|}\right) \\
&= 1 + \frac{\|y\|}{\|x\|} - \frac{\|y\|}{\|x\|} + 1 \\
&= 2 .
\end{aligned}$$

Therefore $\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| = 2$.

By the rotundity of $\| \cdot \|$, we get $\frac{x}{\|x\|} = \frac{y}{\|y\|}$, which means that $x = \lambda y$ for $\lambda = \frac{\|x\|}{\|y\|} > 0$.

(iii) $\Rightarrow$ (i) is obvious.

2. Basic Rotund Renormings

2.1 Lemma

Let X be a Banach space and $|.|$ be an equivalent norm on X^*. Then the following conditions are equivalent:

(i) The norm $|.|$ is a dual norm to some norm $|.|^*$ on X.

(ii) The norm $|.|$ is w^* - lower semicontinuous on X^*.

(iii) The unit ball B of $|.|$ is w^* - closed in X^*.

Proof See [2].

2.2 Theorem

Let X and Y be two Banach spaces such that Y is dual space that admits a dual LUR norm and let $T: Y \to X$ be a bounded linear operator with $T(Y)$ norm dense in X.

(i) If T is weak^* to weak continuous, then X admits an LUR norm.

(ii) If X is a dual space and T is weak^* to weak^* continuous, then X admits a dual LUR norm.

Proof

Let $\|.\|$ be the original norm on X such that $\|.\|$ is a dual norm in case (ii), and let $|.|$ be an equivalent dual LUR norm on Y. For $x \in X$ and $n \in N$, define

$$|x|_n^2 = \inf\left\{\|x - Ty\|^2 + \frac{1}{n}|y|^2 : y \in Y\right\} \qquad (2.1)$$

and

$$|||x|||^2 = \sum_{n=1}^{\infty} 2^{-n}|x|_n^2 . \qquad (2.2)$$

It is easy to check that (2.1) defines a norm on X.

From (2.1), $|x|_n^2 = \|x - Ty\|^2 + \frac{1}{n}|y|^2$ for all $y \in Y$.

Since $0 \in Y$, $|x|_n \le \|x\|$.

For the converse inequality, we consider two cases separately as

$|y| \le \frac{1}{2}\|T\|^{-1}$ and $|y| \ge \frac{1}{2}\|T\|^{-1}$.

For the former case, $\|Ty\| \le \|T\||y| \le \frac{1}{2}$ which implies that

$\|x - Ty\| \ge \|x\| - \|Ty\| \ge \frac{1}{2}$ for $\|x\| = 1$.

Thus $\|x - Ty\|^2 + \frac{1}{n}|y|^2 \ge \|x - Ty\|^2 \ge \frac{1}{4}$ and so

$|x|_n^2 = \inf\left\{\|x - Ty\|^2 + \frac{1}{n}|y|^2 : y \in Y\right\} \ge \frac{1}{4}$.

For the latter case, when $|y| \ge \frac{1}{2}\|T\|^{-1}$,

we have $\|x - Ty\|^2 + \frac{1}{n}|y|^2 \ge \frac{1}{n}\cdot\frac{1}{4}\left(\|T\|^{-1}\right)^2$.

Thus $|x|_n^2 \geq \frac{1}{n} \cdot \frac{1}{4}\left(\|T\|^{-1}\right)^2$.

Let $K = \min\left\{\frac{1}{2}, \frac{1}{2} n^{\frac{-1}{2}} \|T\|^{-1}\right\}$.

Then $|x|_n \geq K$ if $\|x\| = 1$.

If $u \neq 0$, put $x = \frac{1}{\|u\|} u$. Then $\|x\| = 1$ and therefore $\frac{1}{\|u\|}|u|_n \geq K$ and

$|u|_n \geq K\|u\|$ for all $u \in X$.

Therefore $|x|_n \leq \|x\| \leq \frac{1}{K}|x|_n$ for all $x \in X$.

From (2.2)
$$\begin{aligned} \||x\||^2 &= \sum_{n=1}^{\infty} \frac{1}{2^n} |x|_n^2 \\ &\leq \|x\|^2 \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \|x\|^2 . \end{aligned}$$

On the other hand, $\||x\|| \geq \frac{1}{\sqrt{2}} |x|_1 \geq M\|x\|$.

Hence $\||.\||$ and $\|.\|$ are equivalent.

2.3 Theorem

Assume that a Banach space Z admits a rotund norm and that X is a Banach space such that there exists a bounded linear one-to-one operator T of X into Z. Then X admit a rotund norm. If, Moreover $X = Y^*$ for some Y and if the operator T is w^*-w-continuous, then $X = Y^*$ admits a dual round norm.

Proof

Define an equivalent norm for $x \in X$ by $\|x\|^2 = |x|^2 + \||Tx\||^2$, where $|.|$ is the original norm on X and $\||.\||$ is a rotund norm on Z.

Let $x, y \in X$.

Then
$$\begin{aligned} 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = &\left(2|x|^2 + 2|y|^2 - |x+y|^2\right) \\ &+ \left(2\|Tx\|^2 + 2\|Ty\|^2 - \|Tx+Ty\|^2\right) \qquad (2.3) \end{aligned}$$

To show that $\|.\|$ is a rotund norm on X, let x and $y \in X$ be such that

$$2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0 . \qquad (2.4)$$

By using (2.3) and (2.4) it follows

$$2\||Tx\||^2 + 2\||Ty\||^2 - \||Tx+Ty\||^2 = 0 . \qquad (2.5)$$

According to the rotundity of the norm $\||.\||$ on Z, (2.5) implies that $Tx = Ty$ by Proposition 1.10.

Since T is one-to-one, we have $x = y$.

By Proposition 1.10, $\|.\|$ is a rotund norm on X.

Let $X = Y^*$ and T be a bounded linear one-to-one operator of Y^* into Z, which is w^*-w-continuous.

Let the norm $\|.\|$ be defined on Y^* for $f \in Y^*$ by $\|f\|^2 = |f|^2 + \|\|Tf\|\|^2$, where $|.|$ is the canonical norm on Y^* and $\|\|.\|\|$ is a rotund norm on Z.

The same argument as above gives that $\|.\|$ is a rotund norm on Y^*.

Note that $|.|$ is w^*-lower semicontinuous on Y^*, that any norm on Z is w-lower semicontinuous on Z and that T is a w^*-w-continuous operator.

The norm $\|.\|$ is therefore w^*-lower semicontinuous on X and thus it is a dual norm on Y^* by Lemma 2.1.

2.4 Lemma

(i) If X is a separable Banach space, then there is a bounded w^*-w-continuous operator $T : X^* \to l_2(N)$ such that $T^* : l_2^*(N) = l_2(N) \to X$ is w^*-w-continuous and $T^*\left(l_2^*(N)\right)$ is dense in X.

(ii) If X^* is separable, then there is a bounded linear operator $T : X \to l_2(N)$ such that $T^*\left(l_2^*(N)\right) = T^*\left(l_2(N)\right)$ is norm dense in X^*.

Proof See [2].

2.5 Theorem

(i) If X is a separable Banach space, then X admit an LUR norm and X^* admits a dual R norm.

(ii) If X^* is separable, then X^* admits a dual LUR norm.

Proof

(i) Let X be a separable Banach space. Then, by Lemma 2.4 (i), there is a bounded linear w^*-w- continuous operator $T : X^* \to l_2(N)$ such that $T^* : l_2(N) \to X$ is w^*-w- continuous. Therefore X admits an LUR norm by Theorem 2.2 and X^* admits a dual R norm by Theorem 2.3.

(ii) Let X^* be a separable Banach space.

Then by Lemma 2.4 (ii), there is a bounded linear operator $T : X \to l_2(N)$ such that $T^*\left(l_2^*(N)\right) = T^*\left(l_2(N)\right)$ is norm dense in X^*. Therefore by Theorem 2.2, X^* admits a dual LUR norm.

2.6 Theorem

(i) If X^* is separable, then X admits a Fréchet differentiable norm.

(ii) If X is separable, then X admits a Gâteaux differentiable norm.

Proof

(i) Let X^* be separable.

By Theorem 2.5 (ii), X^* admits a dual LUR norm and by Proposition 1.7, $\|.\|$ is Fréchet differentiable.

Therefore (i) is proved.

(ii) Let X be separable. By Theorem 2.5 (i), X^* admits a dual rotund norm.

Therefore by Proposition 1.8 (i), $\|.\|$ is a Gâteaux differentiable norm.

Thus (ii) is proved.

3. The James Tree Space JT

3.1 Definition

Let $(T,<)$ be a partially ordered set. A **chain** is a subset of T which is totally ordered by $<$.

3.2 Definition

A **tree** is a partially ordered set $(T,<)$ such that for every $t \in T$, the set $\{s \in T : s \le t\}$ is well-order by $<$. The elements of T are called **nodes**. The predecessor node of x is the maximal element of $\{y \in T : y < x\}$.

An **antichain** is a subset S of T such that two different elements of S are incomparable.

3.3 Definition

For a tree T, the **James tree space JT** is defined as the completion of $c_{00}(T) = \{f \in \mathbb{R}^T : |\operatorname{supp}(f)| < \omega\}$ endowed with the norm

$$\|f\| = \sup\left\{\left(\sum_{i=1}^{n}\left(\sum_{t\in\sigma_i} f(t)\right)^2\right)^{\frac{1}{2}}\right\},$$

where the supremum runs over all finite families of disjoint segments $\sigma_i, \dots, \sigma_n$ of the tree T. The space JT is l_2-saturated, that is every subspace contains a copy of l_2 and in particular JT does not contain l_1.

3.4 Definition

An element $h^* \in \mathbb{R}^T$ induces a linear map $c_{00}(T) \to \mathbb{R}$ given by $h^*(x) = \sum_{t\in T} h^*(t)x(t)$. When such a linear map is bounded for the norm of JT, then h^* defines an element of the **dual space JT^***.

That is the case when h^* is the characteristic function of a segment σ of the tree, χ_σ^* for which we have indeed $\|\chi_\sigma^*\| = 1$. Namely, if we take an element $x \in c_{00}(T)$ of the norm less than or equal to one we will have, taking only the segment σ in the definition of the norm of JT, that $\left|\sum_{i\in\sigma} x(i)\right| \le 1$, and that is the action of χ_σ^* on x.

3.5 Definition

For each $t \in T$, there exists a unique ordinal $r(t)$ with the same order type as $(0,t)$. A tree T is **Hausdorff** if whenever $r(t)$ is a limit ordinal and $(0,t') = (0,t)$ we have $t = t'$.

3.6 Definition

Let T be a Hausdorff tree. $\mathbf{C_0(T)}$ is the space of real-valued functions f on T, which are continuous for the locally compact topology and are such that for all $\varepsilon > 0$ the set $\{t \in T : |f(t)| \ge \varepsilon\}$ is compact for that topology.

3.7 Notation

James tree spaces JT cannot be used to provide example of separable Banach spaces with non LUR renormable dual. We denote by $\overline{T}$, the completed tree of T, the tree whose nodes are the initial segments of the tree T ordered by inclusion.

We use $T \subset \overline{T}$ by identifying every $t \in T$ with the initial segment $\{s \in T : s \le t\}$. A result if Brackebusch states that for every tree T, JT^{**} is isometric to $J\overline{T}$ where $\overline{T}$ is the completed tree of T.

3.8 Definition

A Banach space X is called **weakly countably determined** (WCD) if there exists a countable collection $\{K_n : n \ge 1\}$ of w^*-compact subset of X^{**} such that for every $x \in X$ and $u \in X^{**} \setminus X$ there exists n_0 such that $x \in K_{n_0}$ and $u \notin K_{n_0}$.

3.9 Definition

A Banach space X is said to be **weakly compactly generated** (WCG) if there exists a weakly compact subset W of X that spans a dense linear subspace in X.

3.10 Theorem

Let T be a tree. The Banach space $C_0(T)$ admits an equivalent norm with the Kadec property if and only if there exists an increasing function $\rho : T \to \mathbb{R}$ with no bad points.

Proof See [3].

3.11 Theorem

Let T be a tree.

(i) If $C_0(\overline{T})$ admits an equivalent strictly convex norm, then JT^* also admits an equivalent strictly convex norm.

(ii) If $C_0(\overline{T})$ admits an equivalent LUR norm, then JT^* also admits an equivalent LUR norm.

Proof

Let $C_0(\overline{T})$ admits an equivalent strictly convex norm.

Then for every element $x^* \in JT^*$ as a continuous function on $\overline{T} \cup \{\infty\}$ vanishing at ∞, and thus to define an operator,

$$F : JT^* \to C_0(\overline{T}).$$

Then F is one-to-one operator and one-to-one operators transfer strictly convex renorming. Therefore JT^* admits an equivalent strictly convex norm.

Moreover, F has the additional property that the dual operator

$$F^* : C_0(\overline{T})^* \to JT^{**} \cong J\overline{T}$$

has dense range, because for every Dirac measure δ_s, $s \in \overline{T}$, we have that $F^*(\delta_s) = e_s$.

One to one operators whose duals have dense range transfer LUR renorming.

Therefore JT^* admits an equivalents LUR norm.

3.12 Theroem

Let X be a Banach space such that X^* is weakly countably determined. Then X admits LUR norm whose dual norm is also LUR.

Proof See [2]

3.13 Proposition

Let T be a tree and X be a separable subspace of JT. Then X^{**} is a subspace of a weakly compactly generated and hence X^{*} admits an equivalent LUR norm.

Proof

Let T_1 be a countable set such that $X \subset \overline{\text{span}}\left(\left\{\chi_{\{t\}} : t \in T_1\right\}\right) \cong JT_1$.

Since T_1 is a countable tree, it has countable height $\text{ht}\left(\overline{T}_1\right) = \alpha < \omega_1$ and the height of the completed tree cannot be larger, $\text{ht}\left(\overline{T}_1\right) \le \alpha + 1 < \omega_1$.

Then $\overline{T}_1$ is a countable union of antichains and $J\overline{T}_1$ is weakly compactly generated.

Finally, $JT_1^{**} \cong J\overline{T}_1$, so X^{**} is a subspace of a weakly compactly generated space.
Therefore, by Theorem 3.12, X^{*} is LUR renormable.

3.14 Theorem

For a tree T the following are equivalent:
(1) JT is weakly compactly generated,
(2) JT is weakly countably determined,
(3) T is the union of countably many antichains,
(4) $T = \bigcup_{n<\omega} S_n$ where for every $n < \omega$, S_n contains no infinite chain.

Proof See [1].

3.15 Lemma

Let T be any tree and suppose that there exists an equivalent Kadec norm on JT. Then there exist
(a) a countable partition of $\overline{T}$, $\overline{T} = \bigcup_{n<\omega} T_n$ and
(b) a function $F : \overline{T} \to 2^{T}$ which associates to each initial segment $\sigma \in \overline{T}$ a finite set $F(\sigma)$ of immediate successors of σ, such that for every $n < \omega$ and for every infinite chain $\sigma_1 < \sigma_2 < \ldots$ contained in T_n there exists $k_0 < \omega$ such that $F(\sigma_k) \cap \sigma_{k+1} \ne \varnothing$ for every $k > k_0$.

Proof See [1].

3.16 Theorem

Let T be a tree which is the union of countably many antichains. The following are equivalent:
(i) $\overline{T}$ is also the union of countably many antichains.
(ii) JT^{*} admits an equivalent Kadec norm.
(iii) JT^{*} admits an equivalent LUR norm.

Proof

By a consequence of the result of Troyanski[4], a Banach space admits an equivalent LUR norm if and only if it admits an equivalent strictly convex norm and also an equivalent Kadec norm.
Therefore (ii) and (iii) are equivalent.

Let T be the tree which is the union of countably many antichains.
Suppose that $\overline{T}$ is also the union of countably many antichains.
By Theorem 3.14, if T is the union of countably many antichains, the JT^* is weakly compactly generated.
For any tree T, JT^{**} is isometric to $J\overline{T}$ where $\overline{T}$ is the completed tree of T.
Then JT^{**} is weakly compactly generated.
By Theorem 3.12, JT^* admits an equivalent LUR norm.
Thus (i) implies (ii).
Now we show that (ii) implies (i).
Assume that T is the union of countably many antichains, $T = \bigcup\limits_{m<\omega} R_m$ and that it verifies the conclusion of Lemma 3.15 for a decomposition $\overline{T} = \bigcup\limits_{n<\omega} T_n$ and a function F.
Then we show that $\overline{T}$ is the union of countably many antichains.
For every $n < \omega$ and every finite subset A of natural numbers, we consider the set

$$S_{n,A} = \left\{ \sigma \in \overline{T} : \sigma \in T_n \text{ and } F(\sigma) \subset \bigcup_{m\in A} R_m \right\}. \tag{3.1}$$

(3.1) gives an expression of $\overline{T}$ as countable union $\overline{T} = \bigcup\limits_{n,A} S_{n,A}$.
By Condition (4) of Theorem 3.14, $T = \bigcup\limits_{n<\omega} S_n$ where for every $n < \omega$, S_n contains no infinite chain.
Suppose by contradiction that we have an infinite chain $\sigma_1 < \sigma_2 < \ldots$ inside a fixed $S_{n,A}$.
Since $S_{n,A} \subset T_n$ there exists k_0 such that $F(\sigma_k) \cap \sigma_{k+1} \neq \emptyset$ for every $k > k_0$, say $t_k \in F(\sigma_k) \cap \sigma_{k+1} \subset \bigcup\limits_{m\in A} R_m$.
Then $t_1 < t_2 < \ldots$ is an infinite chain of T contained in $\bigcup\limits_{m\in A} R_m$ which is a finite union of antichains.
This is a contradiction.
Therefore $\overline{T}$ is also the union of countably many antichains.

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