SOLVABLE GROUPS AND ITS RELATED RESULTS

Dr. Phyu Phyu Khin[15]

ABSTRACT

Some properties related to solvable groups are investigated together with the solvability of symmetric group. Moreover, some solvable groups of order under 120 are also explored.

Introduction

Solvable groups which have many applications, including applications in Galois Theory. In this paper, two parts are organized, the first one concerns commutator subgroup and the second one relates to solvability of symmetric groups. Relevant results are also studied.

Preliminaries

Definition. Let $N \subseteq G$ be a group, and $g \in G$. A ***left coset*** gN and ***right coset*** Ng of N in G are defined, respectively, as $gN = \{gn \mid n \in N\}$ *and* $Ng = \{ng \mid n \in N\}$.

Let G be a group. A ***normal subgroup*** N of G, written $N \lhd G$, is a subgroup of G such that for all $g \in G, gN = Ng$.

The number of distinct right cosets of N in G is called the ***index*** of N in G and it is denoted by $[G:N]$.

Proposition. Let G be a group. A subgroup N of G is normal if and only if for every $g \in G$ and $n \in N, gng^{-1} \in N.$

Proposition. Every subgroup of an abelian group G is normal in G.

Lemma. Let G be a group and N be a normal subgroup of G. Then $gN = N$ *implies* $g \in N$.

Definition. Let S be a nonempty set and Sym(S) is the set of all bijections of S onto itself. Then Sym(S) is a group, called ***symmetric group*** under the operation of composition of functions.

If $S = \{1, 2, \dots, n\}$, then Sym(S) is called the symmetric group on n letters and it is denoted by S_n and the order of S_n *is* $n!$ An element in S_n is called a ***transposition*** if it interchanges two elements, leaving the other fixed.

---

[15] Associate Professor, Department of Mathematics, Yangon University of Economics

The permutation $\sigma \in S_n$ is an ***odd permutation*** if σ is the product or composition of an odd number of transpositions, and is an ***even permutation*** if σ is the product of an even number of transpositions.

S_n has a normal subgroup of A_n, which we called the ***alternating group*** of degree n, which is a group of order $\frac{n!}{2}$ if $n \geq 2$. In fact, A_n was merely the set of all even permutations in S_n.

Example. If *G* is a group and *N* is a subgroup of index 2 in *G*, then we can prove that *N* is a normal subgroup of *G.*

If $N \leq G$ and $[G: N]=2$, there are precisely two right and left cosets of *N* in *G.* The right cosets are *N* and *Nx* where $x \notin N$, and the left cosets are *N* and *xN* where $x \notin N$, but $Nx = G - N$ and $xN = G - N$. Thus *Nx* = *xN.* So every left coset of *N* in *G* is a right coset of *N* in *G.* Then we get $N \lhd G$.

Theorem. If *G* is a finite group and *N* is a subgroup of *G*, then $|N|$, is a divisor of $|G|$.

Example. The converse of above theorem is false.

Let $G = S_3$, where S_3 is symmetric group and let $N = \{e, (12), (123)\}$. Then $|G| = |S_3| = 6$ *and* $|N| = 3$. So $|N|$ is a divisor of $|S_3|$, but *N* is not a subgroup of S_3.

Commutator Subgroup

Definition. Let *G* be a group and *N* be normal subgroup of *G.* The ***quotient group (factor group)*** of *G* with *N,* written G/N, is the set of all cosets of *N* in *G* under the operation $(aN)(bN) = (ab)N$ for all $a,b \in G$. Note that the identity of G/N is simply *N.*

Note: If *G* is abelian, then any factor group G/N is abelian.

Definition. Let *G* be a group. The commutation of two elements $a,b \in G$ is the element $aba^{-1}b^{-1}$. The commutation of two elements is a ***commutator***. $aba^{-1}b^{-1}$ is usually denoted by $[a,b]$.

Let *G* be a group. The ***commutator subgroup*** G' of *G* is defined as $G' = < aba^{-1}b^{-1} : a,b \in G >$

Note: The commutator subgroup is the group generated by all the commutators of G.

Example. The inverse of a commutator is a commutator.

Let *G* be a group and $a,b \in G$, Commutator $[a,b] = aba^{-1}b^{-1} = z$, so $z^{-1}=(aba^{-1}b^{-1})^{-1}=(b^{-1})^{-1}(a^{-1})^{-1}b^{-1}a^{-1}=bab^{-1}a^{-1}=[b,a]$,for all $a,b\in G$.

Example. G is abelian if and only if $G'=\{e\}$.

Suppose G is abelian and $a,b\in G$. As a and b commute, we have ab = ba.

$[a,b]=aba^{-1}b^{-1}=baa^{-1}b^{-1}=b^{-1}eb=bb^{-1}=e$.Then G' is a subgroup of G generated by e and $G'=\{e\}$.

Suppose $G'=\{e\}$, then any commutator $[a,b]=aba^{-1}b^{-1}=e$.

Hence $(aba^{-1}b^{-1})b=eb$

$$(aba^{-1})(b^{-1}b)=b$$
$$(aba^{-1})e=b$$
$$aba^{-1}=b$$
$$aba^{-1}a=ba$$
$$(ab)(a^{-1}a)=ba$$
$$(ab)e=ba$$
$$ab=ba$$

, for all $a,b\in G$. Thus G is abelian.

Example. Let N be a normal subgroup of G. Then G/N is abelian if and only if $[x,y]\in N$ for all $x,y\in G$.

Assume that G/N is abelian and take any x ,y in G .Then the product of the factor groups xN and yN can be written as $(xy)N=(xN)(yN)=(yN)(xN)=(yx)N$.

$(xyx^{-1}y^{-1})N=((xy)(yx)^{-1})N=(xy)N(yx)^{-1}N=(yx)N(yx)^{-1}N=((yx)(yx)^{-1})N=N$.

Therefore $[x,y]=xyx^{-1}y^{-1}\in N$.

Assume that $[x,y]\in N$ for any $x,y\in G$.Since N is normal, then G/N is defined and $[y^{-1},x^{-1}]N=N$. Recall this is the identity element.

so $(xy)N=(xy)N[y^{-1},x^{-1}]N=((xy)[y^{-1},x^{-1}])N=((xy)(y^{-1}x^{-1})(yx))N=(yx)N$.

Hence, $(xN)(yN)=(xy)N=(yx)N=(yN)(xN)$ .Therefore G/N is abelian.

Theorem. Let G be a group and G' be its commutator subgroup then G' is a normal subgroup of G.

Example. If G is a group and G', the commutator subgroup of G.Then G/G' is an abelian.

Let G be a group and elements a,b in G. Then the commutator of a and b is the element

$a^{-1}b^{-1}ab$ in G'.

We have to show that G/G' is an abelian.

Given any two elements aG', bG' in G/G' for some $a, b \in G$.

Then $(aG')(bG') = (ab)G'$ since G' is a normal.

$= (baa^{-1}b^{-1})abG' = (ba)(a^{-1}b^{-1}ab)G'$

$= (ba)G'$ since $a^{-1}b^{-1}ab \in G'$

$= (bG')(aG')$ since G' *is a normal*. Therefore G/G' is an abelian.

Solvability of Symmetric Groups

Definition. A group G is said to be ***solvable*** if we can find a finite chain of subgroups $G = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_k = \{e\}$ where each N_i is a normal subgroup of N_{i-1} and such that every factor group N_{i-1}/N_i is an abelian $(i = 1, 2, \ldots, k)$.

Example. Every abelian group is solvable.

Let G be abelian group.

Let $N_0 = G$ *and* $N_1 = \{e\}$. Then $G = N_0 \supset N_1$.

By proposition, $N_1 \lhd G$. So G has a finite chain and G/N_1 is abelian since G is abelian and G/N_1 is any factor group of G. Thus G is solvable.

Example. The symmetric group S_n is solvable for $n = 1, 2, 3, 4$.

For $n = 1$, $S_1 = \{e\}$.

We know that the set $\{e\}$ is normal subgroup of S_1 and $\{e\}$, identity set is abelian.

Take $N_0 = S_1$ and $N_1 = \{e\}$. Then $S_1 = N_0 \supset N_1 = \{e\}$. So, S_1 has a finite chain and N_0/N_1 is abelian. Hence S_1 is a solvable.

For $n = 2$, let $S_2 = \{e, (12)\}$.

Take $N_0 = S_2$ and $N_1 = \{e\}$. Then $[N_0 : N_1] = 2$ and $N_1 \lhd N_0$.

Then $S_2 = N_0 \supset N_1 = \{e\}$. So S_2 has a finite chain. Hence N_0/N_1 is abelian and S_2 is a solvable.

For $n = 3$, let $S_3 = \{e, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3\}$, and

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

So $|S_3| = 6$. Let $S_3 = N_0$.

Take $N_1 = \{e, \sigma_1, \sigma_2\}$ and $N_2 = \{e\}$. Then $|N_1| = 3$.We know that $N_1$is a cyclic subgroup of S_3.Thus $[N_0 : N_1] = 2$ and $N_1 \lhd N_0$. And then $N_2 \lhd N_1$ and $[N_1 : N_2] = 3$

Therefore $S_3 = N_0 \supset N_1 \supset N_2 = \{e\}$.So S_3has a finite chain and N_{i-1}/N_i is abelian, $i = 1,2$

Thus S_3 is a solvable.

For $n = 4$, let $N_0 = S_4$ be the symmetric group. So $|S_4| = 24$.

Take $N_1 = A_4$,the alternating group and $N_1 = \{e, \sigma_2, \sigma_5, \sigma_8, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8\}$.

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \sigma_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \sigma_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \tau_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

$$\tau_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \tau_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}, \tau_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \tau_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Then $|N_1| = 12$and N_1 is a subgroup ofS_4.Thus$[N_0 : N_1] = 2$ and $N_1 \lhd N_0$.

Take$N_2 = \{e, \sigma_2, \sigma_5, \sigma_8\}$. So $|N_2| = 4$ and we see that N_2 is a subgroup ofN_1 and$N_2 \lhd N_1$ .And then$[N_1 : N_2] = 3$, so $N_1/N_2$is abelian.

Let $N_3 = \{e, \sigma_2\}$ and $|N_3| = 2$ .Then $N_3$is a subgroup ofN_2 and $N_3 \lhd N_2$.And then $[N_2 : N_3] = 2$, so N_2/N_3 is abelian.

Finally, let $N_4 = \{e\}$.Therefore$S_4 = N_0 \supset N_1 \supset N_2 \supset N_3 \supset N_4 = \{e\}$. So S_4has a finite chain with N_{i-1}/N_i is abelian for$i = 1,2,3,4$. Hence $S_4$is a solvable.

Lemma. G is solvable if and only if $G^{(k)} = \{e\}$ for some integer k.

Definition. Let S_n be symmetric group. If $a_1, a_2, ..., a_m$ are distinct integers in $\{1, 2, 3, ..., n\}, (a_1, a_2, ..., a_m)$ stands for the permutation that maps $a_1 \to a_2, a_2 \to a_3, ..., a_{m-1} \to a_m, a_m \to a_1$ and maps every other elements of $\{1, 2, 3, ..., n\}$ onto itself. $(a_1, a_2, ..., a_m)$ is called a ***cycle*** of length m or ***m-cycle***.

Lemma. Let $G = S_n$ where $n \geq 5$, then $G^{(k)}$ for $k = 1, 2, 3, ...$ contains every 3-cycle of S_n

Proof. Let G be an arbitrary group. We know that if N is normal subgroup of G, then N' must also be normal subgroup of G. We claim that if N is a normal subgroup of $G = S_n$ where $n \geq 5$, which contains every 3-cycle in S_n, then N' must also contain every 3-cycle.

For, suppose $a = (123), b = (145) \in N$ (using here that $n \geq 5$).The

$$aba^{-1}b^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix} = (153)$$, as a commutator of elements of N must be in N'.

Since N' is a normal subgroup of G, for any $\pi \in S_n$, $\pi^{-1}(153)\pi$ must also be in N'.

Choose a π in S_n such that $\pi(1) = i_1, \pi(5) = i_2$ *and* $\pi(3) = i_3$ where i_1, i_2, i_3 are any three distinct integers in the range from 1 to n; then $\pi^{-1}(153)\pi = (i_1, i_2, i_3)$ is in N'.

Thus N' contains all 3- cycle. Let $N = G$, which is certainly normal in G and contains all 3-cycle, we get G' contains all 3-cycle.

Since G' is normal in G, $G^{(2)}$ contains all 3- cycles. Since $G^{(2)}$ is normal in G, $G^{(3)}$ contains all 3-cycle. Continuing this way we obtain that $G^{(k)}$ contains all 3- cycles for arbitrary k.

Theorem. S_n is not solvable for $n \geq 5$.

Proof. If $G = S_n$, then by lemma, $G^{(k)}$ contains all 3-cycles in S_n for every k.

Therefore $G^{(k)} \neq \{e\}$ for any k. Hence by Lemma, G cannot be solvable.

Some Basic Theorems on Solvable Groups

Definition. Let p be a prime number that divides the order of G. Let k be the biggest natural number such that p^k divides G. All the subgroups of G with order p^k are called ***p-Sylow subgroups*** of G. We denote their set to be Syl_pG and $Syl_pG = N_p$.

1st Sylow Theorem. If p is a prime number and p^s divides the order of G then G has at least one subgroup of order p^s.

2nd Sylow Theorem. Every two p- Sylow subgroups of G are conjugate.

3rd Sylow Theorem. N_p divides the order of G and it is equivalent to $1 \bmod p$.

Theorem-Tool. If G is a group and N is a normal subgroup of G such that N is solvable and G/N is solvable then G is solvable.

Theorem. If $|G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime is solvable.

Theorem. If $|G| = pqr$ where $p < q < r$ primes then G is solvable.

Some Related Results

(1) If $|G| = 2^k.3$ for $k \geq 2$ then G is solvable.

Proof. By induction on k,

(i) For $k = 1$, $|G| = 2^1.3 = 6$, since then G has only one 3-sylow subgroup H which is normal, cyclic, abelian of order 3 and the quotient G/H is cyclic abelian of order 2. Thus G is solvable.

(ii)Let the above proposition hold for all $k = 1, 2, 3, ..., n$.

(iii)We will prove that it holds for $k = n + 1$.

From Sylow's theorems we know that G contains at least one 2-Sylow subgroup of order 2^{k+1}. Lets call that H which is normal, cyclic, abelian of order 2. Then $i(H) = 3$ thus $2^{k+1}.3$ does not divide $3! = 6$. Thus H contains a normal subgroup of G, say K. But $|K| = 2^m$ thus by theorem, H is solvable. Also $|G/K| = 3.2^{k-m}$ thus by (ii) G/K is solvable. Finally from Tool Theorem, G is solvable.

(2) If $|G| = 3^k.2^2$ then G is solvable.

Proof. By induction on k,

(i) It holds for $k = 1$, then $|G|$=12 since $|G|$=2^2.3 and it is solvable by (1).

(ii) Let the above proposition holds for all $k = 1, 2, 3, ..., n$ with $n \geq 1$.

(iii) We prove that it holds for $k = n+1$. Thus $n+1 \geq 2$. From Sylow's theorem we know that G contains at least on 3-Sylow subgroup of order 3^{n+1}. Lets call that H which is normal, cyclic, abelian of order 3. Then $i(H) = 2^2$.

Thus $3^{n+1}.2^2 = 3^2.2^2.3^{n+1-2} = 36.3^{n+1-2}$ does not divide $2^2! = 24$. Thus H contains a normal subgroup of G, say K. But $|K| = 3^m$ thus by theorem H is solvable. Also $|G/K| = 2^2.3^{k-m}$ thus by (ii) G/K is solvable. Finally by Tool-Theorem G is solvable.

(3) If $|G| = 2^k.5$ then G is solvable.

Proof. By induction on k.

(i) It holds for $k = 1, 2, 3$ where $|G| = 10, 20, 40$ respectively because we can use theorem in this case.

(ii) Let the above proposition holds for all $k = 1, 2, 3, ..., n$ with $n \geq 3$.

(iii) We prove that it holds for $k = n+1$. Thus $k = n+1 \geq 4$. From Sylow's theorems we know that G contains at least one 2-Sylow subgroup of order 2^{n+1}. Lets call that H which is normal, cyclic, abelian of order 2.Then $i(H) = 5$.

Thus $2^{n+1}.5 = 2^4.5.2^{n+1-4} = 80.2^{n+1-4}$ does not divide $5! = 120$. Thus H contains a normal subgroup of G, say K. But $|K| = 2^m$ thus by theorem H is solvable. Also $|G/K| = 5.2^{k-m}$ thus by (ii), G/K is solvable. Finally from Tool-Theorem G is solvable.

Example. Every group of order 56 is solvable.

Since From theorems we know that N_7 is either 1 or 8.If it is 1, G is solvable. So it is 8. Again count all the elements in those groups that are not identity. We get $8.6 = 48$ elements. But again by Sylow theorems we get there exists at least one 2-sylow subgroup of order 8.If add them, $48+8=56$ elements which leaves no room for another 2- sylow subgroup. Thus G is again solvable.

Example. Every group of order 84 is solvable.

Since $|G| = 84 = 7.3.2^2$. Consider the number of the 7-sylow subgroups of G, say N_7. Then $N_7 = 1 \bmod 7$ and N_7 divides 84. Thus $(N_7, 7) = 1$ so N_7 can be only 1.Let N_7 be normal 7-sylow subgroup P. Then P is normal, cyclic of order 7, abelian and solvable. Again $|G/P| = 12$ which is solvable by related result (1). So by tool-theorem, G is solvable.

ACKNOWLEDGEMENTS

I would like to mention my grateful thanks to Professor Dr. Khin Naing Oo, Rector of Yangon University of Economics for her kind patronage. I continue to be indebted to Professor Dr. Tun Aung, Pro-rector of Yangon University of Economics for his kind permission to present this paper. I am also deeply thankful to my Professor Dr. Myint Wai, Head of Mathematics Department, Yangon University of Economics for his kind advice and encouragement.

REFERENCES

1. Baumslag, B.& Chandler, B. **Schaum's Outline Series of Theory and Problems of Group Theory**, Mc-Graw Hill Book Company, New York,1968.
2. Herstein,I.N. **Topic in Algebra**, JohnWiley & Sons, Inc. New York, 1975.
3. Singh S& Zameeruddin Q, **Modern Algebra** UBS Publisher Distributors Ltd, New Delhi, 1999.
4. Wisnesky, R.J. **Solvable Groups**, Harvard University, Cambridge, USA, 2005
5. Szymoniak B, **"On the existence of normal subgroups of prime index"** Mathematics Journal, Rose-Hulman Institute of Technology, Volume 17, No-1, spring 2016