The Eccentricity Sequence of a Graph

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Abstract

There are several interesting nonnegative integer sequences (for example, eccentric sequences, distance-sum sequences, branch-weight sequences) associated with vertices in graphs. In this paper, the eccentricity sequences of graphs are studied and some properties of graphs having preassigned eccentric sequence are expressed. Moreover, the necessary and sufficient condition for a sequence to be eccentric is discussed.

Keywords: Eccentricity, eccentric sequence

Introduction

A graphical sequence, namely, eccentricity sequence which is distance based, is introduced to study various graph properties. The eccentricity sequence is a nonnegative integer sequence associated with vertices in a graph.

In this paper, only undirected connected finite graphs without loops are considered. An eccentricity of a vertex, and a center of a graph are defined. Some useful definitions and their examples are expressed.

Some properties of eccentricity sequence are discussed. The eccentricity sequences of some special classes of graphs are characterized.

The Eccentricity of a Vertex and the Center of a Graph

In this section, some definitions which are useful for the later discussions. For other graph-theoretic terms are not defined but used in this paper, they are referred to Bondy and Murty (1982).

The Eccentricity of a Vertex. Let G be a connected graph with a vertex set $V(G)$ and an edge set $E(G)$. If x and y are two vertices in G, the ***distance*** between x and y is denoted by $d_G(x, y)$ and defined as the length of a shortest path joining them. For a vertex v in G, the ***eccentricity*** $e_G(v)$ of v is the distance between v and the farthest vertex from it in G, more precisely

$$e_G(v) = \max \{d_G(v, x) : x \in V(G)\}.$$

In the sequel if it is not complex, $V(G)$, $E(G)$, $d_G(x, y)$ and $e(u)$ will be denoted by V, E, $d(x, y)$ and $e(u)$ respectively.

Example. The eccentricity of each vertex of a graph G is shown in Figure 1.1 where each number in parenthesis next to each vertex represents its eccentricity.

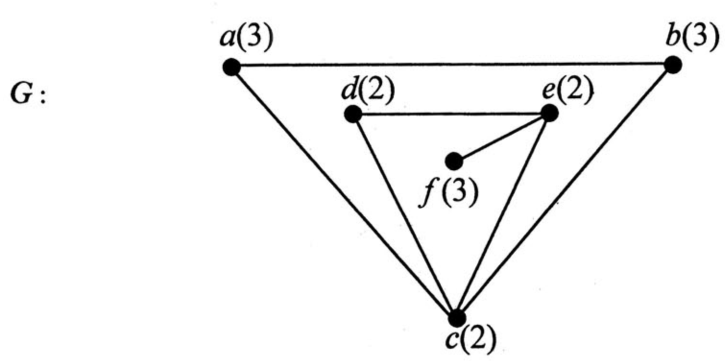

Figure 1.1

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For the vertex a in G , $d(a, b) = 1$, $d(a, c) = 1$, $d(a, d) = 2$, $d(a, e) = 2$, $d(a, f) = 3$. Thus

$$e(a) = \max \{d(a, x) : x \in V\} = 3.$$

Similarly, the eccentricities of remaining vertices of the graph G can be found.

The Center of a Graph. Let G be a connected graph with a vertex set V and an edge set E. The ***center*** $C(G)$ of G is the set of all vertices having minimum eccentricity. Thus

$$C(G) = \{u \in V : e(u) \leq e(v) \text{ for all } v \in V\}.$$

Example. In the graph G of Figure 1.1,

$e(a) = 3$, $e(b) = 3$, $e(c) = 2$,
$e(d) = 2$, $e(e) = 2$, $e(f) = 3$.

Therefore $C(G) = \{c, d, e\}$.

Jordan [1869] obtained the following result on the center of a tree.

Theorem. The center of a tree consists of either a vertex or a pair of adjacent vertices.

Definitions. Let G be a connected graph with the vertex set V and the edge set E. The ***radius*** of G, denoted by $rad\ G$, is the minimum eccentricity of vertices in G. Thus

$$rad\ G = \min \{e(v) : v \in V\}.$$

The ***diameter*** of G, denoted by $diam\ G$, is the maximum eccentricity of vertices in G. Thus

$$diam\ G = \max \{e(v) : v \in V\}.$$

Example. For the graph G of Figure 1.1, $rad\ G = 2$ and $diam\ G = 3$.

The following theorem which can be easily proved is useful for the later discussion.

Theorem. For any connected graph G, the radius and diameter satisfy

$$rad\ G \leq diam\ G \leq 2\ rad\ G.$$

The Eccentricity Sequence of a Graph and Some Properties

In this section, the eccentricity sequences of a connected graph will be investigated.

The Eccentricity Sequence of a Graph. A nondecreasing sequence $S(a_1, a_2, \ldots, a_p)$ of nonnegative integers is called an ***eccentric sequence*** if there exists a connected graph G whose vertices can be labelled $v_1, v_2, \ldots, v_p$ so that $e(v_i) = a_i$ for all i. In this case S is said to be the ***eccentricity sequence*** of G.

Example. The nondecreasing sequence (3, 3, 4, 4, 4, 4, 4, 4, 5, 5) is the eccentricity sequence of the graph G shown in Figure 1.2.

G:

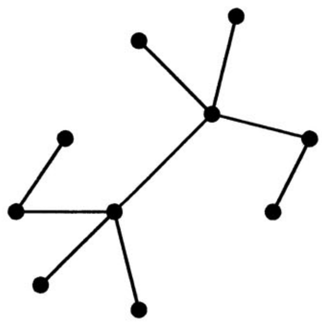


Figure 1.2

Now some properties of eccentric sequences will be discussed.

Theorem. If a nondecreasing sequence $(a_1, a_2, \ldots, a_p)$ where $p \geq 2$ is eccentric, then the following must hold:

(i) $a_1 \leq p/2$;

(ii) If k is any integer with $a_1 < k \leq a_p$, then $a_i = a_{i+1} = k$ for some i $(2 \leq i \leq p-1)$;

(iii) $a_p \leq \min\{(p-1), 2a_1\}$.

Proof. Let G be a connected graph with eccentricity sequence $(a_1, a_2, \ldots, a_p)$.

(i) The vertices of G can be labelled $v_1, v_2, \ldots, v_p$ so that $e(v_i) = a_i$ for all i. Choose a spanning tree T of G which is distance-preserving from v_1. Then $e_G(v_1) = e_T(v_1)$. For $2 \leq i \leq p$, $e_G(v_i) \leq e_T(v_i)$. Therefore if $(a_1^*, a_2^*, \mathrm{K}, a_p^*)$ is the eccentricity sequence of T, $a_1^* = a_1$. Thus it suffices to show that if $(a_1, a_2, \ldots, a_p)$ is the eccentricity sequence of a tree T, then $a_1 \leq p/2$. For $p = 2$, the result holds. Assume that $p \geq 3$. Suppose, to the contrary, that $a_1 \geq (p+1)/2$. Let u be a vertex of T with $e(u) = a_1$. Clearly, u must be a cutvertex of T. Since $e(u) \geq (p+1)/2$, the graph $T - u$ has a component H with $|V(H)| \geq (p+1)/2$. Let v be the unique vertex of H adjacent to u in T. For $w \in V(H)$, $d(v, w) = d(u, w) - 1$ which implies that $d(v, w) < e(u)$. For $w \in V(T) - V(H)$, $d(v, w) = d(u, w) + 1$.

Since $|V(T) - V(H)| \leq (p-1)/2$, it follows that $d(u, w) \leq (p-3)/2$. Thus $d(v, w) \leq (p-1)/2 < e(u)$. But then $e(v) < e(u)$ which is a contradiction.

(ii) To show that if $P : u_0, u_1, \ldots, u_m$ is a path in G with $e(u_0) < e(u_m)$ and k is an integer with $e(u_0) < k \leq e(u_m)$, then there exists an integer j, $0 < j \leq m$, such that $e(u_j) = k$. Since $u_i u_{i+1} \in E(G)$, $0 \leq i \leq m-1$, $e(u_{i+1}) \leq e(u_i) + 1$.

Let $j = 1 + \max\{i \mid e(u_i) < k\}$. Then $e(u_{j-1}) < k$ and so $e(u_j) \leq e(u_{j-1}) + 1 \leq k$. By the choice of j, $e(u_j) \geq k$. Thus $e(u_j) = k$.

In order to complete the proof of part (ii), it suffices to show that if w is a vertex of G with $e(w) > a_1$ and k is an integer such that $a_1 < k \leq e(w)$, then there is a vertex of G other than w with eccentricity equal to k. Let $u \in V(G)$ with

$d(u, w) = e(w)$, let $v \in V(G)$ with $e(v) = a_1$, and let $P : v = u_0, u_1, \ldots, u_m = u$ be a (v, u)-path in G. Note that $m \le a_1$. Now, since

$$e(v) = a_1 < k \le e(w) = d(u, w) \le e(u),$$

by the preceding paragraph there exists an integer j, $0 < j \le m$, such that $e(u_j) = k$. Moreover, since

$$d(u, u_j) < m \le a_1 < e(w) = d(u, w),$$

$u_j \ne w$.

(iii) Clearly, $a_p \le p - 1$. For every connected graph G, $diam\ G \le 2\ rad\ G$. Since $a_1 = rad\ G$ and $a_p = diam\ G$, the result follows. ∎

Remark. The conditions in Theorem (2.3) are necessary for a sequence to be eccentric, but these conditions are not sufficient. For example, the sequence (3, 4, 4, 5, 5, 5) is not eccentric.

The following theorem is a necessary and sufficient condition for a sequence to be eccentric.

Theorem. A nondecreasing sequence $S(a_1, a_2, \ldots, a_p)$ with m distinct values is eccentric if and only if some subsequence of S with m distinct values is eccentric.
Proof. If S is eccentric, then S is an eccentric subsequence of itself with m distinct values.

For the converse, suppose S^* is an eccentric subsequence with m distinct values. Let G be a graph with eccentricity sequence S^* and let $t_1, t_2, \ldots, t_m$ be the distinct values that occur in S^*. For each t_i, $1 \le i \le m$, select a vertex w_i of G whose eccentricity in G is t_i. For each i, $1 \le i \le m$, let n_i equal one more than the number of occurrences of t_i in S less the number of occurrences of t_i in S^*. In G, replace w_1 with K_{n_1} and join each vertex of K_{n_1} to all vertices adjacent to w_1 in G, to obtain a new graph, say G_1. Again in G_1, replace w_2 with K_{n_2} and join each vertex of K_{n_2} to all vertices adjacent to w_2 in G_1, to get a new graph, say G_2. Continue in this way to obtain the graph G_m. It is not difficult to see that S is the eccentricity sequence of G_m. ∎

The proof technique of Theorem (2.5) with an example will be illustrated.
Example. Let $S(2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4)$ be a nondecreasing sequence with three distinct values $t_1 = 2$, $t_2 = 3$ and $t_3 = 4$.

Consider the sequence $S^*(2, 2, 2, 3, 3, 3, 3. 4. 4, 4, 4, 4, 4)$ which is a subsequence of S with three distinct values $t_1 = 2$, $t_2 = 3$ and $t_3 = 4$, also is the eccentricity sequence of the graph G_1 shown in Figure 1.3.

G_1 :

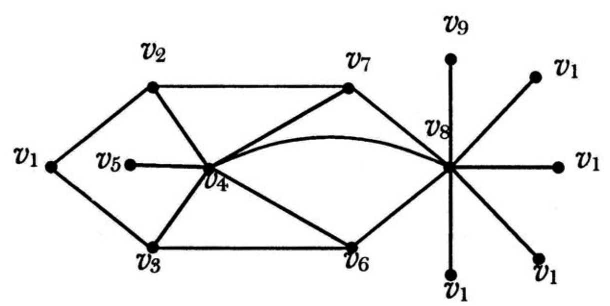


Figure 1.3

In fact

$e(v_4) = e(v_6) = e(v_7) = 2,$
$e(v_2) = e(v_3) = e(v_5) = e(v_8) = 3,$
$e(v_1) = e(v_9) = e(v_{10}) = e(v_{11}) = e(v_{12}) = e(v_{13}) = 4.$

Thus $S^*(2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4)$ is the eccentricity sequence of G_1. For $t_1 = 2$, choose the vertex v_6 with $e(v_6) = 2$.

Then

n_1 = the number of the occurrences of t_1 in S
– the number of the occurrences of t_1 in $S^* + 1$
$= 4 - 3 + 1$
$= 2.$

Replacing the vertex v_6 by K_2 with the vertex set $\{x_1, x_2\}$ and joining each vertex of K_2 to all vertices adjacent to v_6 in G_1, the graph G_2 with the eccentricity sequence (2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4) shown in Figure 1.4 is obtained.

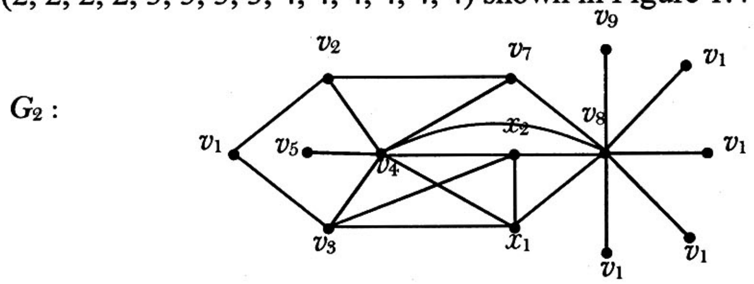


Figure 1.4

For $t_2 = 3$, choose the vertex v_5 with $e(v_5) = 3$. Then $n_2 = 2$. Replacing the vertex v_5 by K_2 with the vertex $\{y_1, y_2\}$ and joining each vertex of K_2 to all vertices adjacent to v_5 in G_2, the graph G_3 with the eccentricity sequence (2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4) as shown in Figure 1.5 is obtained.

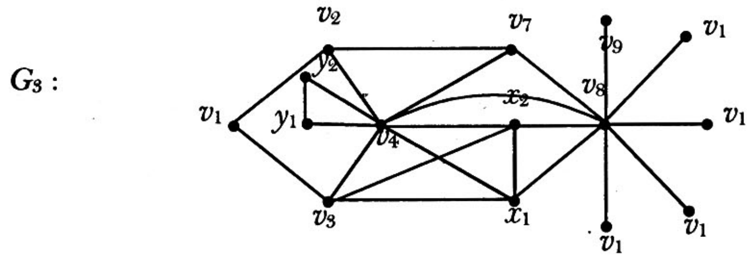


Figure 1.5

For $t_3 = 4$, choose the vertex v_{12} with $e(v_{12}) = 4$. Then $n_3 = 2$.

Replacing the vertex v_{12} by K_2 with the vertex $\{z_1, z_2\}$ and joining each vertex of K_2 to all vertices adjacent to v_{12} in G_3, the graph G_4 with the eccentricity sequence $S(2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4)$ as shown in Figure 1.6 is obtained.

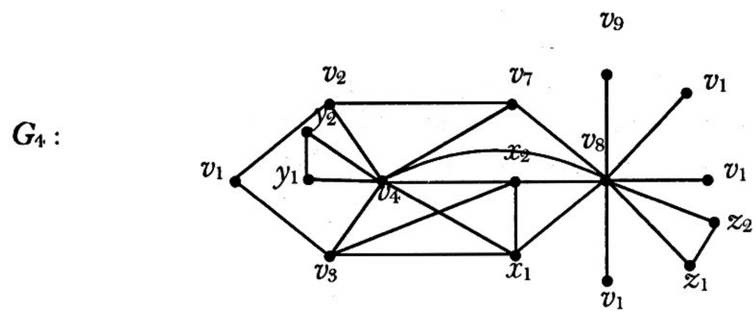


Figure 1.6

Now, a corollary to Theorem (2.5) may be stated.

Corollary. Let $S(a_1, a_2, \ldots, a_p)$ be a sequence of nonnegative integers, where $a_i = a$ for all i. Then S is eccentric if and only if $a = 0$ and $p = 1$ or $1 \leq a \leq p/2$.

Proof. Suppose S is eccentric. If $a \neq 0$, since $0 \leq a \leq p/2$ by Theorem (2.3), it follows that $1 \leq a \leq p/2$.

If $a = 0$, then $p = 1$.

For the converse, the following cases are considered.

Case 1. Suppose $a = 0$ and $p = 1$. In this case, S is the eccentricity sequence of K_1.

Case 2. Suppose $a = 1$. Since $a \leq p/2$, $p \geq 2$. Thus S is the eccentricity sequence of K_p .

Case 3. Suppose $a \geq 2$. Since $a \leq p/2$, $p \geq 2a \geq 4$. Consider the subsequence $S^*(a_1, a_2, \ldots, a_{2a})$, which is the eccentricity sequence of the cycle C_{2a}. By Theorem (2.5), the sequence S is eccentric.

Hence in each possible case, S is eccentric. ■

The Eccentricity Sequences of Some Special Classes of Graphs

In this section the eccentricity sequences of some special classes of graphs, namely complete graphs K_p, complete bipartite graphs $K_{m,n}$, cycles C_n of order n, and paths P_n of order n (here n is the number of vertices in the corresponding graph) will be characterized.

The Eccentricity Sequence of a Complete Graph. The sequence (1,1, …,1) with p terms is the eccentricity sequence of a complete graph $K_p (p \geq 2)$.

Example. The sequence (1,1) is the eccentricity sequence of complete graph K_2 shown in Figure 1.7(a). Similarly, the sequence (1, 1, 1, 1, 1) is the eccentricity sequence of K_5 shown in Figure 1.7(b).

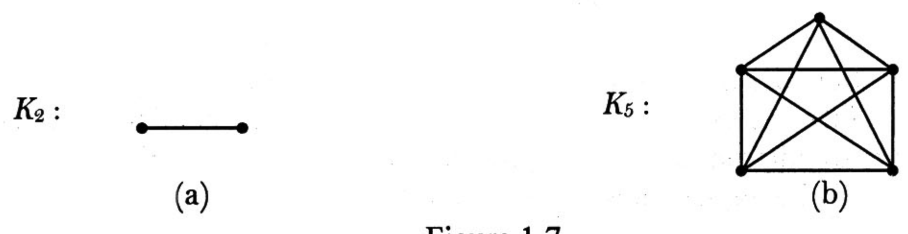


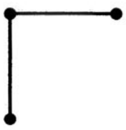
Figure 1.7

The Eccentricity Sequence of a Complete Bipartite Graph. The sequence (1, 2, 2, …, 2) is the eccentricity sequence of a complete bipartite graph $K_{m,n}$ ($m = 1, n \geq 2$). For $2 \leq m \leq n$, the sequence (2, 2, …, 2) with n terms is the eccentricity sequence of $K_{m,n}$.

Example. Figure 1.8(a) shows that the sequence (1, 2, 2) is an eccentricity sequence of the complete bipartite graph $K_{1,2}$ ($m = 1$, $n = 2$).

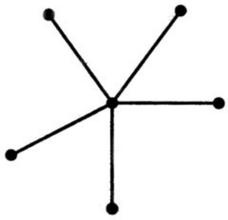
Similarly, the sequence (1, 2, 2, 2, 2, 2) is the eccentricity sequence of $K_{1,5}$ shown in Figure 1.8(b).

$K_{1,2}$:



(a)

$K_{1,5}$:

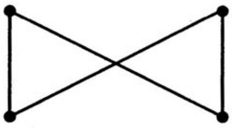


(b)

Figure 1.8

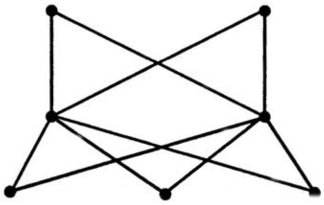
The sequence (2, 2, 2, 2) is the eccentricity sequence of $K_{2,2}$ shown in Figure 1.9(a) and the sequence (2, 2, 2, 2, 2, 2, 2) is also the eccentricity sequence of $K_{2,5}$ as shown in Figure 1.9(b).

$K_{2,2}$:



(a)

$K_{2,5}$:



(b)

Figure 1.9

The Eccentricity Sequence of a Cycle. The sequence $(\frac{n}{2}, \frac{n}{2}, \mathrm{K}, \frac{n}{2})$ with n terms is the eccentricity sequence of a cycle C_n of order n where n is even.

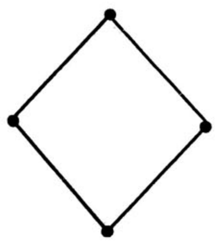
If n is odd, the eccentricity sequence of the cycle C_n is $(\frac{n-1}{2}, \frac{n-1}{2}, \mathrm{K}, \frac{n-1}{2})$ with n terms.

Example. The sequence (1, 1) is the eccentricity sequence of a cycle C_2 shown in Figure 1.10(a). The sequence (2, 2, 2, 2) is the eccentricity sequence of C_4 shown in Figure 1.10(b).

C_2 :

(a)

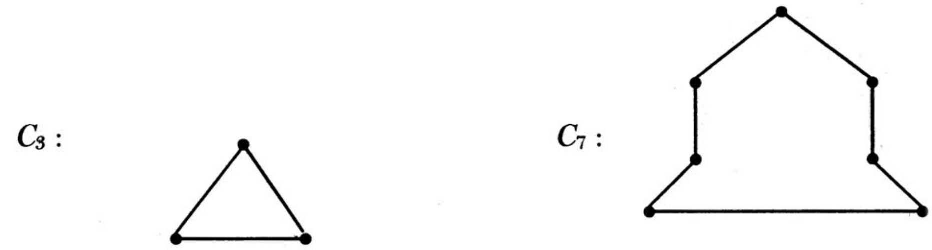
C_4 :



(b)

Figure 1.10

Similarly, the sequence (1, 1, 1) is the eccentricity sequence of cycle C_3 shown in Figure 1.11(a) and the sequence (3, 3, 3, 3, 3, 3, 3) is the eccentricity sequence of C_7 shown in Figure 1.11(b).



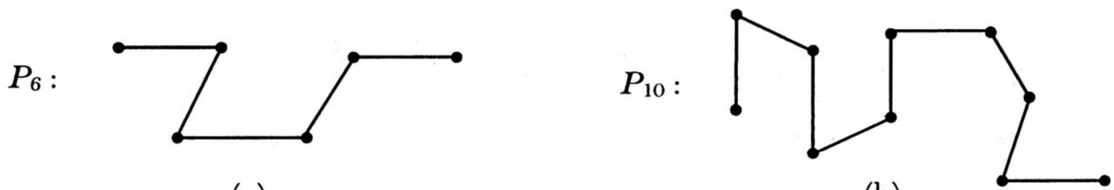
(a) (b)

Figure 1.11

The Eccentricity Sequence of a Path. The sequence $(\frac{n}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+2}{2}, \ldots, \frac{2n-2}{2}, \frac{2n-2}{2})$ is an eccentricity sequence of a path P_n of order n where n is even.

If n is odd, the eccentricity sequence of the path P_n is $(\frac{n-1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+3}{2}, \ldots, \frac{2n-2}{2}, \frac{2n-2}{2})$.

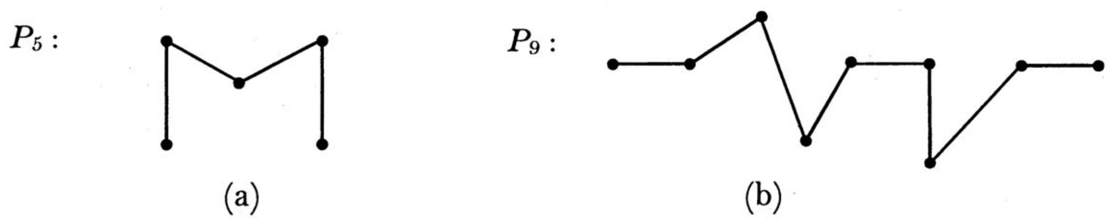
Example. The sequence (3, 3, 4, 4, 5, 5) is the eccentricity sequence of the path P_6 shown in Figure 1.12(a) and the sequence (5, 5, 6, 6, 7, 7, 8, 8, 9, 9) is the eccentricity sequence of P_{10} shown in Figure 1.12(b).



(a) (b)

Figure 1.12

Similarly, the sequence (2, 3, 3, 4, 4) is the eccentricity sequence of path P_5 shown in Figure 1.13(a) and the sequence (4, 5, 5, 6, 6, 7, 7, 8, 8) is also the eccentricity sequence of P_9 shown in Figure 1.13 (b).



(a) (b)

Figure 1.13

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