

Analysis of First Generalized IIR Digital Butterworth Filter Design Using MATLAB

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Abstract— Filtering involves the manipulation of the spectrum of a signal by passing or blocking certain portions of the spectrum, depending on the frequency of those portions. Filters are designed according to what kind of manipulation of the signal is required for a particular application. This paper introduces a first generation IIR digital filters that unifies the classical digital Butterworth filters and the well known maximally at FIR filters. In the design of IIR filters, a commonly used approach is called the bilinear transformation. This design begins with the transfer function of an analog filter, and then performs a mapping from the s-domain to the z-domain. New closed form expressions are provided, and a straight forward design technique is described. The first generation IIR digital filters have more zeros than poles (away from the origin) and their (monotonic) square magnitude frequency responses are maximally flat at $\omega = 0$ and at $\omega = \pi$. Another result of the paper is that, for a specified cut-off frequency and a specified number of zeros, there is only one valid way in which to split the zeros between $z = -1$ and the pass band. This technique also permits continuous variation of the cut-off frequency. IIR filters having more zeros than poles are of interest, because often, to obtain a good trade-off between performance and implementation complexity, just a few poles are best. The results have been demonstrated with the aid of a MATLAB script. The design process of a digital filter is long and tedious if done by hand.

Keywords— IIR, Filter Design, Butterworth Design, MATLAB and First Generalization

I. INTRODUCTION

Probably the best known and most commonly used method for the design of IIR digital filters is the bilinear transformation of the classical analog filters (the Butterworth, Chebyshev I and II, and Elliptic filters). One advantage of this technique is the existence of formulas for these filters. However, the numerator and denominator of such IIR filters have equal degree. It is sometimes desirable to be able to design filters having more zeros than poles (away from the origin) to obtain an improved compromise between performance and implementation complexity.

The first generalized formulas used in this paper unify the classical digital Butterworth filters and the well known maximally at FIR filters described by Herrmann. The new maximally at low pass IIR filters have an unequal number of zeros and poles and possess a specified half magnitude frequency. It is worth noting that not all the zeros are restricted to lie on the unit circle, as is the case for some previous design techniques for filters having an unequal number of poles and zeros. The method consists of the use of a formula and polynomial root finding. No phase approximation is done; the approximation is in the magnitude squared as are the classical IIR filters types.

Another result of the paper is that for a specified number of zeros and a specified half magnitude frequency, there is only one valid way to divide the number of zeros between $z = -1$ and the pass band.

Given a half-magnitude frequency ω_0 , the filters produced by the formulas described below are optimal (maximally at) in the following sense: the maximum number of derivatives at $\omega_0 = 0$ and $\omega_0 = \pi$ are set to zero, under the constraint that the filters possesses the half-magnitude frequency ω_0 and a monotonic frequency response magnitude.

II. GENERAL INSTRUCTIONS

Let $H(z) = B(z)/A(z)$ denote the transfer function of a digital filters. Its frequency response magnitude is given by $|H(e^{j\omega})|$. Throughout this paper, the degree of $B(z)$ will be denoted by $L+M$, where L is the number of zeros at $z = -1$ and M is the number of remaining zeros. The zeros at $z = -1$ produce a flat behavior in the frequency response magnitude at $\omega = \pi$, while the remaining zeros, together with the poles, are used to produce a flat behavior at $\omega = 0$. The half-magnitude frequency is that frequency at which the magnitude equals one half. Like the 3 dB point, it indicates the location of the transition band. The meanings of the parameters are shown in Table 1. It should be noted that by 'degree of flatness', the author means the number of derivatives that are made to match the desired response, including the 0th derivative.

TABLE I
NOTATION

Parameters	
L+M	Total number of zeros
L	Number of zeros at $z=-1$
M	Number of zeros contributing to the pass band
N	Total number of poles
w_0	Half-magnitude frequency
x_0	$\frac{1}{2}(1-\cos w_0)$
Flatness	
L+M+N	Total degree of flatness
M+N	Degree of flatness at $w=0$
L	Degree of flatness at $w=\pi$

III. DESIGN FORMULAS

The approach described below uses the mapping $x = 1/2(1 - \cos w)$ and provides formulas for two nonnegative polynomials $P(x)$ and $Q(x)$. A stable IIR filters $B(z)/A(z)$ is obtained having a magnitude squared frequency response $|H(e^{jw})|^2$ given by

$$|H(e^{jw})|^2 = \frac{P(\frac{1}{2} - \frac{1}{2}\cos w)}{Q(\frac{1}{2} - \frac{1}{2}\cos w)} \quad (1)$$

Accordingly, $F(x) = P(x)/Q(x)$ is designed to approximate a low pass response over $x \in [0, 1]$. $B(z)$ and $A(z)$ are most conveniently found by first computing the roots of $P(x)$ and $Q(x)$ and by then mapping those roots to the z -plane via

$$z = 1 - 2x \pm \sqrt{1 - 2x} \quad (2)$$

For stable minimum-phase solutions, take the sign of the square root yielding points inside the unit circle. The author begins with the classical digital Butterworth filter. This establishes notation and makes the generalization more clear.

A. Classical Digital Butterworth Filter

Assume $L \leq N$ and $M = 0$, then the rational function $F(x) = P(x)/Q(x)$ is given by

$$F(x) = \frac{(1-x)^L}{(1-x)^L + cx^N} \quad (3)$$

The classical Butterworth filters is obtained when $N = L$. Note that $|H(e^{jw/2})|^2 = F(1/2) = 1/(1+c \cdot 2^{-L-N})$. Clearly, c should be chosen so that this value lies between 0 and 1. Therefore, c must be greater than zero.

Choosing c to achieve a specified half-magnitude frequency is straight-forward. The equation $|H(e^{jw_0})| = 1/2$

becomes $F(x_0) = 1/4$ where $x_0 = 1/2(1-\cos w_0)$. Solving this equation for c , one obtains $c = 3(1-x_0)^L/x_0^N$. Because this expression is positive for all $x_0 \in (0, 1)$, any half-magnitude $w_0 \in (0, \pi)$ is achievable when $L \leq N$ and $M = 0$.

B. First Generalization

For the first generalization, assume $L > N$, $M = 0$. Then, introducing the notation T_N for polynomial truncation (discarding all terms beyond the N^{th} power), $F(x)$ can be written as

$$F(x) = \frac{(1-x)^L}{T_N\{(1-x)^L\} + cx^N} \quad (4)$$

The term, c , is the free parameter that, as in the classical case, can be chosen to achieve a specified half-magnitude frequency, and must be chosen to lie within an appropriate range. The allowable ranges for c are given in Table 2. When c is chosen to lie in the ranges shown in the table, $F(x)$ is obtained as shown in the following range: $0 < F(x) < 1$ for $x \in (0, 1)$.

To choose c to achieve a specified half-magnitude frequency w_0 , solve $F(x_0) = 1/4$ for c . This yield

$$c = \frac{4(1-x_0)^L \cdot T_N\{(1-x)^L\}(x_0)}{x_0^N} \quad (5)$$

The value of this expression given for c may or may not lie in the appropriate range shown in Table 2. If it does not, then the specified half-magnitude frequency is too high for the current choice of L and N . It should be noted that, although the pass band can be made arbitrarily narrow, it cannot be made arbitrarily wide for a fixed L and N (when $L > N$).

TABLE II
PERMISSIBLE RANGES FOR c FOR THE FIRST GENERALIZATION

<i>N even</i>	$C \geq 0$
<i>N odd</i>	$C \geq \binom{L-1}{N}$

TABLE III
THE NUMBER AND LOCATIONS OF THE REAL ROOTS OF $T_N\{(1-x)^L\} + cx^N - 4(1-x)^L = 0$ FOR $L > N > 0$

	L even	L odd
N even $C=0$	2 real roots: $x_1 \in (0, 1), x_2 > 1$	1 real root: $x_1 \in (0, 1)$
N odd $c = \binom{L-1}{N}$	2 real roots: $x_1 \in (0, 1), x_2 = 1$	3 real roots: $x_1 \in (0, 1), x_2 = 1, x_3 > 1$

The greatest half-magnitude frequency achievable for a fixed L and N can be found by setting c equal to the appropriate value

